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ON REFINEMENTS OF CERTAIN INEQUALITIES FOR MEANS

J. Sándor

Abstract. In this paper we obtain certain refinements (and new proofs) for inequalities involving means, results attributed to Carlson; Leach and Sholander; Alzer; Sándor; and Vamanamurthy and Vuorinen.

1. The logarithmic and identric means of two positive numbers a and b are defined by

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a} \qquad \text{for} \quad a \neq b;\ L(a,a) = a$$

and

$$I = I(a,b) := \frac{1}{e}(b^b/a^a)^{1/(b-a)} \quad \text{for} \quad a \neq b;\ I(a,a) = a\,,$$

respectively.

Let $A = A(a,b) := \frac{a+b}{2}$ and $G = G(a,b) := \sqrt{ab}$ denote the arithmetic and geometric means of a and b, respectively. For these means many interesting inequalities have been proved. For a survey of results, see [1] and [6].

The aim of this note is to indicate some connections between the following inequalities. B. C. Carlson [3] proved that

$$L < \frac{2G+A}{3} \tag{1}$$

(where, as in what follows, $L = L(a,b)$, etc, and $a \neq b$) while E. B. Leach and M. C. Sholander [4] showed that

$$L > \sqrt[3]{G^2 A} \tag{2}$$

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These two inequalities appear in many proofs involving means. H. Alzer [1], [2] has obtained the following inequalities:

$$A \cdot G < L \cdot I \quad \text{and} \quad L + I < A + G \tag{3}$$

$$\sqrt{G \cdot I} < L < \frac{G+I}{2} \tag{4}$$

J. Sándor [7] has proved that the first inequality of (3) is weaker that the left side of (4), while the second inequality of (3) is stronger than the right side of (4). In fact, the above statement are consequences of

$$I > \sqrt[3]{A^2 G} \tag{5}$$

and

$$I > \frac{2A+G}{3} \tag{6}$$

Clearly, (6) implies (5), but one can obtain different methods of proof for these results (see [7]). In [6] J. Sándor has proved (Relation 21 in that paper) that

$$ln \frac{I}{L} > 1 - \frac{G}{L} \tag{7}$$

2. Particularly, as applications of (7), one can deduce (1) and the right side of (4). First we note that (e.g. from (1), (2) and (5))

$$G < L < I < A \tag{8}$$

Let $x > 1$. Then $L(x,1) > G(x,1)$ implies $ln\, x < (x-1)/\sqrt{x}$ which applied to $x = \frac{I}{L} > 1$ gives, in view of (7):

$$(I - L)\sqrt{L} > (L - G)\sqrt{I} \tag{9}$$

This inequality contains a refinement of the right side of (4), for if we put $a = \sqrt{I}/\sqrt{L} > 1$, (9) gives

$$L < \frac{I + aG}{1 + a} < \frac{I + G}{2} \tag{10}$$

since the function $a \mapsto (I + aG)/(1 + a)$ $(a \geqslant 1)$ is strictly decreasing. Now, inequality $L(x,1) < A(x,1)$ for $x > 1$ yields $ln\, x > 2(x-1)/(x+1)$. Since $ln \frac{I}{G} = \frac{A-L}{L}$ (which can be obtained immediately by simple computations) and $ln \frac{I}{L} = ln \frac{I}{G} - ln \frac{L}{G}$, from $ln \frac{L}{G} > 2 \cdot \frac{L-G}{L+G}$ and (7) one obtains

$$2 \cdot \frac{L - G}{L + G} < \frac{A + G}{L} - 2 \tag{11}$$

By $L > G$ this refines Carlson's inequality (1), since by $L + G < 2L$ one has $2(L - G)/(L + G) > 1 - G/L$, so by (11) one can derive

$$0 < \frac{(L - G)^2}{L(L + G)} < \frac{A + 2G}{L} - 3 \tag{12}$$

3. Inequality (6) and (1) improves also the right side of (4). This follows by

$$I > \frac{2A + G}{3} > 2L - G \tag{13}$$

where the second relation is exactly (1). We note that $\sqrt{G \cdot I} > \sqrt[3]{G^2 \cdot A}$ follows by (5), so from the left side of (4) one can write:

$$L > \sqrt{G \cdot I} > \sqrt[3]{G^2 \cdot A} \tag{14}$$

improving inequality (2). The left side of (4) can be sharpened also, if we use the second inequality of (3). Indeed, by the identity $ln \frac{I}{G} = \frac{A-L}{L}$ and $ln\, x < (x-1)/\sqrt{x}$ applied with $x = \frac{I}{G} > 1$ one can deduce

$$\sqrt{IG} < \frac{I - G}{A - L} \cdot L < L \tag{15}$$

Remark. Identity $ln \frac{I}{G} = \frac{A-L}{L}$ is due to H.-J. Seiffert [9]. For this and similar identities with applications, see [8].

4. Inequality (6) with (1) can be written also as

$$I > \frac{2A + G}{3} > \frac{A + L}{2} \tag{16}$$

Relation

$$I > \frac{A + L}{2} \tag{17}$$

appears also in [6], inequality (9). Since $\frac{A+L}{2} > \sqrt{AL}$, one has

$$I > \sqrt{AL} \tag{18}$$

For a simple method of proof of (18), see [7]. As an application of (18) we note that in a recent paper M. K. Vamanamurthy and M. Vuorinen [11] have proved, among other results, that for the arithmetic-geometric men M of Gauss we have

$$M < \sqrt{AL} \tag{19}$$

$$M < I \tag{20}$$

Now, by (18), relation (20) is a consequence of (19). In the above mentioned paper [11] the following open problem is stated:

Is it true that $I < \left(\frac{a^t+b^t}{2}\right)^{1/t} = S(t)$ for some $t \in (0,1)$?

We note here that by a result of A. O. Pittinger [5] this is true for $t = ln\, 2$. The reversed inequality $I > S(t)$ is valid for $t = \frac{2}{3}$ as has been proved by K. B. Stolarsky [10]. The values given by Pittinger and Stolarsky are best possible, so I and $S(t)$ are not comparable for $t < ln\, 2$ and $t > \frac{2}{3}$, respectively.

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