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SPECTRAL INVARIANT OF THE ZETA FUNCTION OF THE LAPLACIAN ON Sp(r + 1)/Sp(1) × Sp(r)

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Abstract. The aim of this paper is to compute a spectral invariant of the Zeta function $\zeta(\Delta, s)$ at $s = 0$ of the Laplace Beltrami operator Δ acting on 1-forms on $Sp(r+1)/Sp(1)\, xSp(r)$.

Key words. Eigenvalues of the Laplace Beltrami operator, Zeta functions, spectral invariant.

MS Classification. 58 C 40.

1. INTRODUCTION

When A is a self adjoint positive elliptic pseudo-differential operator of order $m(> 0)$ acting on a compact n-dimensional manifold X, its eigen values are $\lambda > 0$. Its Zeta function $\zeta(A, s)$ is defined as

$$\zeta(A, s) = \text{Trace } A^{-s} = \sum_{\lambda > 0} \lambda^{-s}. \tag{1}$$

Here each eigenvalue repeats as many times as its multiplicity. This series converges to $Re(s) > \frac{n}{m}$ [4] and gives a holomorphic function of the complex variables in this half plane. Moreover $\zeta(A, s)$ can be analytically continued to the whole of s plane as a meromorphic function with simple poles. When the operator A is not necessarily positive, we define its eta function as $\eta(A, s) = \sum_{\lambda \neq 0} \text{Sign } \lambda \mid \lambda \mid^{-s}$, where each eigenvalue λ of A repeats as many times as its multiplicity.

The real valued invariants of the metric satisfying the condition that it is a continuous function of the metric can be obtained by evaluating ζ at some point where it is known to be finite. For a positive operator A, $\eta(A, s) = \zeta(A, s)$ is finite at $s = 0$ by the results in [7]. In [2] and [3], the spectral asymmetry of certain selfadjoint elliptic operators arising in Riemannian geometry was studied. In particular an expression for $\eta(0)$ was given in [2] for the selfadjoint operator $B = \pm({}^*d - d^*)$ acting on even forms on the boundary Y of a $4k$ dimensional compact oriented manifold X. Here B^2, the square of B, is the usual Laplace Beltrami operator Δ. The function $\zeta(A, s)$ for arbitrary selfadjoint operators was studied in detail in [4]. In [10], we gave a general method of analytic continuation to compute the spectral invariant of the Zeta function $\zeta(\Delta, s)$ at $s = 0$ of the Laplace Beltrami operator Δ acting on 2-forms on sphere S^{4r-1} and computed explicitly the value of $\zeta(\Delta, 0)$ for S^{4r-1}. The aim of this paper is to compute the spectral invariant of the Zeta function $\zeta(\Delta, s)$ at $s = 0$ of the Laplacian Δ acting on forms of degree 1 on $Sp(r+1)/Sp(1)\,xSp(r)$. The method we have used here is similar to the method that we used in [10].

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2. SPECTRA OF THE LAPLACIAN Δ ON $Sp(r+1)/Sp(1)\,xSp(r)$

Let G be a compact connected semisimple Liegroup and K be a closed subgroup of it. We consider the space $M = G/K$. The Laplace Beltrami operator or Laplacian is defined as $\Delta = d\delta + \delta d$, where d is the exterior differentiation defined on $C^\infty(\Lambda^p M)$, the vector space of the smooth sections of the p^{th} exterior power of the complexified cotangent bundle $\Lambda^p M$ on M and δ is the operator adjoint to d. When B is the Killing form of the Lie algebra $\mathfrak{J}$ of G, the casimir element is $C = \sum_{1 \le i,j \le N} C^{ij} X_i \,.\, X_j$, where $\{X_1, \dots, X_N\}$ is a basis of $\mathfrak{J}$ and $C^{ij} = (B(X_i, X_j))^{-1}$.
We have the Cartan decomposition

$$\mathfrak{J} = k \oplus m$$

(direct sum), where k is Lie algebra of K and m is the orthogonal complement to k in $\mathfrak{J}$ with respect to the Killing form B. Restricting the Killing form sign changed to m, we get a bi-variant Riemannian metric on $M = G/K$ and the identity $\Delta = -C$ [5]. Using this result, the eigenvalues and its multiplicities for Δ acting on p-form were computed for some p and some r for the space $Sp(r+1)/Sp(1)\,xSp(r)$ in [13]. In [9], we computed all eigenvalues of Δ acting on 1-form on $Sp(r+1)/Sp(1)\,xSp(r)$ without using the identity $\Delta = -C$ and our results coincide with the results in [13]. We state the theorem A that we have proved in [9].

