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DIRECT FACTORS OF MULTILATTICE GROUPS

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Dedicated to Academician Otakar Borůvka on his 90th birthday

Abstract. Subgroups of a directed distributive multilattice group **G** are characterized which are direct factors of **G**. The main result is formulated in Theorem 1.1.

Key words. Partially ordered group, multilattice group, direct product, distributivity.

MS Classification. 06 F 15.

Let $\mathbf{P} = (P; \leqq)$ be a partially ordered set (p. o. set). A subset $A \subset P$ is said to be convex if $a, b \in A$, $c \in P$ and $a \leqq c \leqq b$ imply $c \in A$. A is connected if for each $a, b \in A$ there is a sequence $a = x_0, x_1, \ldots, x_n = b$, $x_i \in A$, such that x_i and x_{i+1} are comparable for each $i \in \{0, 1, \ldots, n-1\}$.

Given $a, b \in P$, denote $(a] = \{x \in P\colon x \leqq a\}$, $[a) = \{x \in P\colon a \leqq x\}$, $l(a, b) = (a] \cap (b]$ and $u(a, b) = [a) \cap [b)$. **P** is called directed if for any $a, b \in P$ the sets $l(a, b)$ and $u(a, b)$ are not empty. Call **P** a multilattice [2] if for any $a, b, c \in P$ such that $c \in u(a, b)$, the set $u(a, b) \cap (c]$ has a minimal element, and dually for $c \in l(a, b)$. Denote by $a \vee b$ the set of all minimal elements of $u(a, b)$. If $c \in u(a, b)$, $(a \vee b)_c$ will denote the set $(a \vee b) \cap (c]$. $a \wedge b$ and $(a \wedge b)_c$ have dual meanings.

A multilattice **P** is said to be distributive [2] if for each $a, b, c \in P$ the relations $(a \vee b) \cap (a \vee c) \neq \emptyset$, $(a \wedge b) \cap (a \wedge c) \neq \emptyset$ together imply $b = c$.

A partially ordered group [3] (p. o. group) $\mathbf{G} = (G; +, \leqq)$ is said to be a multilattice group if the p. o. set $(G; \leqq)$ is a multilattice. **G** is called distributive if the multilattice $(G; \leqq)$ is.

Let **G** be a p. o. group. We say that a subset C of G forms a direct factor of **G** whenever a direct product decomposition $f\colon \mathbf{G} \cong \mathbf{A} \times \mathbf{B}$ exists such that $f^{-1}(\{(a, 0)\colon a \in A\}) = C$. The main result of the present note is the following.

1.1. Theorem. *Let* **G** *be a directed distributive multilattice group. A subset* $C \subset G$ *forms a direct factor of* **G** *iff it satisfies the following conditions.*

(1) *$(C; +)$ is a subgroup of $(G; +)$.*
(2) *C is convex and connected in $(G; \leqq)$.*
(3) *For each $a \in G^+$ the set $C \cap [0, a]$ has a greatest element.*

2. PROOF OF THE MAIN THEOREM

Before to prove Theorem 1.1 some auxiliary results will be presented.

2.1. Let **P** be a p. o. set with the least element 0 and let f: $\mathbf{P} \cong \mathbf{A} \times \mathbf{B}$ be a direct, (cardinal) product decomposition of **P**. It can be easily checked that the sets $C = f^{-1}(\{(a, 0)\colon a \in A\})$ and $D = f^{-1}(\{(0, b)\colon b \in B\})$ have the following properties.

(i) C and D are convex subsets of P.

(ii) $C \cap D = \{0\}$.

(iii) For any $a \in C$ and $b \in D$, $\sup\{a, b\}$ exists in P.

(iv) For any $c \in P$ there are $a \in C$ and $b \in D$ such that $c = \sup\{a, b\}$.

(v) If $a, a' \in C$ and $b, b' \in D$ then $a' \leqq \sup\{a, b\}$ implies $a' \leqq a$ and $b' \leqq \sup\{a, b\}$ implies $b' \leqq b$.

