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Transition from decay to blow-up in a parabolic system

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Abstract. We show a locally uniform bound for global nonnegative solutions of the system $u_t = \Delta u + uv - bu$, $v_t = \Delta v + au$ in $(0, +\infty) \times \Omega$, $u = v = 0$ on $(0, +\infty) \times \partial\Omega$, where $a > 0$, $b \geq 0$ and Ω is a bounded domain in $\mathbb{R}^n$, $n \leq 2$. In particular, the trajectories starting on the boundary of the domain of attraction of the zero solution are global and bounded.

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1 Introduction

In many parabolic problems possessing blowing-up solutions, there also exist global bounded solutions. The large-time behavior of solutions lying on the borderline between global existence and blow-up may be quite complicated and its knowledge may be useful e.g. in the study of stationary solutions of these problems (see [8]).

Let us consider first the scalar problem

$$\left.\begin{aligned} u_t &= \Delta u + u|u|^{p-1} + f(x,t,u,\nabla u), && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_o(x), && x \in \Omega, \end{aligned}\right\} \tag{P}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $p > 1$ and f represents a perturbation term. If $f \equiv 0$, $0 \not\equiv U_o \geq 0$ is a smooth function, $\lambda > 0$ and $u_o = \lambda U_o$

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then the solution u_λ of (P) exists globally and $u_\lambda(t) \to 0$ as $t \to +\infty$ for λ small while u_λ blows up in finite time in the $L^\infty(\Omega)$-norm if λ is large. If we put $\lambda_o = \sup\{\lambda\,;\, u_\lambda \text{ exists globally}\}$ and if we consider only radially decreasing solutions in a ball then it is known (see [4], [5]) that the solution u_{λ_o}

- is global and bounded for p subcritical, i.e. $p < (n+2)/(n-2)$ if $n > 2$,
- is global and unbounded for p critical,
- blows up in finite time for p supercritical (and $n \le 10$).

Similarly, if $n = 1$ and $f(x,t,u,u_x) = \varepsilon(u^m)_x$, where $\varepsilon > 0$ and $m > 1$ then the solution u_{λ_o}

- is global and bounded (at least for some) $p > 2m - 1$,
- cannot be global and bounded if $p \le 2m - 1$ and ε is "large".

Sufficient conditions for global existence and boundedness of the solution u_{λ_o} for $f \not\equiv 0$ and a more detailed discussion of the above facts can be found in [7].

In the present note we study the system

$$\left.\begin{aligned} u_t &= \Delta u + uv - bu, && x \in \Omega,\ t > 0,\\ v_t &= \Delta v + au, && x \in \Omega,\ t > 0,\\ u &= v = 0, && x \in \partial\Omega,\ t > 0,\\ u(x,0) &= u_o(x) \ge 0, && x \in \Omega,\\ v(x,0) &= v_o(x) \ge 0, && x \in \Omega, \end{aligned}\right\} \tag{S}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $n \le 2$, $a > 0$ and $b \ge 0$. It was shown in [6] that the system (S) possesses a positive stationary solution. Moreover, any positive stationary solution $(\tilde u, \tilde v)$ of (S) represents a threshold between blow-up and decay to zero provided Ω is a ball. More precisely,

- if $\lambda < \mu \le 1$, $0 \le u_o \le \lambda\tilde u$ and $0 \le v_o \le \mu\tilde v$ then the solution of (S) exists globally and tends to zero as $t \to \infty$,
- if $\lambda, \mu > 1$, $u_o \ge \lambda\tilde u$ and $v_o \ge \mu\tilde v$ then the solution of (S) blows up in finite time.

