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# PRINCIPAL SOLUTIONS AND TRANSFORMATIONS OF LINEAR HAMILTONIAN SYSTEMS

Ondřej Došlý

*Dedicated to Professor M. Novotný on the occasion of his seventieth birthday*

Abstract. Sufficient conditions are given which guarantee that the linear transformation converting a given linear Hamiltonian system into another system of the same form transforms principal (antiprincipal) solutions into principal (antiprincipal) solutions.

## 1. Introduction.

Consider a linear Hamiltonian system

$$Y' = A(t)Y + B(t)Z, \quad Z' = -C(t)Y - A^T(t)Z, \tag{1.1}$$

where $A$, $B$, $C$ are $n \times n$ matrices of continuous, real valued functions, $t \in I = [a, \infty)$, $B$, $C$ are symmetric, i. e., $B^T = B$, $C^T = C$, and $Y, Z$ are $n \times n$ matrices. If the matrices $B, C$ are nonnegative definite, it is known that (1.1) is nonoscillatory at $\infty$ (for terminology see Section 2) if and only if the so-called reciprocal system

$$U' = -A^T(t)U + C(t)V, \quad V' = -B(t)U + A(t)V \tag{1.2}$$

is nonoscillatory at $\infty$, see [2,5,8,9].

Recently the author established a more general duality in oscillation behaviour of various linear Hamiltonian systems which may be described in the following way. If we set

$$U = H(t)Y + M(t)Z, \quad V = K(t)Y + N(t)Z, \tag{1.3}$$

where $H, K, M, N$ are $n \times n$ matrices of continuously differentiable real-valued functions such that $M(t)$ is nonsingular on $I$ and the $2n \times 2n$ matrix

$$R(t) = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix} \tag{1.4}$$

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is $J$-unitary, i.e.,

$$R^T(t)JR(t) = J, \tag{1.5}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, $I_n$ being the $n \times n$ identity matrix, then $(U, V)$ is also a solution of a linear Hamiltonian system which is under certain additional assumptions (corresponding to nonnegativeness of $B$ and $C$ in the case of reciprocal systems) nonoscillatiory at $\infty$. Obviously, if $H = 0 = N$, $M = I_n$, $K = -I_n$, the duality in oscillation behaviour of mutually reciprocal systems (1.1) and (1.2) follows from this result.

Ahlbrandt derived in [1] conditions under which a principal (antiprincipal) solution $(Y, Z)$ of (1.1) at $\infty$ is also coprincipal (anticoprincipal) at $\infty$, i. e., the solution $(U, V) = (Z, -Y)$ is a principal (antiprincipal) solution of (2.1) at $\infty$. Here we generalize this result giving conditions which quarantee that a principal (antiprincipal) solution of (1.1) is transformed by (1.3) into a principal ( antiprincipal ) solution of the new Hamiltonian system.

## 2. Definitions and preliminary results.

Simultaneously with the matrix system (1.1) consider its vector modification

$$y' = A(t)y + B(t)z, \quad z' = -C(t)y - A^T(t)z, \tag{2.1}$$

where $y, z$ are $n$-dimensional vectors. Throughout the paper we shall suppose that all differential systems are *identically normal* on $I$ (a linear Hamiltonian system of the form (2.1) is said to be identically normal on $I$ whenever the trivial solution $(y, z) \equiv (0, 0)$ is the only solution for which $y(t) \equiv 0$ on a nondegenerate subinterval of $I$).

Oscillation and nonoscillation of (2.1) are defined by means of the concept of *conjugate points*. Two points $t_1, t_2$ are said to be *conjugate* relative to (2.1) if there exists a solution $(y, z)$ of (2.1) such that $y(t_1) = 0 = y(t_2)$ and $y(t)$ is not identically zero between $t_1$ and $t_2$. System (2.1) is said to be *conjugate* on an interval $I$ whenever there exist $t_1$, $t_2 \in I$ which are conjugate relative to (2.1), in the opposite case (2.1) is said to be *disconjugate* on $I$. If there exists $c \in I$ such that (2.1) is disconjugate on $(c, \infty)$ then (2.1) is said to be *nonoscillatory* at $\infty$, in the opposite case (2.1) is said to *oscillatory* at $\infty$. In the sequel the concepts oscillatory and nonoscillatory mean always oscillatory or nonoscillatory at $\infty$.