Conversely, if C and D are subsets of P with the properties (i)–(v) then $c \mapsto (a, b)$, where $a \in C$, $b \in D$ and $\sup\{a, b\} = c$, is an isomorphism $\mathbf{P} \cong (C; \leqq) \times (D; \leqq)$.

In such a case we write $\mathbf{P} = \mathbf{C} \cdot \mathbf{D}$ ("inner direct product"). (In [4] an analogous characterization of such products is given.)

2.2. The following theorems will be used in what follows.

A [5; 3.4.1]. *There is a bijective correspondence between direct product decompositions of a quasi-ordered set* **P** *into two factors and pairs of equivalence relations* Θ_1, Θ_2 *on* **P**, *satisfying the conditions*

(i) $\Theta_1 \cap \Theta_2 = \mathrm{id}_P$,

(ii) $\Theta_1 \vee \Theta_2 = P \times P$,

(iii) Θ_1 *and* Θ_2 *are permutable,*

(iv) *if* $a \leqq b$, $a\Theta_i a'$, $b\Theta_i b'$, $a'\Theta_j b'$ *for* $i \neq j$ *then* $a' \leqq b'$.

The correspondence is as follows. Given a direct product decomposition f: $\mathbf{P} \cong \mathbf{A}_1 \times \mathbf{A}_2$ *then* $a\Theta_i b$ *iff* $\pi_i f(a) = \pi_i f(b)$, *where* π_i *is the projection* $A_1 \times A_2 \to A_i$ $(i = 1, 2)$. *Given a pair* (Θ_1, Θ_2) *then* $a \mapsto ([a]\,\Theta_1, [a]\,\Theta_2)$ *is an isomorphism* $\mathbf{P} \to \mathbf{P}/\Theta_1 \times \mathbf{P}/\Theta_2$.

Note that $[a]\,\Theta_i \leqq [b]\,\Theta_i$ in $\mathbf{P}/\Theta_i$ means that $x \leqq y$ for some $x \in [a]\,\Theta_i$ and $y \in [b]\,\Theta_i$.

B [4; Th. 2]. Let **G** be a directed p. o. group and let $(G^+; \leqq) = (C; \leqq) \cdot (D; \leqq)$. Then there is a direct product decomposition $\mathbf{G} \cong \mathbf{A} \times \mathbf{B}$, where **A** and **B** are p. o. subgroups of **G** and $A^+ = C$, $B^+ = D$.

(We use the notations: $G^+ = \{a \in G\colon 0 \leqq a\}$, $G^- = \{a \in G\colon a \leqq 0\}$.

2.3. ***Theorem* 2.2.*A*** *remains true when* (iv) *is replaced by the following two conditions.*

(v) $a\Theta_i b$, $b\Theta_j c$, $i \neq j$ *and* $a \leqq c$ *imply* $a \leqq b \leqq c$.

(vi) *If* $a \leqq b$, $i \in \{1, 2\}$ *and* $a\Theta_i a'$ *then* b' *exists such that* $b\Theta_i b'$ *and* $a' \leqq b'$.

Proof. Suppose Θ_1 and Θ_2 satisfy (i), (ii), (iii), (v), (vi), and let the supposition of (iv) be fulfilled. According to (vi), b_1 exists such that $a' \leqq b_1$ and $b\Theta_i b_1$. Using (v) we get $a' \leqq b' \leqq b_1$. Conversely, if (i)–(iv) hold then there is a direct product decomposition $\mathbf{P} \cong \mathbf{A}_1 \times \mathbf{A}_2$ whence (v) and (vi) can be easily checked.

2.4. *In a distributive multilattice the relations* $u \in a \wedge b$, $v \in a \vee b$, $b \leqq d \leqq v$ *and* $h \in (a \wedge d)_u$ *imply* $d \in h \vee b$.

The proof is easy.