Theorem. *The spectrum of the de Rham Hodge operator* $\square$ *(or Laplacian* Δ*) acting on forms of degree* 1 *on* $Sp(r+1)/Sp(1)\,x Sp(r)$ $(r \geqq 1)$ *is the union of the sets*

$$\left\{\frac{1}{2(r+2)}(n^2 + 2nr + n);\ n \in N^*\right\},$$

$$\left\{\frac{1}{2(r+2)}(n^2 + 2nr + 3n + 2r + 4);\ n \in N\right\}$$

and

$$\left\{\frac{1}{2(r+2)}(n^2 + 2nr + 2n + 2r + 1);\ n \in N^*\right\}.$$

Using the notations in [9] if Λ_ϱ is the highest weight of the irreducible representation ϱ intervening in the C^∞ sections of the complexfied cotangent bundle on m, then the multiplicity of an eigenvalue ω is the product of the multiplicity of Λ_ϱ and the dimension of the representation ϱ. By the results in [9], the multiplicity of Λ_ϱ is 1 in each case. The dimension of the representation can be computed using Weyl's formula:

$$\frac{\pi\langle \Lambda_\varrho + \delta, \alpha\rangle}{\alpha > 0\langle \delta, \alpha\rangle}.$$

Here $\alpha > 0$ are the positive roots in $Sp(r+1)$ and δ is half the sum of the positive roots in $Sp(r+1)$. So we get the following table.

Highest weight Λ_ϱ	Multiplicity of the eigenvalue
$n(\lambda_1 + \lambda_2)$	$K_1 \,.\, a(n, 2r) \,.\, a(n, 2r-1)\ (n \in N^*)$
$(n+2)\,\lambda_1 + n\lambda_2$	$K_2 \,.\, a(n+2, 2r) \,.\, a(n, 2r-1)\ (n \in N)$
$(n+1)\,\lambda_1 + n\lambda_2 + \lambda_3$	$K_3 \,.\, a(n+1, 2r+1) \,.\, a(n, 2r)\ (n \in N^*)$

Here

$$K_1 = \frac{2n + 2r + 1}{(2r+1)(n+1)},$$

$$K_2 = \frac{3(2n + 2r + 3)}{(2r+1)(n+3)},$$

$$K_3 = \frac{4nr(2r-2)(2n+2r+2)}{(n+1)(n+3)(n+2r-1)(n+2r+1)}$$

and

$$a(r, s) = \binom{r+s}{s}.$$

3. ZETA FUNCTION OF THE LAPLACIAN Δ ON $Sp(r+1)/Sp(1)\, x Sp(r)$ AND THE SPECTRAL INVARIANT

As we consider the positive operator Δ, we get $\eta(\Delta, s) = \zeta(\Delta, s)$ and

$$\zeta(\Delta, s) = \sum_{n=1}^{\infty} \left\{ \frac{2n+2r+1}{(2r+1)(n+1)} \binom{n+2r}{2r} \binom{n+2r-1}{2r-1} \times \left\{ \frac{n^2+2nr+n}{2(r+2)} \right\}^{-s} \right\} +$$
$$+ \sum_{n=0}^{\infty} \frac{3(2n+2r+3)}{(2r+1)(n+3)} \binom{n+2r-1}{2r-1} \binom{n+2r+2}{2r} \times \left\{ \frac{n^2+2nr+3n+2r+4}{2(r+2)} \right\}^{-s} \Big\} +$$
$$+ \sum_{n=1}^{\infty} \frac{4nr(2r-2)(2n+2r+2)}{(n+1)(n+3)(n+2r-1)(n+2r+1)} \binom{n+2r+2}{2r+1} \binom{n+2r}{2r} \times$$
$$\times \left\{ \frac{n^2+2nr+2n+2r+1}{2(r+2)} \right\}^{-s} \Big\}.$$

First we explain below the method of analytic continuation which we use to compute $\zeta(\Delta, s)$ at $s = 0$. Let $S = \sum_{n-1}^{\infty} g(n) \{f(n)\}^{-s}$ be a series, where $g(n)$ and $f(n)$ are two primitive polynomials of degrees q and k respectively. This series S has analytic continuation in the entire complex plane as

$$S = \sum_{n=1}^{\infty} g(n) \{n^k + P(n))^{-s} - n^{-ks}\} + \sum_{n=1}^{\infty} g(n)\, n^{-ks}.$$