A solution $(Y, Z)$ of (1.1) is said to be *self-conjugate* (another terminology is *prepared* [7], *self-conjoined* [9], *isotropic* [4]) if $Y^T(t)Z(t) \equiv Z^T(t)Y(t)$. Two solutions $(Y_1, Z_1)$, $(Y_2, Z_2)$ are said to be *linearly independent* if any solution $(Y, Z)$ of (2.1) can be expressed in the form $(Y, Z) = (Y_1C_1 + Y_2C_2, Z_1C_1 + Z_2C_2)$, where $C_1, C_2$ are constant $n \times n$ matrices. If $(Y_1, Z_1), (Y_2, Z_2)$ are self-conjugate then they are linearly independent if and only if the (constant) matrix $Y_1^T(t)Z_2(t) - Z_1^T(t)Y_2(t)$ is nonsingular. A self-conjugate solution $(Y_0, Z_0)$ is said to be *principal* at $\infty$ if $Y_0(t)$ is nonsingular for large $t$ and for any solution $(Y, Z)$, linearly idependent of

$(Y_0, Z_0)$, with $Y$ nonsingular for large $t$, we have $\lim_{t\to\infty} Y^{-1}(t)Y_0(t) = 0$. Any solution linearly idependent of $(Y_0, Z_0)$ is said to be *antiprincipal* at b (another terminology is *nonprincial*, see [7]). Equivalently, the solutions $(Y_0, Z_0)$, $(Y_1, Z_1)$ are principal resp. antiprincipal at $\infty$ if $Y_0, Y_1$ are nonsingular for large $t$ and

$$\lim_{t\to\infty} \left( \int^t Y_0^{-1}(s)B(s)Y_0^{T-1}(s)ds \right)^{-1} = 0$$

resp.

$$\lim_{t\to\infty} \left( \int^t Y_1^{-1}(s)B(s)Y_1^{T-1}(s)ds \right)^{-1} = L,$$

$L$ being a nonsingular $n \times n$ matrix.

Recall that a principal resp. nonprincipal solution of (1.1) at $\infty$ exist whenever (2.1) is nonoscillatory at $\infty$ and the principal solution is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix.

**Lemma 1.** *Let $(Y, Z)$ be a self-conjugate solution of (1.1) such that $Y(t)$ is nonsingular on $I_0 \subseteq I$. Then*

$$\begin{aligned} \tilde{Y}(t) &= Y(t) \int_c^t Y^{-1}(s)B(s)Y^{T-1}(s)ds, \quad c \in I_0 \\ \tilde{Z}(t) &= Z(t) \int_c^t Y^{-1}(s)B(s)Y^{T-1}(s)ds + Y^{T-1}(t) \end{aligned}$$

*is also a self-conjugate solution of (1.1) which is linearly independent of $(Y, Z)$. If $(Y, Z)$ is antiprincipal at $\infty$ then*

$$\begin{aligned} Y_0(t) &= Y(t) \int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)ds \\ Z_0(t) &= Z(t) \int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)ds - Y^{T-1}(t) \end{aligned}$$

*is the principal solution at $\infty$.*

**Proof.** [4, Chap. II]

Let $(Y, Z)$ be a solution of (1.1) such that $Y$ is nonsingular on $I$ then $W = ZY^{-1}$ is a solution of the Riccati equation

$$W' + WB(t)W + WA(t) + A^T(t)W + C(t) = 0. \tag{2.2}$$

If $(Y, Z)$ is principal at $\infty$ then $W$ is said to be distinquished solution of (2.2) at $\infty$, this solution is determined uniquely. If $\tilde{W}$ is another solution of (2.2) which exists on the whole interval $[c, \infty)$, $c \geq a$, then $\tilde{W}(t) \geq W(t)$ on $[c, \infty)$ (this inequality means that the matrix $\tilde{W}(t) - W(t)$ is nonnegative definite).