We are interested in the behavior of all "threshold trajectories", i.e. trajectories starting on the boundary ∂D_A of the domain of attraction of the zero solution

$$\begin{aligned} D_A = \{&(u_o, v_o) \in H_0^1(\Omega)^+ \times H_0^1(\Omega)^+\,;\\ &\text{the solution } (u,v) \text{ of (S) exists globally and } (u(t),v(t)) \to 0 \text{ as } t \to \infty\}, \end{aligned}$$

where $H_0^1(\Omega)^+$ is the positive cone of the usual Sobolev space $H_0^1(\Omega)$. We shall prove the boundedness of any non-negative global trajectory of (S). Since the corresponding bound is locally uniform with respect to the initial values (u_o, v_o), this result implies global existence and boundedness of all trajectories starting on ∂D_A.

Our proof is based on a non-trivial generalization of *a priori* estimates for stationary solutions in [6] (based on the method of Brézis and Turner [1]) to *a priori* estimates for all global solutions of (S). Such generalization sometimes may yield satisfactory results (see e.g. the optimal result in [4] for the problem (P) with $f \equiv 0$, $u_o \ge 0$ based on the method of *a priori* estimates of Gidas and Spruck); in general, it usually requires additional assumptions. This is also the case of our proof: the *a priori* estimates in [6] were shown for a general domain $\Omega \subset \mathbb{R}^n$ if $n \leq 3$. For technical reasons, we had to restrict ourselves to the case $n \leq 2$.

Finally let us note that the boundedness of global solutions of problems of the type (P) is well known in the case where $f(x,t,u,\nabla u)$ is independent of t and ∇u (see e.g. [2], [3] and the references therein). Then the problem has variational structure, i.e. it admits a Lyapunov functional. A perturbation result for f depending on t and ∇u can be found in [7]. Anyhow, in our situation the system (S) does not seem to be "close" to any problem with variational structure.

2 Results and proofs

Throughout the rest of this paper we shall assume that the initial couple $(u_o, v_o) \in H_0^1(\Omega)^+ \times H_0^1(\Omega)^+$ is such that the corresponding solution (u,v) of (S) exists globally (in the classical sense). Moreover, we shall assume $u_o \not\equiv 0$ and we denote by λ_1 and φ_1 the first eigenvalue and the corresponding (positive) eigenfunction of the problem $-\Delta\varphi = \lambda\varphi$ in Ω, $\varphi = 0$ on $\partial\Omega$. We denote by $\|\cdot\|_p$ and $\|\cdot\|_{H^1}$ the norm in $L^p(\Omega)$ and $H^1(\Omega)$, respectively, and we put $\|\cdot\| := \|\cdot\|_2$. We shall also briefly write $u(t)$ instead of $u(\cdot,t)$ and $\int_\Omega u\,dx$ instead of $\int_\Omega u(x,t)\,dx$. Our main result is the following theorem.

Theorem 1. *There exists a constant $C_1 = C_1(\|\nabla u_o\|, \|\nabla v_o\|)$ such that*

$$\|\nabla u(t)\| + \|\nabla v(t)\| \leq C_1 \quad \text{for any } t \geq 0.$$

The proof of Theorem 1 will follow from the following series of lemmata (see Lemma 8 and Lemma 9).

Lemma 2. *There exists a constant $C_2 = C_2(\|u_o\|, \|v_o\|)$ such that*

$$\int_\Omega v(x,t)\varphi_1(x)\,dx \leq C_2 \quad \text{for any } t \geq 0.$$

Proof. Multiplying the equations in (S) by φ_1 and integrating by parts yields

$$\Big(\int_\Omega u\varphi_1\,dx\Big)_t = -(\lambda_1 + b)\int_\Omega u\varphi_1\,dx + \int_\Omega uv\varphi_1\,dx, \tag{1}$$

$$\Big(\int_\Omega v\varphi_1\,dx\Big)_t = -\lambda_1\int_\Omega v\varphi_1\,dx + a\int_\Omega u\varphi_1\,dx. \tag{2}$$