Here $f(n) = n^k + P(n)$, $P(n)$ being a polynomial of degree $k - 1$. Let

$$(\mathrm{I}) = \sum_{n \leq c} g(n) \{n^k + P(n))^{-s} - n^{-ks}\},$$

$$(\mathrm{II}) = \sum_{n > c} g(n)\, n^{-ks} \left\{ \left(1 + \frac{P(n)}{n^k}\right)^{-s} - 1 \right\},$$

$$(\mathrm{III}) = \sum_{n=i}^{\infty} g(n)\, n^{-ks}.$$

The positive number c is chosen such that $\left| \frac{Pu.p(n)}{n^k} \right| < 1$. So, S has analytic continuation in the entire complex plane with

$$S = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).$$

Now, (I) is an entire function whose value at $s = 0$ is 0. In (II), $\left(1 + \frac{Pu.p(n)}{n^k}\right)^{-s}$ can be expanded using binomial theorem because $\left| \frac{Pu.p(n)}{n^k} \right| < 1$.

So

$$(\text{II}) = \sum_{n>c} g(n)\, n^{-ks} \left\{ \frac{-sP(n)}{n^k} + \frac{s(s+1)}{2}\left(\frac{P(n)}{n^k}\right)^2 \ldots \text{ to } \infty \right\},$$

when $s \to 0$, (II) gives some constans due to first few terms as $s\zeta(s+1) \to 1$ and all the other terms in (II) will tend to zero. Here ζ is the ordinary Riemann–Zeta function. Moreover the sum in (II) taken over any finite rectangle also tends to zero as $s \to 0$.

Let $g(n) = \sum_{i=0}^{q} a_i n^i$ where

$$a_i \; (0 \leqq i \leqq q)$$

are constans with

$$a_q = 1.$$

Then

$$(\text{III}) = \sum_{i=0}^{q} a_i \zeta(ks - i),$$

when $s \to 0$, (III) will contribute some constans to $\zeta(\Delta, 0)$. Now we compute $\zeta(\Delta, s)$ at $s = 0$ for $Sp(r+1)/Sp(1)\, x Sp(r)$. Let us make the following assumptions:

$$g_1(n) = \frac{2n+2r+1}{(2r+1)(n+1)} \binom{n+2r}{2r} \binom{n+2r-1}{2r-1} = \sum_{i=0}^{4r-1} a_i n^i,$$

$$g_2(n) = \frac{3(2n+2r+3)}{(2r+1)(n+3)} \binom{n+2r-1}{2r-1} \binom{n+2r+2}{2r} = \sum_{i=0}^{4r-1} b_i n^i,$$

$$g_3(n) = \frac{4nr(2r-2)(2n+2r+2)}{(n+1)(n+3)(n+2r-1)(n+2r+1)} \times \binom{n+2r+2}{2r+1} \binom{n+2r}{2r} = \sum_{i=0}^{4r-1} c_i n^i,$$

$$f_1(n) = \frac{n^2+2rn+n}{2(r+2)},$$

$$f_2(n) = \frac{n^2+2nr+3n+2r+4}{2(r+2)},$$

$$f_3(n) = \frac{n^2+2nr+2n+2r+1}{2(r+2)}.$$

Then

$$\zeta(\Delta, s) = \sum_{n=1}^{\infty} g_1(n)\{f_1(n)\}^{-s} + \sum_{n=0}^{\infty} g_2(n)\{f_2(n)\}^{-s} + \sum_{n=1}^{\infty} g_3(n)\{f_3(n)\}^{-s},$$

$$\zeta(\Delta, s)_{(s=0)} = (r+1)(2r+3) + \{\sum_{n=1}^{\infty} g_1(n)\{f_1(n)\}^{-s} + \sum_{n=1}^{\infty} g_2(n)\{f_2(n)\}^{-s} +$$

$$+ \sum_{n=1}^{\infty} g_3(n)\{f_3(n)\}^{-s}\} \quad (s=0).$$

We first find the contribution to $\zeta(\Delta, 0)$ from

$$S_1 = \sum_{n=1}^{\infty} g_1(n)\{f_1(n)\}^{-s}.$$

This series has analytic continuation in the entire complex plane as

$$S_1 = \text{(I)} + \text{(II)} + \text{(III)},$$

where

$$\text{(I)} = \sum_{n=1}^{2r+1} g_1(n)\left\{\left(\frac{n^2}{2(r+2)} + \frac{2nr+n}{2(r+2)}\right)^{-s} - \left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s}\right\},$$

$$\text{(II)} = \sum_{n=2r+2}^{\infty} g_1(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s}\left\{\left(1+\frac{2r+1}{n}\right)^{-s} - 1\right\},$$

$$\text{(III)} = \sum_{n=1}^{\infty} g_1(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s},$$