**Lemma 2.** *Let $W_0$ and $\tilde{W}$ be distinquished solutions at $\infty$ of (2.2) and*

$$W' + WB(t)W + WA(t) + A^T(t)W + \tilde{C}(t) = 0, \tag{2.3}$$

*respectively. If $\tilde{C}(t) \geq C(t)$ on $I$ then $\tilde{W}(t) \geq W_0(t)$ on $I$.*

**Proof.** [4, Chap. II]

Now recall some results concerning transformations of linear Hamiltonian systems. Let $R(t)$ be a $2n \times 2n$ $J$-unitary matrix of the form (1.4), then substituting into (1.5) and the equivalent relation $RJR^T = J$ we get

$$\begin{aligned} &H^TK = K^TH, \quad M^TN = N^TM, \quad H^TN - K^TM = I_n, \\ &HM^T = MH^T, \quad KN^T = N^KT, \quad HN^T - MK^T = I_n. \end{aligned} \tag{2.4}$$

The transformation (1.3) transforms (1.1) into the system

$$U' = \bar{A}(t)U + \bar{B}(t)V, \quad V' = -\bar{C}(t)U - \bar{A}^T(t)V \tag{2.5}$$

and the matrices $\bar{A}, \bar{B}, \bar{C}$ are related to $A, B, C$ by the equalities

$$\begin{aligned} A &= N^T(-H' + \bar{A}H + \bar{B}K) + M^T(K' + \bar{C}H + \bar{A}^TK), \\ B &= N^T(-M' + \bar{A}M + \bar{B}N) + M^T(N' + \bar{C}M + \bar{A}^TN), \\ C &= H^T(K' + \bar{C}H + \bar{A}^TK) + K^T(-H' + \bar{A}H + \bar{B}K), \end{aligned} \tag{2.6}$$

see, e.g., [3].

The main results of [6] are summarized in the next theorem.

**Theorem A.** *Suppose that the matrix $R(t)$ given by (1.4) is $J$-unitary, the matrix $M$ is nonsingular on $I$ and the matrices $B(t)$, $\bar{B}(t)$ are nonnegative definite on $I$. Then system (1.1) is nonoscillatory if and only (2.5) is nonoscillatory.*

Finally, for the later comparison, recall the results of [1] which were the main motivation for our investigation.

**Theorem B.** *Let $D(t)$ be the fundamental matrix of the equation $D' = A(t)D$. Suppose that the matrices $B(t)$, $C(t)$ are nonnegative definite in $I$, both systems (1.1) and (1.2) are identically normal on this interval and*

$$\lim_{t\to\infty} \left[\int^t D^{-1}(s)B(s)D^{T^{-1}}(s)ds\right]^{-1} = 0.$$

*If $(Y_0, Z_0)$ is the principal solution of (1.1) at $\infty$ then $(U_0, V_0) = (Z_0, -Y_0)$ is the principal solution of (1.2) at $\infty$. Moreover, a solution $(Y_1, Z_1)$ of (1.1) is antiprincipal at $\infty$ if and only if $(U_1, V_1) = (Z_1, -Y_1)$ is an antiprincipal solution of (1.2) at $\infty$.*

## 3. Main results.

Our main results are based on the following lemma which generalizes a similar result of [1].

**Lemma 3.** *Let $(Y,Z)$ and $(U,V)$ be self-conjugate solutions of (1.1) and (1.2), respectively, related by (1.3), such that $Y$ and $U$ are nonsingular. If $M(t)$ is nonsingular and (2.4) holds (i.e., $R(t)$ given by (1.4) is $J$-unitary in $I$), then*