Differentiating (2), using (1), (2), $au = v_t - \Delta v$ and integration by parts we get

$$\begin{aligned}
\Big(\int_\Omega v\varphi_1\,dx\Big)_{tt} &= -\lambda_1\Big(\int_\Omega v\varphi_1\,dx\Big)_t + a\int_\Omega(\Delta u + uv - bu)\varphi_1\,dx \\
&= -\lambda_1\Big(\int_\Omega v\varphi_1\,dx\Big)_t - a(\lambda_1 + b)\int_\Omega u\varphi_1\,dx + a\int_\Omega uv\varphi_1\,dx \\
&\geq -(2\lambda_1 + b)\Big(\int_\Omega v\varphi_1\,dx\Big)_t - \lambda_1(\lambda_1 + b)\int_\Omega v\varphi_1\,dx \\
&\quad + \frac{1}{2}\Big(\int_\Omega v^2\varphi_1\,dx\Big)_t + \frac{\lambda_1}{2}\int_\Omega v^2\varphi_1\,dx,
\end{aligned}$$

where in the last step we have used

$$\int_\Omega (-\Delta v)v\varphi_1\,dx = \int_\Omega \nabla v\cdot\nabla(v\varphi_1)\,dx$$
$$= \int_\Omega |\nabla v|^2\varphi_1\,dx + \frac{1}{2}\int_\Omega \nabla v^2\cdot\nabla\varphi_1\,dx \ge \frac{\lambda_1}{2}\int_\Omega v^2\varphi_1\,dx.$$

Hence, denoting

$$w := w(t) := \int_\Omega v(x,t)\varphi_1(x)\,dx,$$
$$y := y(t) := w'(t) + (\lambda_1+b)w(t) - \frac{1}{2}\int_\Omega v^2(x,t)\varphi_1(x)\,dx,$$

we obtain $y_t \ge -\lambda_1 y$ so that $y(t)\ge e^{-\lambda_1 t}y(0)\ge -c_0$ for some $c_0>0$. Since

$$\frac{1}{2}\int_\Omega v^2(x,t)\varphi_1(x)\,dx \ge c_1\int_\Omega v^2(x,t)\varphi_1^2(x)\,dx \ge c_2 w^2(t) \qquad \text{for some } c_1,c_2>0,$$

we have

$$-c_0 \le y \le w' + (\lambda_1+b)w - c_2w^2 \le w' - c_3w^2 + c_4 \qquad \text{for some } c_3,c_4>0,$$

hence $w' \ge c_3w^2 - (c_0+c_4)$. Since $w(t)$ exists globally, the last inequality implies $w(t)\le\sqrt{(c_0+c_4)/c_3}$ (where $c_0=c_0(v_o)$ and c_3,c_4 do not depend on v).

Lemma 3. *There exists a constant $C_3 = C_3(\|u_o\|,\|v_o\|)$ such that*

$$\int_\Omega u(x,t)\varphi_1(x)\,dx \le C_3 \quad \textit{for any } t\ge 0. \tag{3}$$

Proof. Multiplying the first equation in (S) by φ_1, integrating over Ω and over $(t,t+\theta)$, using $u=\frac{1}{a}(v_t-\Delta v)$ and Lemma 2 we get

$$\int_\Omega u\varphi_1\,dx\Big|_t^{t+\theta} \ge -(\lambda_1+b)\int_t^{t+\theta}\int_\Omega u\varphi_1\,dx\,dt$$
$$= -\frac{\lambda_1+b}{a}\int_\Omega v\varphi_1\,dx\Big|_t^{t+\theta} - \frac{\lambda_1(\lambda_1+b)}{a}\int_t^{t+\theta}\int_\Omega v\varphi_1\,dx\,dt \ge -\tilde{c},$$

where $\tilde{c}=\tilde{c}(C_2)$ does not depend on t and $\theta\in(0,1]$. Integrating the last inequality over $\theta\in(0,1)$ and using $u=\frac{1}{a}(v_t-\Delta v)$ again we obtain

$$\int_\Omega u(x,t)\varphi_1(x)\,dx - \tilde{c} \le \int_t^{t+1}\int_\Omega u\varphi_1\,dx\,dt$$
$$= \frac{1}{a}\int_\Omega v\varphi_1\,dx\Big|_t^{t+1} + \frac{\lambda_1}{a}\int_t^{t+1}\int_\Omega v\varphi_1\,dx\,dt \le C_2\frac{\lambda_1+1}{a},$$

which concludes the proof.

