Pavol Hell; Jaroslav Nešetřil
Universality of directed graphs of a given height

Archivum Mathematicum, Vol. 25 (1989), No. 1-2, 47--54

Persistent URL: http://dml.cz/dmlcz/107338

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 25, No. 1–2 (1989), 47–54

UNIVERSALITY OF DIRECTED GRAPHS OF A GIVEN HEIGHT

PAVOL HELL and JAROSLAV NEŠETŘIL

(Received April 20, 1988)

Dedicated to the memory of Milan Sekanina

Abstract. We consider the classes of directed graphs which are determined by the existence of a homomorphism into (or from) a fixed graph. We completely answer the question when a class of this type is universal.

MS Classification. 05 C 20

1. INTRODUCTION

In this paper we deal with directed graphs (without loops and multiple arcs) Graphs may be infinite.

Given graphs $G = (V, E)$, $H = (W, F)$, a *homomorphism* $f : G \to H$ is a mapping $V \to W$ which satisfies $(f(x), f(y)) \in F$ for every $(x, y) \in E$. We also may say that *G maps into H* and we denote it by $G \to H$.

Denote by *GRA* the category of all graphs and all their homomorphism. Any category $\mathscr{K}$ for which there exists an embedding of *GRA* into $\mathscr{K}$ is said to be *universal* (binding), see [5], [3]. A universal category is very rich in the sense that every concrete category may be embedded into it.

One of the main streams in the study of universal categories is formed by efforts to find simple examples of universal categories, see [1], [2], [5], [8], [9] for numerous examples in various areas of mathematics.

In this context perhaps it is worth to mention the following. Some time ago M. Sekanina and the second author investigated the universality of classes of graphs related to Sekanina's characterization of Hamiltonian powers of graphs [12]:

Supported by Sonderforschungsbereich 303(DFG), Institut für Operations Research, Universität Bonn, W. Germany and the Alexander v. Humboldt Stiftung.
Edited jointly with KAM-Series 88-80, Department of Applied Mathematics, Charles University, Prague.

Let k be a positive integer, let $G = (V, E)$ be an undirected graph. Denote by $G^{(k)} = (V, E^{(k)})$ the graph defined by

$$[x, y] \in E^{(k)} \quad \text{iff} \quad x \neq y \quad \text{and} \quad d_G(x, y) \leqq k.$$

Here $d_G(x, y)$ is the distance of x and y in G. We call $G^{(k)}$ the *k-th power* of G. Among the rèsult which Sèkanina and Nešetřil obtained and which were not yet published is the following:

1.1. Theorem. *Let k be a positive integer. Then the class $Gra^{(k)}$ of all k-th powers is a universal category.*

In this note we consider the following classes of graphs from the point of view of their universality. Let A be a graph. We introduce the following special subclasses of the class *Gra*:

$$\begin{aligned} \to A \; &= \{G; G \to A\}, \\ \not\to A \; &= \{G; G \not\to A\}, \\ A \to \; &= \{G; A \to G\}, \\ A \not\to \; &= \{G; A \not\to G\}. \end{aligned}$$

These classes were investigated previously in various context: in [10] from the point of view of algorithmic complexity and in [15] from the point of view of algebraic properties (such as the existence of products).

In [2] and [1] we considered the classes of undirected graphs which contain a given graph as a subgraph. As an easy modification we get from this the following:

1.2. Proposition. *For every graph A the classes $\not\to A$ and $A \to$ are universal.*

For the remaining two cases we do not get always an affirmative answer and we give a full solution in this paper. This is stated below as Theorem 3.1 and 3.2.

The motivation of this paper is two fold: First we want to complement the research for undirected graphs [1], [2]. Secondly the questions considered in this paper naturally arised in the study of directed rigid graphs, see our companion paper [4]. Our results support the common belief that the directed graphs although sometimes easier to construct are in the context of categorial representations mostly more difficult to analyse.