2.5. *Let* $(M; \leqq)$ *be a directed multilattice and suppose* B *is a non-empty convex and connected subset of* M *satisfying the condition*

(*) *for each* $a \in B$ *and* $b \in M$ *with* $a \leqq b$ *the set* $B \cap [a, b]$ *has a greatest element, and the condition dual to* (*).

We shall successively prove:

a) $B \cap (a \vee b) \neq \emptyset$ and $B \cap (a \wedge b) \neq \emptyset$ for any $a, b \in B$.

b) $a \vee b \subset B$ and $a \wedge b \subset B$ whenever $a, b \in B$.

c) For each $a \in M$ the set $B \cap (a]$ has a greatest element whenever it is not empty. For $B \cap [a)$ the dual assertion holds.

Proof. a) We prove the assertion for $a \vee b$. Given $a, b \in B$, there is a sequence

$$(**) \qquad a = a_0, a_1, \ldots, a_n = b$$

of elements of B such that a_i and a_{i+1} are comparable for each $i < n$. The assertion is trivial if $n = 1$. Suppose the assertion true for sequences of the length $n - 1$ and consider the sequence (**). Then there exists $s \in B \cap (a \vee a_{n-1})$. If $b = a_n \leqq \leqq a_{n-1}$ then $(a \vee b)_s \subset [b, s] \subset B$. In the case $a_{n-1} < a_n$ take $t \in s \vee a_n$. If $m = = \max B \cap [a_{n-1}, t]$ then $a \leqq s \leqq m$, $b \leqq m$ hence $(a \vee b)_m \subset B$.

b) By a) there exists $u \in B \cap (a \wedge b)$. Let $v \in a \vee b$ and $m = \max B \cap [u, v]$. Then $a \leqq m \leqq v$, $b \leqq m$ hence $v = m$, so that $v \in B$. Using duality we get $a \wedge b \subset B$.

c) Let $B \cap (a] \neq \emptyset$ and $u \in B \cap (a]$. Then there exists $\max B \cap [u, a] = m$. If b is an arbitrary element of $B \cap (a]$ then $u \vee b \subset B$ by b). Take $s \in (u \vee b)_a$. Then $u \leqq s \leqq a$, $s \in B$ hence $s \leqq m$, so that $b \leqq m$.

2.6. *Let* a, b, t *be elements of a multilattice group and* $t \in l(a, b)$ ($t \in a \wedge b$). *Then* $a - t + b$ *and* $b - t + a$ *belong to* $u(a, b)$ ($a \vee b$, *respectively*).

The proof is straightforward.

2.7. In this paragraph $\mathbf{G}$ denotes a directed multilattice group and C a subset of G with the properties (1), (2), (3) in Theorem 1.1.

Denote $A = C \cap G^+$. We are going to show that A forms a direct factor of $(G^+; \leqq)$ whenever the multilattice $(G; \leqq)$ is distributive.

2.7.1. *If* $a \in C$, $b \in G$ *and* $a \leqq b$ *then the set* $C \cap [a, b]$ *has a greatest element.*

Proof. $0 \leqq b - a$ hence there exists $\max C \cap [0, b - a] = m$. Then $m + a$ is the greatest element in $C \cap [a, b]$.

2.7.2. Obviously C has also properties dual to (3) and to that proved in 2.7.1. Using 2.5 we get that for any $a \in G$ the set $C \cap (a]$ has a greatest element whenever it is not empty. We adopt the notation a_C for $\max C \cap (a]$. Obviously $a_C = a$ iff $a \in C$, and $a \leqq b$ implies $a_C \leqq b_C$ (b_C exists whenever a_C does). The element a_C is defined for all $a \in G^+$ hence we get a surjective mapping $a \mapsto a_C$ from G^+ onto A.

In what follows whenever the symbol x_C ($x \in G$) is used we suppose $C \cap (x] \neq \emptyset$ without mention it.