$$[(Y^TM^{-1}U)^{-1}]' = -Y^{-1}BY^{T-1} + U^{-1}\bar{B}U^{T-1}$$

**Proof.** We have

$$\begin{aligned}
&[(Y^TM^{-1}U)^{-1}]' = \\
&-(Y^TM^{-1}U)^{-1}[Y^{T\prime}M^{-1}U - Y^TM^{-1}M'M^{-1}U + Y^TM^{-1}U'](Y^TM^{-1}U)^{-1} = \\
&-(Y^TM^{-1}U)^{-1}[Y^TA^T + Z^TB)M^{-1}(HY+MZ) - Y^TM^{-1}M'M^{-1}(HY+ \\
&BZ) + Y^TM^{-1}(\bar{A}U + \bar{B}V)](Y^TM^{-1}U)^{-1} = -(Y^TM^{-1}U)^{-1}[Y^TA^TM^{-1}HY+ \\
&Y^TA^TZ + Z^TBM^{-1}HY + Z^TBZ - Y^TM^{-1}M'M^{-1}HY - Y^TM^{-1}M'M^{-1}BZ+ \\
&Y^TM^{-1}\bar{A}(HY+MZ) + Y^TM^{-1}\bar{B}(KY+NZ) - U^TM^{T-1}BM^{-1}U+ \\
&U^TM^{T-1}BM^{-1}U](Y^TM^{-1}U)^{-1} = -(Y^TM^{-1}U)^{-1}[Y^T(A^TM^{-1}H- \\
&M^{-1}M'M^{-1}H + M^{-1}\bar{A}H + M^{-1}\bar{B}NH^TM^{T-1} - H^TM^{T-1}BM^{-1}H)Y+ \\
&Y^T(A^T - M^{-1}M' + M^{-1}\bar{A}M + M^{-1}\bar{B}N - H^TM^{T-1}B)Z + Z^T(-BM^{-1}H+ \\
&BM^{-1}H)Y + Z^T(B-B)Z](Y^TM^{-1}U)^{-1} - (Y^TM^{-1}U)^{-1}[U^TM^{T-1}BM^{-1}U- \\
&Y^TM^{-1}BM^{T-1}Y](Y^TM^{-1}U)^{-1} = -U^{-1}(MA^T - M' + \bar{A}M + \bar{B}N- \\
&HB)H^TU^{T-1} - (MA^T - M' + \bar{A}M + \bar{B}N - HB)ZY^{-1}M^TU^{T-1}- \\
&Y^{-1}BY^{T-1} + U^{-1}\bar{B}U^{T-1},
\end{aligned}$$

where the relations (2.4) and the symmetry of the matrix $Y^TM^{-1}U = Y^T(M^{-1}H + ZY^{-1})Y$ has been used. Computing the expression $MA^T - M' + \bar{A}M + \bar{B}N - HB$, using (2.5) and (2.6), we get

$$\begin{aligned}
&MA^T - M' + \bar{A}M + \bar{B}N - HB = M(-H^{T\prime} + H^T\bar{A}^T + K^T\bar{B})N + M(K^{T\prime}+ \\
&H^T\bar{C} + K^T\bar{A})M + \bar{A}M + \bar{B}N - M' - HN^T(-M' + \bar{A}M + \bar{B}N) - HM^T(N'+ \\
&\bar{C}M + \bar{A}^TN) = M(-H^{T\prime} + H^T\bar{A}^T + K^T\bar{B})N + M(K^{T\prime} + H^T\bar{C}+ \\
&K^T\bar{A})M + \bar{A}M + \bar{B}N - M' + M' - MK^TM' + MH^TN' - MK^T\bar{A}M- \\
&MK^T\bar{B}N - HM^T\bar{C}M - HM^T\bar{A}^TN - \bar{A}M - \bar{B}N = M(-H^{T\prime} + H^T\bar{A}^T+ \\
&K^T\bar{B})N + M(K^{T\prime} + H^T\bar{C} + K^T\bar{A})M + M(-K^{T\prime}N + H^{T\prime}N) - MK^T\bar{A}M- \\
&MK^T\bar{B}N - HM^T\bar{C}M - HM^T\bar{A}^TN = M(-H^{T\prime} + H^T\bar{A}^T + K^T\bar{B})N+ \\
&M(K^{T\prime} + H^T\bar{C} + K^T\bar{A})M - M(-H^{T\prime} + H^T\bar{A}^T + K^T\bar{B})N+ \\
&M(K^{T\prime} + H^T\bar{C} + K^T\bar{A})M = 0
\end{aligned}$$

which completes the proof.