The key to our analysis is the study of balanced graphs. This is contained in Section 2 where we define invariants $\lambda(G)$ and $\Lambda(G)$; $\Lambda(G)$ is called the *height* of G. In Section 3 we state our main results. It appears that it suffices to consider the case $\to A$ as the case $A \not\to$ is a byproduct of our proof.

A bit surprisingly the universality of a class $\to A$ is fully characterized by a fact whether it contains (just) two mutually rigid graphs. A graph G is *rigid* if the identity is the only homomorphism $G \to G$. Two rigid graphs G and H are said to be *mutually rigid* if they are rigid and there are no homomorphisms $G \to H$ and $H \to G$.

2. BALANCED GRAPHS

Definition 2.1. *A cycle is balanced if it has the same number of arcs going one way as going the other way (with respect to a fixed transversal of the cycle). A directed graph $G = (U, E)$ is balanced if each of its cycles is balanced. The net length of a path is the number of arcs going forward minus the number going backwards. A directed path of length n (i.e. with $n + 1$ vertices) will be denoted by $\vec{P}_n$. Finally, $\vec{P}_\infty$ denotes the doubly infinite directed path.*

Proposition 2.2. *For a directed graph G the following two statements are equivalent;*
1. G is balanced,
2. there is a homomorphism $G \to \vec{P}_\infty$.

Proof. Since the homomorphic image of an unbalanced cycle must contain an unbalanced cycle, it suffices to prove that 1. implies 2. Without loss of generality let G be a connected balanced graph. Any two paths with a fixed beginning and a fixed end have the same net length. Let x be a fixed vertex of G and let $f(y)$ be the net length of any path from x to y. One can check that f is a homomorphism $G \to \vec{P}_\infty$.

□

This leads to the following:

Definition 2.3. *Let G be a balanced graph. Let $\Lambda(G)$ be the minimum n such that there exists a homomorphism $G \to \vec{P}_n$. (Possibly $n = \infty$). We call $\Lambda(G)$ the height of G. Denote also $\lambda(G)$ the maximum n such that there exists a homomorphism $\vec{P}_n \to G$. Clearly $\lambda(G) \leqq \Lambda(G)$.*

Let us remark that it follows from the above proof of Proposition 2.2 that for a connected graph G a homomorphism $f: G \to \vec{P}_\infty$ is uniquely determined by the value $f(x)$ for any one vertex x of G. It follows that for a connected balanced G with finite height Λ there exists unique homomorphism $f: G \to \vec{P}_\Lambda$. This homomorphism will also be denoted by Λ. By convention, we let Λ denote an arbitrary homomorphism $G \to \vec{P}_\infty$ if G has infinite height.

This has several corollaries. We want to mention the following results explicitely as we shall need them later:

Lemma 2.4. *Let G be a connected balanced graph with finite $\Lambda(G)$. Then $\Lambda(x) = \max\{\Lambda(P) \mid P$ is a path in G which terminates in $x\}$.*

□

Lemma 2.5. *Let G and H be balanced, $f: G \to H$ a homomorphism. Then $\Lambda(G) \leqq \Lambda(H)$.*

□

Lemma 2.6. *Let G and H be connected balanced graphs with $\Lambda(G) = \Lambda(H) < \infty$, and let $f: G \to H$ be a homomorphism. Then f preserves Λ. (Explicitely $\Lambda_H(f(x)) = \Lambda_G(x)$ for every $x \in V(G)$.)*

□

Finally we have

Proposition 2.7. *Let G be a rigid balanced graph with finite $\Lambda(G)$. Then G contains a rigid path P with $\Lambda(G) = \Lambda(P)$.*

Proof. Let P be a shortest path (i.e., having the fewest arcs) with $\lambda(P) = \lambda(G)$. (It exists by 2.4). Then P can be seen to be rigid by 2.6.

□

Remark. Of course 2.7 need not hold for infinite Λ.

An antidirected path is a path P with $\lambda(P) = 1$. Denote by $a(G)$ the maximal length (number of arcs) of an antidirected path in G. We put $a(G) = \infty$ if there are arbitrarily long antidirected paths. As we shall see below the numbers $a(G)$ may be used for testing the existence a homomorphism.