2.7.3. *If* $a \in C$ *and* $b \in G^+$ *then* $(a + b)_C = a + b_C$ *and* $(b + a)_C = b_C + a$.

Proof. $a + b_C \in C \cap (a + b]$ hence $a + b_C \leqq (a + b)_C$ and $b_C \leqq -a + (a + b)_C$. On the other hand $(a + b)_C \leqq a + b$ hence $-a + (a + b)_C \leqq b$ so that $-a + (a + b)_C \leqq b_C$. It follows $-a + (a + b)_C = b_C$. The proof of the second equality is similar.

2.7.4. $(a - a_C)_C = 0$ *for each* $a \in G^+$.

Proof. Using 2.7.3. to $a = (a - a_C) + a_C$ we get $a_C = (a - a_C)_C + a_C$.

2.7.5. *Let* $a \in G^+$, a_C *exist and* $a \leqq u$. *Then* $a_C \in a \wedge u_C$.

Proof. Obviously $a_C \leqq a$ and $a_C \leqq u_C$. If $a_C \leqq d \leqq a$ and $d \leqq u_C$ then $d \in C$ (C is convex) hence $d \leqq a_C$ so that $d = a_C$.

2.7.6. We shall use the following equivalence relations on G^+.

$$a\Theta b \text{ iff } a_C = b_C, \qquad a\Phi b \text{ iff } a - b \in C.$$

It can be easily checked that the blocks $[a]\,\Theta$ and $[a]\,\Phi$ ($a \in G^+$) are convex.

2.7.7. *Let* $a, a', b \in G^+$, $a \leqq b$ *and* $a\Theta a'(a\Phi a')$. *Then* $b' \in G^+$ *exists such that* $a' \leqq b'$ *and* $b\Theta b'(b\Phi b')$.

Proof. If $a\Theta a'$ take $b' = a' - a'_C + b_C$. Obviously $a' \leqq b'$. Using 2.7.3. and 2.7.4. we get $b'_C = (a' - a'_C)_C + b_C = b_C$. If $a\Phi a'$ then $b' = a' - a + b$ will do.

2.7.8. $\Theta \cap \Phi = \mathrm{id}_{G^+}$.

Proof. Let $a\Theta \cap \Phi b$. Then $a - b = e \in C$ and $a_C = b_C$. The element $-e + a_C$ belongs to C and $-e + a_C \leqq -e + a = b$. Since b_C is the greatest element of $C \cap (b]$, $-e + a_C \leqq b_C$, hence $a_C \leqq e + b_C \leqq e + b = a$. But $e + b_C \in C$ hence $a_C = e + b_C$ so that $e = 0$ and $a = b$.

2.7.9. *Let* $a, b, u \in G^+$, $u \in l(a, b)$ *and* $a\Phi b$. *Then* $v \in G^+$ *exists such that* $u \leqq v \in l(a, b)$ *and* $a\Phi v$.

Proof. First let $u = 0$. From $a\Phi b$ we get $a - b = e \in C$, $e \leqq a$, hence $e \leqq a_C \leqq a$, $a = e + b \leqq a_C + b$, $0 \leqq -a_C + a \leqq b$. Hence $-a_C + a$ is the desired element v.

Now let u be an arbitrary element of G^+ with $u \in l(a, b)$. Then $0 \in l(a - u, b - u)$ and $(a - u)\,\Phi(b - u)$. Hence $w \in G^+$ exists such that $w \in l(a - u, b - u)$ and $(a - u)\,\Phi w$. Then $u \leqq w + u \in l(a, b)$ and $a\Phi(w + u)$.

2.7.10. *Let* $a, b \in G^+$, $a\Phi b$ *and* $v \in u(a, b)$. *Then* v' *exists such that* $v \geqq v' \in u(a, b)$ *and* $a\Phi v'$.