**Theorem 1.** *Let $D$ be the fundamental matrix of the equation*

$$D' = (-B(t)M^{-1}(t)H(t) + A(t))D. \tag{3.2}$$

*Suppose that the matrices $B(t)$, $\bar{B}(t)$ are nonnegative definite,*

$$\lim_{t\to\infty} \left[\int^t D^{-1}(s)B(s)D^{T-1}(s)\, ds\right]^{-1} = 0 \tag{3.3}$$

*and both systems (1.1) and (1.2) are identically normal on $I$. If $(Y, Z)$ is a principal solution of (1.1) at $\infty$, then $(U, V)$ given by (1.3) is a principal solution of (2.5) at $\infty$.*

**Proof.** By Lemma 3

$$\int_a^t Y^{-1}(s)B(s)Y^{T-1}(s)\, ds + (Y^T(s)M^{-1}(s)U(s))^{-1}|_a^t = \int_a^t U^{-1}\bar{B}(s)U^{T-1}(s)\, ds \tag{3.4}$$

If $(Y, Z)$ is a principal solution, by definition

$$\lim_{t\to\infty}\left(\int_a^t Y^{-1}BY^{T-1}\, ds\right)^{-1} = 0,$$

hence all eigenvalues of the matrix $\int_a^t Y^{-1}BY^{T-1}ds$ tend to $\infty$ as $t \to \infty$. Consequently, to prove the theorem it suffices to show that the (symmetric) matrix

$$Y^{T-1}(t)M^{-1}(t)U(t)$$

is nonnegative definite for large $t$, i. e., the matrix $M^{-1}H + ZY^{-1} = M^{-1}H + W_0$, $W_0$ being the distinquished solution of (2.2) at $\infty$, has this property. Since the matrix $\bar{B}$ is nonnegative definite, by (2.6) $MCM^T \geq -HM^{T\prime} + HAM^T - HBN^T + MH^{T\prime} + MA^TN^T$, hence $C \geq -M^{-1}HM^{T\prime}M^{T-1} + M^{-1}HA - M^{-1}HBN^TM^{T-1} + H^{T\prime}M^{T-1} + A^TN^TM^{T-1} =: \tilde{C}$. Using the symmetry of the matrix $M^{-1}H$, one can directly verify that $\tilde{W} = -M^{-1}H$ is a solution of (2.3). Let $D$ be a solution of (3.2) and $F = \tilde{W}D$. Then $(F, D)$ is a solution of (1.1) with $\tilde{C}$ instead of $C$ and (3.3) implies that this solution is principal at $\infty$. Consequently, $\tilde{W} = FD^{-1} = -M^{-1}H$ is the distinquished solution of (2.3) at $\infty$ and by Lemma 2 $W_0 \geq -M^{-1}H$, i. e., the matrix $Y^TM^{-1}U$ is nonnegative definite for large $t$ and the proof is complete.

**Lemma 4.** *Let the assumptions of Theorem 1 hold. If $(Y, Z)$ is an antiprincipal solution of (1.1) and $U$ is given by (1.3), then*

$$\lim_{t\to\infty} U^{-1}(t)M(t)Y^{T-1}(t) = 0 \tag{3.5}$$

**Proof.** Let

$$Y_2(t) = Y(t)\int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)\,ds,$$
$$Z_2(t) = Z(t)\int_t^\infty Y^{-1}(s)B(s)Y^{T-1}(s)\,ds - Y^{T-1}(t).$$

According to Lemma 1 $(Y_2, Z_2)$ is the principal solution of (1.1) at $\infty$ and by Theorem 1 $(U_2, V_2) = (HY_2 + MZ_2, KY_2 + NZ_2)$ is the principal solution of (2.5). It follows $\lim_{t\to\infty} U^{-1}(t)U_2(t) = 0$. Substituting for $U_2$, we have

$$U^{-1}U_2 =$$
$$U^{-1}HY\int_t^\infty Y^{-1}BY^{T-1}\,ds - U^{-1}MY^{T-1} + U^{-1}MZ\int_t^\infty Y^{-1}BY^{T-1}\,ds,$$

hence

$$U^{-1}MY^{T-1} = -U^{-1}U_2 + U^{-1}(HY + MZ)\int_t^\infty Y^{-1}BY^{T-1}\,ds =$$
$$-U^{-1}U_2 + \int_t^\infty Y^{-1}BY^{T-1}\,ds,$$

i. e., (3.5) holds.

**Theorem 2.** *Suppose that the assumptions of Theorem 1 hold. A solution $(Y, Z)$ of (1.1) is antiprincipal at $\infty$ if and only if the solution $(U, V)$ of (2.5) given by (1.3) is antiprincipal at $\infty$.*

**Proof.** The statement follows immediately from (3.4), the previous lemma and the definition of the antiprincipal solution.