We begin our investigation of balanced rigid graphs of small height with an analysis of rigid trees.

Denote by T_a the path of length $2a + 3$ which contains an antidirected path of length $2a + 1$ and does not contain directed path of length 3. It is easy to see that T_a is uniquely determined (up to isomorphism). The path T_3 is depicted in Fig. 1 (where all arcs are directed upwards).

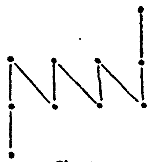

Fig. 1

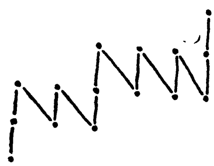

Fig. 2

Similarly, $T_{a,b}$ will denote a path of length $2a + 2b + 4$ and height 4, as illustrated in Fig. 2.

Proposition 2.8. *For a fixed Λ the following two statements are equivalent*

1. There are mutually rigid trees T, T', $\Lambda(T) = \Lambda(T') = \Lambda$,

2. $\Lambda \geqq 4$.

Proof. 2. $\Rightarrow$ 1. Consider trees $T_{a,b}$.

Then, using 1.6 there exists a homomorphism $f: T_{ab} \to T_{a'b'}$ if and only if $a \leqq a'$, $b \leqq b'$ Thus $T_{1,2}$ and $T_{2,1}$ are mutually rigid. It is easy to extend these to $T'_{2,1}$ and $T'_{1,2}$ respectively, so that $T'_{2,1}$ and the $T'_{1,2}$ remain mutually rigid, and have the required Λ.

1. $\Rightarrow$ 2. Exhausting a few cases one can check that all rigid trees with $\Lambda \leqq 3$ are

directed paths with $\Lambda \leqq 2$ and the graphs T_a, $a \geqq 1$. (In the only non-trivial case $\Lambda = 3$, this also follows from the next Proposition.)

□

The next result characterises rigid graphs with height $\leqq 3$. Recall that a *retract* of a graph G is a subgraph H of G such that there exists a homomorphism $G \to H$ with $f(H) = h$ for all $h \in V(H)$.

Proposition 2.8. (1) *Let G be connected and balanced,* $\Lambda(G) = 3$. *Then there exists an a such that G has a retract isomorphic to* T_a. (2) *Let G be connected and balanced,* $\Lambda(G) = i$, $i = 0, 1, 2$. *Then G has a retract isomorphic to* $\vec{P}_i$.

Proof. Let a be the minimal such that T_a is a subgraph of G. Put $V(T_a) = x_0$, $x_{1(0)}, x_{2(0)}, x_{1(1)}, \ldots, x_{1(a)}, x_{2(a)}, x_3$. We show that T_a is a retract of G. Define $r : G \to T_a$ by the following:

$r(z)$ = the unique vertex ξ of T_a with $\lambda(\xi) = \lambda(z)$ and with the distance (# arcs) to x_0 at least $\min((2a + 3), d_z)$ (where d_z is the minimum distance between z and any vertex v with $\Lambda(v) = 0$ in G).

This r maps all z with $\Lambda(z) = 0$ to x_0, all z with $\Lambda(z) = 3$ to x_3 (by minimality of a) and all other vertices "as far away from x_0 as possible". It is easy to see that r is a homomorphism, and that $r(z) = z$ if $z \in T_a$.

The proof of (2) is easy. Since $\vec{P}_i \to G \to P_i$ is rigid, $G \to \vec{P}_i$ must be a retraction.

□

3. MAIN RESULTS

Now we can formulate our main results:

Theorem 3.1. *For a directed graph A the following three statements are equivalent:*

1. Either A is unbalanced or $\Lambda(A) \geqq 4$;

2. There are two mutually rigid paths P_1 *and* P_2 *of height* 4 *which admit homomorphism into A;*

3. The class $\to A$ *is universal.*

Theorem 3.2. *For a directed graph A the following two statements are equivalent:*

1. Either A is unbalanced or $\Lambda(A) \geqq 3$;

2. The class $A \nrightarrow$ *is universal.*

First, we shall prove Theorem 3.1, Theorem 3.2 will be proved similarly.