(I) is entire function whose value at $s = 0$ is 0. As

$$\left|\frac{2r+1}{n}\right| < 1 \qquad \text{for } n \geqq 2r+2,$$

$$\text{(II)} = \sum_{n=2r+2}^{\infty} g_1(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s}\left\{-s\frac{2r+1}{n} + \frac{s(s+1)}{2}\left(\frac{2r+1}{n}\right)^2 - \dots \text{ to } \infty\right\}.$$

Denoting by (II)_i, the i-th term of (II) in above expansion, we get

$$\text{(II)}_1 = -s\sum_{n=2r+2}^{\infty} g_1(n)\left\{\frac{n}{\sqrt{2(r+2)}}\right\}^{-2s} \cdot \frac{2r+1}{n} =$$

$$= \frac{-s(2r+1)}{(\sqrt{(2(r+2))})^{-2s}} \sum_{n=2r+2}^{\infty} g_1(n)\, n^{-(2s+1)},$$

$$\text{(II)}_1 \qquad (\text{at } s=0) = -(2r+1)\{s\sum_{i=0}^{4r-1} a_i\zeta(2s+1-i)\} = \frac{-(2r+1)}{2}a_0$$

$$(\text{at } s = 0).$$

Similarly

$$(\text{II})_2 \quad (\text{at } s = 0) = \frac{(2r+1)^2}{4} \cdot a_1,$$

$$(\text{II})_3 \quad (\text{at } s = 0) = \frac{-(2r+1)^3}{6} \cdot a_2$$

and so on. Finally

$$(\text{II})_{4r-1} \quad (\text{at } s = 0) = \frac{(-1)^{4r-1}(2r+1)^{4r-1}}{2(4r-1)} \cdot a_{4r-2}$$

and

$$(\text{II})_{4r} \quad (\text{at } s = 0) = \frac{(-1)^{4r}(2r+1)^{4r}}{2(4r)} \cdot a_{4r-1}.$$

All the other terms in (II) will tend to zero as s tends to zero.

$$(\text{III}) = \sum_{n=1}^{\infty} g_1(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s} = \left(\frac{1}{\sqrt{2(r+2)}}\right)^{-2s} \sum_{n=1}^{\infty} g_1(n)\, n^{-2s},$$

$$(\text{III}) \quad (\text{at } s = 0) = \sum_{i=0}^{4r-1} a_i \zeta(2s - i) \qquad (\text{at } s = 0) =$$

$$= a_0\zeta(0) + a_1\zeta(-1) + a_3\zeta(-3) + \ldots + a_{4r-1}\zeta(1-4r) =$$

$$= \frac{-a_0}{2} - \frac{a_1 B_1}{2} + \frac{a_3 B_2}{4} - \frac{a_5 B_3}{6} + \ldots + a_{4r-1}\frac{B_{2r}}{4r}.$$

Here we used the formulae:

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta(-2m) = 0$$

and

$$\zeta(1-2m) = \frac{(-1)^m B_m}{2m} \qquad \text{for } m = 1, 2, 3, \ldots,$$

in which $B_1, B_2, \ldots$ are Bernoulli's numbers [12].

Let us now find out the contribution to $\zeta(\Delta, 0)$ from

$$S_2 = \Sigma g_2(n)\,\{f_2(n)\}^{-s}.$$

S_2 has analytic continuation in the entire complex plane as $S_2 = (\text{IV}) + (\text{V}) +$ $+ (\text{VI})$ where

$$(\text{IV}) = \sum_{n=1}^{c} g_2(n)\left\{\left(\frac{n^2}{2(r+2)} + \frac{2nr+3n+2r+4}{2(r+2)}\right)^{-s} - \left(\frac{n}{\sqrt{2(r+2)}}\right)\right\}^{-2s},$$

$$(\text{V}) = \sum_{n=c+1}^{\infty} g_2\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s}\left\{\left(1 + \frac{2nr+3n+2r+4}{n^2}\right)^{-s} - 1\right\},$$

$$(\mathrm{VI}) = \sum_{n=1}^{\infty} g_2(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s}.$$