Proof. $a\Phi b$ implies $a - b \in C$. Let $v' \in (a \vee b)_V$. Then $v' - b \in (a - b) \vee 0 \subseteq C$ (by 2.5). Hence $v'\Phi b$. It implies (together with $b\Phi a$) $a\Phi v'$.

2.7.11. $\Theta \,.\, \Phi = \Phi \,.\, \Theta$.

Proof. It suffices to show $\Theta \,.\, \Phi \leqq \Phi \,.\, \Theta$. Suppose $a\Theta t\Phi b$ $(a, b, t \in G^+)$. Then $a_C = t_C$ and $b - t = d \in C$. Using 2.7.3 we get $b_C = d + t_C$. Since $d \leqq d + a$, $(d + a)_C$ exists and $(d + a)_C = d + a_C = b_C$ hence $(d + a)\,\Theta b$. But $(d + a)\,\Phi a$ hence $a\Phi \,.\, \Theta b$.

2.7.12. $\Theta \vee \Phi = G^+ \times G^+$.

Proof. $a\Theta a_C\Phi b_C\Theta b$.

2.7.13. In the following propositions 2.7.13.1.—2.7.13.5. we suppose that G is distributive.

2.7.13.1. *If* $a, b \in G^+$, $a_C = b_C$ *and* $u \in a \vee b$ *then* $u_C = a_C$.

Proof. According to 2.7.5 $a_C \in a \wedge u_C$ and $a_C \in b \wedge u_C$. Choose $d \in (b \vee u_C)_u$, $v \in (a \wedge b)_{a_C}$, $g \in (v \vee u_C)_d$ and $e \in (a \wedge d)_v$. By 2.4, $d \in e \vee b$ and, obviously, $v \in b \wedge e$. According to the assertion dual to 2.4, $v \in g \wedge b$. We get $d \in (b \vee e) \cap (b \vee g)$ and $v \in (b \wedge e) \cap (b \wedge g)$ hence $e = g$ because of the distributivity. Then $u_C \leqq e$ which together with $e \leqq a$ gives $u_C \leqq a$, hence $u_C \leqq a_C$. But $a \leqq u$ implies $a_C \leqq u_C$ so that $u_C = a_C$.

2.7.13.2. *Let* $a, b, c \in G^+$ *and* $a \leqq b$. *Then both* $a\Theta c\Phi b$ *and* $a\Phi c\Theta b$ *imply* $a \leqq c \leqq b$.

Proof. a) Let $a\Theta c\Phi b$. Using the directedness of G and 2.7.10 we get that $v \in u(b, c)$ exists such that $b\Phi v\Phi c$. Let $u \in (a \vee c)_v$. By 2.7.13.1 $u\Theta c$ hence $u\Theta \cap \Phi c$ (the blocks of Φ are convex), so that $u = c$ and $a \leqq c$. Using 2.7.9. we get that $d \in l(b, c)$ exists such that $a \leqq d$ and $c\Phi d$. Then $c\Theta \cap \Phi d$ hence $c = d$, so that $c \leqq b$.

b) Suppose now $a\Phi c\Theta b$. Let $u \in b \vee c$. Then $u\Theta c$ by 2.7.13.1. Using 2.7.10 we get an element $v \in u(a, c)$ such that $v \leqq u$ and $v\Phi c$, hence $v\Theta \cap \Phi c$ and $v = c$, so that $a \leqq c$. Take $t \in (b \wedge c)_a$. Then $c\Phi t$ i.e. $c - t \in C$. According to 2.6. the element $c - t + b = s$ belongs to $b \vee c$. Moreover $s - b \in C$ hence $s\Phi b$. Using 2.7.13.1 we get $s\Theta b$ so that $s = b$, hence $c \leqq b$.

2.7.13.3. *$a \mapsto ([a]\,\Theta,\ [a]\,\Phi)$ is an isomorphism $(G^+;\ \leqq) \cong (G^+;\ \leqq)/\Theta \times (G^+;\ \leqq)/\Phi$.*

Proof. From 2.7.8, 2.7.11, 2.7.12, 2.7.13.2 and 2.7.7 we infer that the equivalence relations Θ, Φ satisfy the conditions of the proposition 2.3.