We shall make use of the following:

Lemma 3.3. *Let P be a rigid finite path,* $\lambda(P) \geqq 4$. *Then there are mutually rigid paths* P_1, P_2 *such that P is a homomorphic image of both* P_1 *and* P_2.

Proof. Put $a(P) = k$. An antidirected path in P is called of type 1 (type 2, respectively) if it contains only vertices x with $\Lambda(x) = 1$ and 2 ($\Lambda(x) = 2$ and 3, respectively). (Note that every antidirected path contains vertices with two values of Λ only.) Let P_1 (P_2 respectively) be the path which is obtained from P replacing every antidirected path of length a of type 1 (type 2 respectively) by an antidirected path of length $k + a$. It is easy to check (using 2.6) that P_1, P_2 are rigid, that there is no homomorphism $P_1 \to P_2$ and $P_2 \to P_1$, and that P is a homomorphic image of both P_1 and P_2.

□

Proof *of Theorem 3.1.* 1. $\Leftrightarrow$ 2. is a combination of Lemma 3.3 and Proposition 2.8. Next, we prove $3 \Rightarrow 2$, which is easier. Of course it follows from universality that there are 2 mutually rigid graphs G_1, G_2 which admit homomorphisms to H. Using Proposition 2.8 and Lemma 2.6 we get $\Lambda(H) \geqq 4$. Combining Proposition 2.7 with Lemma 3.3 yields 2.

Now we prove 2. $\Rightarrow$ 3.

Let P_1, P_2 be two mutually-rigid paths of height 4. Explicitely, let $P_i = (V_i, E_i)$, $i = 1, 2$. Let $a_i^0, a_i \in V_i$ satisfy $\Lambda(a_i^0) = 0$, $\Lambda(a_i) = 3$, $i = 1, 2$. Let $k \geqq \max\{a(P_1)\, a(P_2)\}$ be a fixed odd number. Let $G = (V, E)$ be a given antisymmetric digraph (i.e. such that $(x, y) \in E \Rightarrow (y, x) \in E$).

We shall construct a directed graph $G^* = (V^*, E^*)$ as follows:

$$V^* = (V \times V_1) \cup (E \times V_2) \cup (E \times \{a_1, \ldots, a_k, b_1, \ldots, b_k\}).$$

The set of arcs consists of the following arcs:

$$((v, v_1), (v, v_1')) \quad \text{where} \quad (v_1, v_1') \in E_1,$$

$$((e, v_2), (e, v_2')) \quad \text{where} \quad (v_2, v_2') \in E_2.$$

Furthermore, for any $e = (v, v') \in E$, let the vertices $(v, a_1^0), (e, a_1), (e, a_2), \ldots, (e, a_k), (e, a_2^0)$ and the vertices $(e, a_2^3), (e, b_1), (e, b_2), \ldots, (e, b_k), (v', a_1^3)$ form an antidirected path of length $k + 1$ with $((v, a_1^0), (e, a_1)) \in E^*$ and $((e, a_2^3), (e, b_1)) \in E^*$.

Thus the graph G^* is obtained from G by replacing every vertex by a copy of P_1 and every edge of G by a copy of P_2 and by joining appropriate copies by "long" antidirected paths. Obviously G^* admits a homomorphism to H. See also Fig. 3 (again all arrows upwards):