Here c is chosen such that $\left|\dfrac{2nr + 3n + 2r + 4}{n^2}\right| < 1.$

For example when $r = 2$, c can be chosen to be equal to 8. (IV) is entire function whose value at $s = 0$ is 0.

$$(\mathrm{V}) = \sum_{n=c+1}^{\infty} g_2(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s}\left\{-s.\frac{2nr+3n+2r+4}{n^2} + \right.$$
$$\left. + \frac{s(s+1)}{1.2}\left(\frac{2nr+3n+2r+4}{n^2}\right)^2 - \ldots \text{ to } \infty\right\}.$$

Let

$$P_2(n) = 2nr + 3n + 2r + 4$$

and

$$g_2(n)\,(P_2(n)^t = \sum_{i=0}^{4r+t-1} b_{(t,i)}n^i \qquad \text{for } 1 \leqq t \leqq 4r.$$

Denoting by $(\mathrm{V})_i$, the i-th term of (V), we get

$$(\mathrm{V})_1 = \frac{-s}{(\sqrt{2(r+2)})^{-2s}} \sum_{n=c+1}^{\infty} g_2(n)\,P_2(n)\,n^{-(2s+2)},$$

$$(\mathrm{V}_1) \quad (\text{at } s=0) = -s\sum_{i=0}^{4r} b_{(1,i)}\zeta(2s+2-i) = -\frac{1}{2}b_{(1,1)}$$

$$(\text{at } s = 0).$$

Similarly

$$(\mathrm{V})_2 = \frac{1}{4}\,.\,b_{(2,3)} \qquad \text{and so on.}$$

Finally

$$(\mathrm{V})_{4r} = \frac{(-1)^{4r}}{8r}\,.\,b_{(4r,8r-1)}.$$

Now

$$(\mathrm{VI}) = \sum_{n=1}^{\infty} g_2(n)\left(\frac{n}{\sqrt{2(r+2)}}\right)^{-2s} = \left(\frac{1}{\sqrt{2(r+2)}}\right)^{-2s} \sum_{i=0}^{4r-1} b_i\zeta(2s-i),$$

$$(\mathrm{VI}) \quad (\text{at } s=0) = -\frac{b_0}{2} - \frac{b_1B_1}{2} + \frac{b_3B_2}{4} + \ldots + \frac{(-1)^{2r}\,b_{4r-1}B_{2r}}{4r}.$$

Similarly to find the contribution from $S_3 = \Sigma g_3(n)\,\{f_3(n)\}^{-s}$, we assume that

$$P_3(n) = 2nr + 2n + 2r + 1$$

and

$$g_3(n)\,(P_3(n))^t = \sum_{i=0}^{4r+t-1} c_{(t,i)} n^i \qquad \text{for } 1 \leqq t \leqq 4r.$$

As we have done for S_2, the contribution from S_3 can be found out.

So we have now proved the following theorem:

Theorem. *A spectral invariant of the Zeta function $\zeta(\Delta, s)$ at $s = 0$ of the Laplace Beltrami operator Δ acting on forms of degree* 1 *on* $Sp(r+1)/Sp(1)\,xSp(r)$ is

$$\sum_{t=1}^{4r} \frac{(-1)^t(2r+1)^t}{2t} \cdot a_{t-1} - \frac{a_0}{2} + \sum_{t=0}^{2r} \frac{(-1)^t a_{2t-1} B_t}{2t} +$$
$$+ \sum_{t=1}^{4r} \frac{(-1)^t}{2t} \cdot b_{(t,2t-1)} - \frac{b_0}{2} + \sum_{t=1}^{2r} \frac{(-1)^t b_{2t-1} B_t}{2t} +$$
$$+ \sum_{t=1}^{4r} \frac{(-1)^t}{2t} c_{(t,2t-1)} - \frac{c_0}{2} + \sum_{t=1}^{2r} \frac{(-1)^t c_{2t-1} B_t}{2t} + (r+1)(2r+3)$$

The meanings of the above symbols were already explained.

Remarks. Using the same method and using the results in [8], a spectral invariant for $SO(n+2)/SO(2)\,xSO(n)$ can be computed [11].

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