In what follows we shall exploit the following well known result (used also in [1] and [6]).

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain. For any $u \in H_0^1(\Omega)$, we have*

$$\|\frac{u}{\delta^r}\|_p \le C_4 \|\nabla u\|, \tag{4}$$

where $\delta = \delta(x) = dist\,(x, \partial\Omega)$, $r \in [0,1]$ and $p \le \frac{2n}{n-2(1-r)}$ $(=\frac{2}{r}$ if $n = 2)$.

Since $\delta(x) \le C_\varphi \varphi_1(x)$ for some $C_\varphi > 0$, it is now easy to show the next three lemmata.

Lemma 5. *There exists a constant $C_5 = C_5(\|u_o\|, \|v_o\|)$ such that*

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + b\|u\|^2 = \int_\Omega u^2 v\,dx \le C_5 \|\nabla u\|^{4/3}\|\nabla v\|. \tag{5}$$

Proof. The equality in (5) can be obtained by multiplying the first equation in (S) by u and integrating over Ω. Now the Hölder inequality, Lemmata 3, 4 and any choice of $\alpha, \alpha' > 1$ with $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ imply

$$\begin{aligned}
\int_\Omega u^2 v\,dx &\le \Big(\int_\Omega u\delta\,dx\Big)^{2/3}\Big(\int_\Omega u^4 v^3 \delta^{-2}\,dx\Big)^{1/3} \\
&\le (C_\varphi C_3)^{2/3}\Big(\int_\Omega \Big(\frac{u}{\delta^{1/(2\alpha)}}\Big)^{4\alpha} dx\Big)^{1/(3\alpha)}\Big(\int_\Omega \Big(\frac{v}{\delta^{2/(3\alpha')}}\Big)^{3\alpha'} dx\Big)^{1/(3\alpha')} \\
&\le (C_\varphi C_3)^{2/3} C_4^{7/3} \|\nabla u\|^{4/3}\|\nabla v\|.
\end{aligned}$$

Lemma 6. *There exists a constant $C_6 = C_6(\|u_o\|, \|v_o\|)$ such that*

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 = a\int_\Omega uv\,dx \le C_6 \|\nabla u\|^{1/2}\|\nabla v\|. \tag{6}$$

Proof. The equality in (6) follows from the second equation in (S). Now, similarly as in the proof of Lemma 5 we obtain

$$\begin{aligned}
\int_\Omega uv\,dx &\le \Big(\int_\Omega u\delta\,dx\Big)^{1/2}\Big(\int_\Omega uv^2\delta^{-1}\,dx\Big)^{1/2} \\
&\le (C_\varphi C_3)^{1/2}\Big(\int_\Omega \Big(\frac{u}{\delta}\Big)^2 dx\Big)^{1/4}\Big(\int_\Omega v^4\,dx\Big)^{1/4} \le C_6 \|\nabla u\|^{1/2}\|\nabla v\|,
\end{aligned}$$

since $H^1(\Omega)$ is imbedded in $L^p(\Omega)$ for any $p \ge 1$.