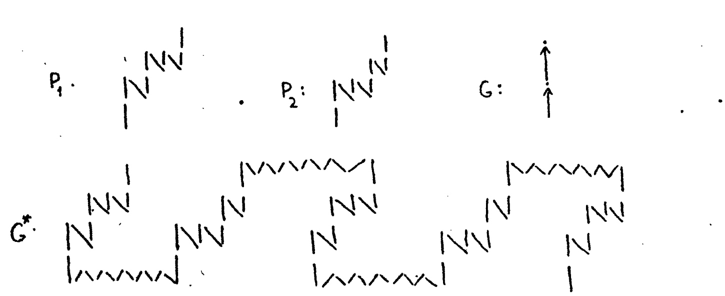

Fig. 3

Now it should be clear that if $G = (V, E)$ and $G' = (V', E')$ are directed graphs and $f : G \to G'$ is a homomorphism then f induces a homomorphism $f^* : G^* \to G^*$. The mapping f^* may be defined by

$$f^*(v, v_1) = (f(v), v_1),$$
$$f^*((v, v'), x) = ((f(v), f(v')), x).$$

On the other hand, if $g : G^* \to G'^*$ is a homomorphism, then (using the mutual rigidity of P_1 and P_2 and the assumption on k) we have

$$g(\{v\} \times V_1) = \{v\} \times V_1,$$
$$g(\{e\} \times V_2) = \{e'\} \times V_2.$$

Put $\bar{v} = f(v)$. It is also clear from construction that $e' = (f(v), f(v'))$ if $e = (v, v')$. Thus $g = f^*$.

Consequently the homomorphisms between graphs G^* and G'^* are in $1 - 1$ correspondence with homomorphisms between G and G'. This correspondence establishes the desired embedding of the category of all antisymmetric graphs into the category of all digraphs which admit homomorphisms to H.

□

Proof *of Theorem 3.2.* We do not need to worry about homomorphic image. Thus let P be a path indicated on Fig. 4:

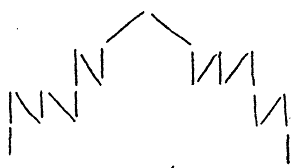

Fig. 4

It is easy to show that P is a rigid graph. For a given antisymmetric graph $G + (V, E)$ we can construct a directed graph $G^* = (V^*, E^*)$ by replacing every edge of G by a copy of the path P. It is a routine to check that every homomorphism between G^* and H^* is induced by a homomorphism between G and H. This is similar (in fact easier) to the above proof, we leave the details.

□

REFERENCES

[1] M. E. Adams, J. Nešetřil and J. Sichler, ***Quotients of rigid graphs.*** **J. Comb. Th. (B), 30, 3 (1981), 351–359.**

[2] L. Babai and J. Nešetřil, ***High chromatic rigid graphs I.*** **In: Coll. Math. Soc. Janos Bolyai H. Combinatorics, North Holland (1978), 53–60.**

[3] Z. Hedrlin and A. Pultr, ***Symmetric relations (undirected graphs) with given semigroups.*** **Monatsh. für Math. 69 (1965), 318–322.**

[4] P. Hell and J. Nešetřil, Images of rigid graphs (to appear).
[5] P. Hell and J. Nešetřil, *Homomorphisms of graphs and of their orientations*. Monatsh. für Math. 85 (1978), 39–48.
[6] P. Hell and J. Nešetřil, *Graphs and k-societies*. Canad. Math. Bull. 13, 3 (1970), 375–381.
[7] P. Hell and J. Nešetřil, *Complexity of H-colorings* (to appear).
[8] E. Mendelsohn, *On a technique for representing semigroups as endomorphism semigroups of graphs with given properties*. Semigroup Forum 4 (1972), 283–294.
[9] J. Nešetřil, *On symmetric and antisymmetric relations*. Monatsh. für Math. 76 (1972) 323–327.
[10] J. Nešetřil, *Graph theory*. SNTL (Prague), 1979 (in Czech).
[11] J. Nešetřil and A. Pultr, *On classes of relations and graphs determined by subobjects and factorobjects*. Discrete Math. 22 (1978), 287–300.
[12] M. Sekanina, *On an ordering of the vertices of a connected graph*. Publ. Fac. Sci. Univ. Brno, No. 412 (1960), 137–142.

P. Hell
Simon Fraser University
Burnaby
B.C. 25263
Canada

J. Nešetřil
KAM MFF UK
Department of Applied Mathematics
Charles University
Malostranské nám. 25
118 00 Praha 1
Czechoslovakia