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ON SOME INEQUALITIES CHARACTERIZING THE EXPONENTIAL FUNCTION

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Abstract. Some inequalities relating the slope of a function and mean values are completely solved. Characterizations of the exponential function are obtained.

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Recently B. Poonen [2] has characterized (up to multiplicative constants) the exponential function e^x in terms of the system of (simultaneous) inequalities

$$\min(f(x), f(y)) \leqq \frac{f(y) - f(x)}{y - x} \leqq \max(f(x), f(y)).$$

In this paper we treat separately the above two inequalities as well as others in close connection (as we shall see) with these. As a result we obtain nondifferentiable and also discontinuous solutions and sharper bounds for the particular case of the exponential functions.

Our first aim, to begin with, is to study functions $f: R \to R$ satisfying, for all $x < y$

$$\frac{f(y) - f(x)}{y - x} \leqq \frac{f(x) + f(y)}{2}. \tag{1}$$

Examples of such functions are given in the following:

Lemma 1. *Given any function $g: R \to R$ which is non-increasing then $f(x) = g(x)\, e^x$ is a solution of* (1).

Proof. First we show that the exponential function satisfies (1), i.e., for all $x < y$

$$\frac{e^y - e^x}{y - x} \leqq \frac{e^x + e^y}{2}. \tag{2}$$

In fact, consider $h : [0, \infty) \to R$ given by $h(t) = (t - 2) e^t + t + 2$. Since h has non-negative derivative for $t \geqq 0$ and vanishes at $t = 0$ we have $0 = h(0) \leqq h(t)$, for $t > 0$, and from this taking $t = y - x$ the inequality (2) follows at once. Now let us take $g : R \to R$ to be any non-increasing function and consider $f(x) = g(x) e^x$. If $x < y$ then $g(x) = f(x) e^{-x} \geqq f(y) e^{-y} = g(y)$, i.e., $f(x) \geqq f(y) e^{x-y}$. Therefore by (2) we have

$$\frac{f(y) - f(x)}{y - x} \leqq \frac{f(y) - f(y) e^{x-y}}{y - x} = \frac{f(y)}{e^y} \frac{e^y - e^x}{y - x} \leqq \frac{f(y)}{e^y} \frac{e^x + e^y}{2} =$$
$$= \frac{f(y) e^{x-y} + f(y)}{2} \leqq \frac{f(x) + f(y)}{2},$$

i.e., (1) holds.

We will show that the example given above constitutes, in fact, the general solution of (1).

Theorem 1. *A function $f: R \to R$ satisfies* (1) *if and only if f can be represented in the form $f(x) = g(x) e^x$ where $g: R \to R$ is a non-increasing function.*

Proof. Sufficiency follows from Lemma 1. To prove necessity, if f satisfies (1) let us take $h \in (0,2)$ and $y = x + h$ in (1), i.e.

$$\frac{f(x + h) - f(x)}{h} \leqq \frac{f(x) - f(x + h)}{2},$$

or equivalently

$$f(x + h) \leqq f(x) \frac{2 + h}{2 - h},$$

and by iteration

$$f(x + nh) \leqq f(x) \left(\frac{2 + h}{2 - h}\right)^n, \qquad \text{for } n = 1, 2, \dots \tag{3}$$

If $t > 0$ is fixed there exists a large n_0 such that for $n > n_0$ we have $h_n = t/n \in (0,2)$ and by (3) if we let n tend to infinity we eventually get

$$f(x + t) \leqq f(x) e^t,$$

whence

$$f(x + t) e^{-x-t} \leqq f(x) e^{-x},$$

i.e., $g(x) = f(x) e^{-x}$ is non-increasing.

Using the previous theorem it is easy to construct non-monotonic and discontinuous solutions of (1). As corollaries we will present the following results needed in the sequel:

Corollary 1. *A continuous function $f: R \to R$ satisfies* (1) *if and only if there exists a continous non-increasing function $g : R \to R$ such that $f(x) = g(x) e^x$.*

Corollary 2. *A differentiable function* $f : R \to R$ *satisfies* (1) *if and only if* $f(x) = g(x)\, e^x$, *where* $g : R \to R$ *is any differentiable non-increasing function. In this case* (1) *is equivalent to the inequality*

$$f'(x) \leqq f(x), \qquad \text{for all } x \text{ in } R. \tag{4}$$

Remark 1. The procedure used in the proof of Theorem 1 can also be applied to the more general inequality

$$\frac{f(y) - f(x)}{y - x} \leqq \max\,(f(x), f(y)), \tag{1$'$}$$

for $x < y$, yielding actually the same result. Indeed, (1)′ can be rewritten, setting $h = y - x$, as either

$$f(x + h) \leqq (1 + h)\, f(x)$$

or

$$f(x + h) \leqq \frac{1}{1 - h} f(x), \qquad \text{for } h \in (0, 1),$$

according as $\max\,(f(x), f(x + h))$ is respectively $f(x)$ or $f(x + h)$.