Lemma 7. *There exists a constant $C_7 = C_7(\|u_o\|, \|v_o\|)$ and for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\begin{aligned}
\|u\| \le C_7 \|\nabla u\|^{2/3}, &\qquad \|v\| \le C_7 \|\nabla v\|^{2/3}, \\
\|uv\| \le C_\varepsilon(\|\nabla u\|^{2/3+\varepsilon} &+ 1)\|\nabla v\|.
\end{aligned} \tag{7}$$

Proof. Denoting $w := u$ or $w := v$ and $C_{23} := \max(C_2, C_3)$ we get

$$\int_\Omega w^2\,dx \le \Big(\int_\Omega w\delta\,dx\Big)^{2/3}\Big(\int_\Omega\Big(\frac{w}{\delta^{1/2}}\Big)^4 dx\Big)^{1/3} \le (C_\varphi C_{23})^{2/3} C_4^{4/3}\|\nabla w\|^{4/3}.$$

Putting $K_\varepsilon = \frac{2(2+\varepsilon)}{\varepsilon}$ and using $\|w\|_p \le c_p\|\nabla w\|$ for any $p \ge 1$ we obtain

$$\begin{aligned}\int_\Omega u^2v^2\,dx &\le \Big(\int_\Omega u^{2+\varepsilon}\,dx\Big)^{2/(2+\varepsilon)}\Big(\int_\Omega v^{K_\varepsilon}\,dx\Big)^{2/K_\varepsilon}\\ &\le c_{K_\varepsilon}^2\|\nabla v\|^2\Big(\int_\Omega u^2\,dx\Big)^{(2-\varepsilon)/(2+\varepsilon)}\Big(\int_\Omega u^4\,dx\Big)^{\varepsilon/(2+\varepsilon)}\\ &\le c_{K_\varepsilon}^2 c_4^{4\varepsilon/(2+\varepsilon)} C_7^{2(2-\varepsilon)/(2+\varepsilon)}\|\nabla v\|^2\|\nabla u\|^{4/3+\varepsilon'},\end{aligned}$$

where $\varepsilon' < 2\varepsilon$.

Lemma 8. *There exists a constant $C_8 = C_8(\|\nabla v_o\|, \|\nabla u_o\|)$ such that*

$$\|\nabla v(t)\| \le C_8 \max_{0\le\tau\le t}\|\nabla u(\tau)\|^{1/2} \qquad \textit{for any } t \ge 0. \tag{8}$$

Proof. If $\frac{d}{dt}\|v(t)\|^2 \ge -\|\nabla v(t)\|^2$ then (6) implies

$$\|\nabla v(t)\| \le 2C_6\|\nabla u(t)\|^{1/2} \tag{9}$$

and we are done. Hence, let $\frac{d}{dt}\|v(t)\|^2 < -\|\nabla v(t)\|^2$. Then

$$\|\nabla v(t)\|^2 < -\frac{d}{dt}\|v\|^2 \le 2\|v\|\cdot\|v_t\| \le 2C_7\|\nabla v\|^{2/3}\cdot\|v_t\|,$$

so that

$$\|\nabla v\|^{4/3} \le 2C_7\|v_t\|. \tag{10}$$

Multiplying the second equation in (S) by v_t and integrating over Ω we get

$$\|v_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 = a\int_\Omega uv_t\,dx \le \frac{1}{2}\|v_t\|^2 + \frac{a^2}{2}\|u\|^2,$$

which together with (7) yields

$$\|v_t\|^2 + \frac{d}{dt}\|\nabla v\|^2 \le a^2\|u\|^2 \le (aC_7)^2\|\nabla u\|^{4/3}. \tag{11}$$

Now (10) and (11) imply

$$\frac{1}{(2C_7)^2}\|\nabla v\|^{8/3} + \frac{d}{dt}\|\nabla v\|^2 \le (aC_7)^2\|\nabla u\|^{4/3}. \tag{12}$$

If $\|\nabla v\| \leq (2aC_7^2)^{3/4}\|\nabla u\|^{1/2}$ then we are done. Otherwise the inequality (12) implies $\frac{d}{dt}\|\nabla v\|^2 < 0$ and putting

$$t_1 := \inf\{\tau > 0\,;\ \frac{d}{dt}\|\nabla v\|^2 < 0 \ \text{ on } (\tau, t]\}$$

we have $\|\nabla v(t)\| < \|\nabla v(t_1)\|$.