2.7.13.4. *$(G^+;\ \leqq) = (A;\ \leqq)\,.\,(B;\ \leqq)$ where $A = C \cap G^+ = [0]\,\Phi$ and $B = [0]\,\Theta$.*

Proof. Let f: $a \mapsto ([a]\,\Theta,\ [a]\,\Phi)$ be the isomorphism in 2.7.13.3. Using the considerations in 2.1. we get that $(G^+;\ \leqq) = (E;\ \leqq)\,.\,(D;\ \leqq)$ where $E = f^{-1}(\{([x]\,\Theta,\ [0]\,\Phi)\colon x \in G^+\})$ and $D = f^{-1}(\{([0]\,\Theta,\ [y]\,\Phi)\colon y \in G^+\})$. But $a \in E$ iff $[a]\,\Phi = [0]\,\Phi$ so that $E = A$. Analogously, $D = B$.

2.7.13.5. Proof of Theorem 1.1. Let C satisfy (1), (2), (3). According to 2.7.13.4 $(G^+;\ \leqq) = (A;\ \leqq)\,.\,(B;\ \leqq)$. By 2.2.*B* there is a direct product decomposition of the multilattice group **G**,

$$\mathbf{G} \cong \mathbf{K} \times \mathbf{L},$$

where **K** and **L** are p.o. subgroups of **G** and $K^+ = A$, $L^+ = B$. Obviously **K** is a convex multilattice subgroup of **G**. It follows $K^+ = C^+$ and from this we infer that $K^- = C^-$. Now let $a \in C$. By 2.5, $a \vee 0 \subset C^+$ and $a \wedge 0 \subset C^-$, hence $a \vee 0 \subset K^+$ and $a \wedge 0 \subset K^-$, too. It follows that $a \in K$, thus $C \subset K$. By the same reasoning we get $K \subset C$. This proves that C forms a direct factor of **G**.

Conversely, if C forms a direct factor of **G**, i.e. $\mathbf{G} = \mathbf{C}\,.\,\mathbf{D}$ for some **D**, then obviously C has the properties (1), (2) and (3) stated in Theorem 1.1. (If in the above isomorphism $a \mapsto (x, y)$ and $C \cap (a] \neq \emptyset$ then $a_C \mapsto (x, 0)$.)

2.7.14. Example. The following example of a multilattice group, occurring in [1], shows that in Theorem 1.1 the condition of distributivity cannot be omitted. Let $G = Z \times Z \times Z$ where Z is the additive group of integers with the natural order, and let $H = \{(a, b, c) \in Z \times Z \times Z\colon a + b + c \text{ even}\}$.

Define in H the operation $+$ and the order relation $\leqq$ componentwise. Then $(H;\ +,\ \leqq)$ is a non-distributive multilattice group. The subset $C = \{(a, 0, 0)\colon a \text{ even}\}$ of H has the properties (1), (2), (3) in Theorem 1.1 but it does not form a direct factor of H. For suppose $H = C\,.\,D$. Then any element $x \in H$ can be uniquely represented in the form $x = c + d$, $c \in C$, $d \in D$. Then $(1, 1, 0) = (a, 0, 0) + (1 - a, 1, 0)$, a even, $0 \leqq a$, $0 \leqq 1 - a$ hence $a = 0$ and $(1, 1, 0) \in D$.

Similarly we get that $(1, 0, 1)$ and $(0, 1, 1)$ belong to D. Then $(1, 0, -1) = (1, 1, 0) - (0, 1, 1) \in D$, $(2, 0, 0) = (1, 0, -1) + (1, 0, 1) \in D$, but $(2, 0, 0) \in C$ – a contradiction.

Added in Proof: Recently J. Lihová generalized the main result of this paper without the assumption of distributivity.

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