By iterating we obtain in all cases

$$f(x + nh) \leqq \frac{(1 + h)^r}{(1 - h)^{n-r}} f(x), \tag{3$'$}$$

for $h \in (0,1)$ and some integer r between 0 and n. But

$$(1 + h)^n \leqq \frac{(1 + h)^r}{(1 - h)^{n-r}} \leqq \frac{1}{(1 - h)^n} = (1 + h + h^2 + \ldots)^n,$$

and setting now, as in Theorem 1, $h_n = t/n \in (0,1)$ for large n, and observing that the outside expressions of the preceding chain of inequalities both tend to e^t, we eventually obtain, as in Theorem 1, for any positive t

$$f(x + t) \leqq e^t f(x).$$

In particular we see that the more generality of inequality (1)′ with respect to (1) is actually apparent, corollaries 1 and 2 can also be stated for (1)′ in place of (1), and, at the same time, (1) gives a sharper bound than (1)′ for exponential functions.

Remark 2. If we now play the same game this time with (1)′ replaced by

$$\min\,(f(x), f(y)) \leqq \frac{f(y) - f(x)}{y - x} \tag{1$''$}$$

for $x < y$, we obtain exactly (3)′ with the inequality sign reversed. This entails $f(x + t) \geqq e^t f(x)$ so that (1)′ and (1)″ together imply $f(x + t) = e^t f(x)$. Setting

$x = 0$, we see that the only solutions of (1)′ and (1)″ are exactly the functions $f(x) = Ke^x$ for any constant K. This is just another proof of the result of B. Poonen [1] answering a problem proposed by D. Shelyupsky [2].

Now we fix our attention in the inequalities, for all $x < y$

$$0 \leqq \frac{f(y) - f(x)}{y - x} \leqq \frac{f(x) + f(y)}{2}. \tag{5}$$

Conditions (5) are equivalent to the fact that f satisfies (1) and f is nondecreasing. If this is the case then we have.

Theorem 2. *A function $f: R \to R$ satisfies* (5) *if and only if f can be represented in the form $f(x) = g(x)\, e^x$, where $g: R \to R$ is a continuous non-increasing function such that for all x in R and for all $t > 0$*

$$g(x + t) \geqq e^{-t} g(x). \tag{6}$$

Proof. In view of the previous results we just need to prove that any solution f of (5) is continuous. In fact, as f is increasing, if x_0 happens to be a discontinuity point for f, taking $x = x_0 - h$ and $y = x_0 + h$, for small positive h say $h \leqq 1$, then by (5) we would have

$$0 \leqq \frac{f(x_0 + h) - f(x_0 - h)}{2h} \leqq \frac{f(x_0 + h) + f(x_0 - h)}{2} \leqq f(x_0 + h) \leqq f(x_0 + 1).$$

Then the term $(f(x_0 + h) - f(x_0 - h))/2h$ tends to infinity when we let h go to zero from the right, while this same term remains bounded by $f(x_0 + 1)$. Thus f must be continuous. Note that (6) follows from Theorem 1 and the fact that f is increasing.

Corollary 3. *A differentiable function $f: R \to R$ satisfies* (5) *if and only if, for all x*

$$0 \leqq f'(x) \leqq f(x); \tag{7}$$

and this holds if and only if f can be represented in the form $f(x) = g(x)\, e^x$ where $g: R \to R$ is a differentiable function such that $g(x) \geqq -g'(x) \geqq 0$.

Remark 3. Both Theorem 2 and corollary 3 obviously hold if we replace $\dfrac{f(x) + f(y)}{2}$ in (5) by max $(f(x), f(y))$, or, more generally, by any mean lying between these two explicit ones.

Theorem 3. *Let M be a continuous two-place function from $R^+ \times RR^+$ into R^+ such that $M(x, x) = x$, for all $x \geqq 0$. A function $f: R \to R^+$ satisfies the inequalities*

(7) $$M(f(x), f(y)) \leqq \frac{f(y) - f(x)}{y - x} \leqq \frac{f(x) + f(y)}{2},$$

for all $x < y$, if and only if

$$f(x) = Ke^x, \qquad \text{where } K \geqq 0$$

and M satisfies

(8) $$M(x, y) \leqq \frac{y - x}{\ln y - \ln x} \qquad \text{for all } x < y;$$

Proof. If f satisfies (7), since $M(f(x), f(y)) \geqq 0$ we have that f satisfies (5) and therefore, by Theorem 2, f is continuous. In view of (7) and the continuity of f and M the differentiability of f follows at once and, moreover, $f'(x) = f(x)$, i.e., $f(x) = Ke^x$, with $K \geqq 0$. Then substituting $f(x) = Ke^x$ into (7) we obtain (8).

Remark 4. Theorem 3 obviously holds if we replace $\frac{f(x) + f(y)}{2}$ in (7) by any mean lying between it and $\max(f(x), f(y))$.

Corollary 4. *A function f: $R \to R^+$ satisfies*

$$\sqrt{f(x) f(y)} \leqq \frac{f(y) - f(x)}{y - x} \leqq \frac{f(x) + f(y)}{2},$$

if and only if $f(x) = Ke^x$, where $K \geqq 0$.

Proof. We have to observe just that the geometric mean satisfies (8), i.e., is bounded above by the logarithmic mean, and this follows because of the inequality $e^{h/2} \leqq (e^h - 1)/h$ for $h > 0$, which can easily be checked by looking at the corresponding series expansions.

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