If $t_1 = 0$ then $\|\nabla v(t)\| < \|\nabla v(0)\| \leq C_0\|\nabla u(0)\|^{1/2}$ for some $C_0 > 0$. Hence, we may assume $t_1 > 0$.

If $\frac{d}{dt}\|v(t_1)\|^2 \geq -\|\nabla v(t_1)\|^2$ then the inequality (9) (with t replaced by t_1) implies

$$\|\nabla v(t)\| < \|\nabla v(t_1)\| \leq 2C_6\|\nabla u(t_1)\|^{1/2}.$$

If $\frac{d}{dt}\|v(t_1)\|^2 < -\|\nabla v(t_1)\|^2$ then the inequality (12) (with t replaced by t_1) implies

$$\|\nabla v(t)\| < \|\nabla v(t_1)\| \leq (2aC_7^2)^{3/4}\|\nabla u(t_1)\|^{1/2},$$

since the definition of t_1 implies $\frac{d}{dt}\|\nabla v(t_1)\|^2 = 0$ if $t_1 > 0$.

Lemma 9. *There exists a constant $C_9 = C_9(\|\nabla u_o\|, \|\nabla v_o\|)$ such that*

$$\|\nabla u(t)\| \leq C_9 \qquad \textit{for any } t \geq 0.$$

Proof. We may suppose $\|\nabla u(0)\| < \sup_{t\geq 0}\|\nabla u(t)\|$ (otherwise we are done). Let $t_o > 0$ be such that

$$\|\nabla u(t_o)\| = \max_{0\leq t\leq t_o}\|\nabla u(t)\|. \tag{13}$$

If $\frac{d}{dt}\|u(t_o)\|^2 \geq -\|\nabla u(t_o)\|^2$ then (5), Lemma 8 and (13) imply

$$\|\nabla u(t_o)\|^2 \leq 2C_5\|\nabla u(t_o)\|^{4/3}\|\nabla v(t_o)\| \leq 2C_5C_8\|\nabla u(t_o)\|^{11/6},$$

hence

$$\|\nabla u(t_o)\| \leq (2C_5C_8)^6.$$

Consequently, we may assume

$$\frac{d}{dt}\|u(t_o)\|^2 < -\|\nabla u(t_o)\|^2.$$

This implies

$$\|\nabla u(t_o)\|^2 < -\frac{d}{dt}\|u\|^2 \leq 2\|u\|\cdot\|u_t\| \leq 2C_7\|\nabla u\|^{2/3}\|u_t\|,$$

so that

$$\|\nabla u(t_o)\|^{4/3} \leq 2C_7\|u_t(t_o)\|. \tag{14}$$

Multiplying the first equation in (S) by u_t and integrating over Ω we obtain

$$\begin{aligned}\|u_t(t_o)\|^2 \leq \|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 &= -b\int_\Omega uu_t\,dx + \int_\Omega uvu_t\,dx \\ &\leq \frac{1}{2}\|u_t\|^2 + \|uv\|^2 + b^2\|u\|^2,\end{aligned}$$

where the inequality $\frac{d}{dt}\|\nabla u(t_o)\|^2 \geq 0$ follows from (13). Now the last inequality together with (14) and Lemmata 7, 8 imply

$$\begin{aligned}\frac{1}{(2C_7)^2}\|\nabla u(t_o)\|^{8/3} \leq \|u_t(t_o)\|^2 &\leq 2\|uv(t_o)\|^2 + 2b^2\|u(t_o)\|^2 \\ &\leq \tilde{C}_\varepsilon(\|\nabla u(t_o)\|^{4/3+2\varepsilon}+1)(\|\nabla v(t_o)\|^2+1) \\ &\leq \tilde{C}'_\varepsilon(\|\nabla u(t_o)\|^{7/3+2\varepsilon}+1),\end{aligned}$$

so that the choice $\varepsilon < 1/6$ yields the desired estimate for $\|\nabla u(t_o)\|$.

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