**4. Remarks.** i) If $H(t) \equiv 0$ in Theorems 1,2, then the statements of these theorems comply with the results of Ahlbrandt [1] given in Theorem B.

ii) Consider the second order system

$$(R(t)Y')' + P(t)Y = 0, \tag{4.1}$$

where $R, P$ are symmetric $n \times n$ matrices, $R$ is positive definite and let $\Gamma$ be a constant symmetric $n \times n$ matrix such that $P(t) + \Gamma R(t)\Gamma =: Q(t)$ is positive definite. Then the combination $U = RY' + \Gamma Y$ is a solution of the system

$$\begin{aligned}
&[Q^{-1}U' + Q^{-1}PR^{-1}\Gamma Q^{-1}U]' - Q^{-1}\Gamma R^{-1}PQ^{-1}U' + \\
&Q^{-1}[PR^{-1}P - \Gamma(R^{-1})'P - P(R^{-1})'\Gamma + \Gamma R^{-1}PR^{-1}\Gamma + PR^{-1}\Gamma(P' + \\
&\Gamma(R^{-1})'\Gamma Q^{-1} + Q^{-1}(P' + \Gamma(R^{-1})'\Gamma)PR^{-1}\Gamma - \\
&(\Gamma R^{-1}P - PR^{-1}\Gamma)Q(PR^{-1}\Gamma - \Gamma R^{-1}P)]Q^{-1}U = 0
\end{aligned} \tag{4.2}$$

which is nonoscillatory if and only if (4.1) is nonoscillatory, see [6]. If $Y$ is the principal solution of (4.1) and $\Gamma$ is such that $W_0 = -\Gamma$ is the distinquished solution of the Riccati equation $W' + WR^{-1}W - \Gamma R^{-1}\Gamma = 0$ then $U = RY' + \Gamma Y$ is the principal solution of (4.2).

iii) Let $(Y, Z)$ be the principal solution of (1.1) at $\infty$, i. e., all eigenvalues of the matrix $\int^t Y^{-1}(s)B(s)Y^{T-1}(s)\,ds$ tend to $\infty$ as $t \to \infty$. In order to show that the matrix $\int^t U^{-1}(s)\bar{B}(s)U^{T-1}(s)\,ds$ also has this property (i. e., that $(U, V)$ given by (1.3) is the principal solution of (2.5)), we used the idea suggested in [1], we proved that under the assumptions of Theorem 1 the matrix $U^{-1}(t)M(t)Y^{T-1}(t)$ is nonnegative definite for large $t$. To the same end it suffices to prove that the last matrix is "bounded below", i. e., $c^T U^{-1}(t)M(t)Y^{T-1}(t)c$ is bounded from bellow for every $c \in \mathbb{R}^n$. We hope to follow this more general (but also more difficult) idea elsewhere.

## References

[1] Ahlbrandt C.D., *Principal and antiprincipal solutions of selfadjoint diferential systems and their reciprocals*, Rocky Mountain J. Math. **2** (1972), 169-189.

[2] Ahlbrandt C.D., *Equivalent boundary value problems for self-adjoint differential systems*, J. Diff Equations **9** (1971), 420-435.

[3] Ahlbrandt C.D., Hinton D.B., Lewis R.T., *The effect of variable change on oscillation and disconjugacy criteria with application to spectral theory and asymptotic theory*, J. Math. Anal. Appl. **81** (1981), 234-277.

[4] Coppel W.A., *Disconjugacy*, Lecture Notes in Math. No. 220 (1971), Berlin – New York – Heidelberg.

[5] Došlý O., *On transformation of self-adjoint linear differential systems and their reciprocals*, Annal. Pol. Math. **50** (1990), 223-234.

[6] Došlý O., *Transformations of linear Hamiltonian system preserving oscillatory behaviour*, Arch. Math. **27** (1991), 211-219.

[7] Hartman P., *Self-adjoint, non-oscillatory systems of ordinary, second order linear differential equations*, Duke J. Math. **24** (1956), 25-35.

[8] Rasmussen C.H., *Oscillation and asymptotic behaviour of systems of ordinary linear differential equations*, Trans. Amer. Math. Soc. **256** (1979), 1-49.

[9] Reid W.T., *Sturmian Theory for Ordinary Differential Equations*, Springer Verlag, New York – Berlin – Heidelberg, 1980.

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