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ON CONSTANT SOLUTIONS OF THE YANG-MILLS EQUATIONS

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Abstract. The problem is posed to solve the Yang–Mills equations for constant gauge potentials. We show that for certain Lie algebras and underlying vector spaces with a scalar product only flat fields come out. For certain other cases we construct some nonflat constant Yang–Mills fields.

Key words. Yang–Mills equations, constant solutions, Lie algebras.

MS Classification. 81 E 13, 17 B 99.

INTRODUCTION

Yang–Mills theory attracts increasing attention of both physicists and mathematicians. It provides models for all four fundamental types of interactions – strong, weak, electromagnetic, and gravitational – and also candidates for grand unification. There have been spectacular successes in constructing exact solutions of the classical (i.e. unquantized) Yang–Mills equations which exhibit self-duality or a related property, namely instantons, periodic instantons, monopoles, and dyons. Other types of Yang–Mills fields which are physically relevant as well as mathematically interesting are those which obey some constancy condition, let us mention constant potentials, constant field strengths, parallel field strengths as considered in [3], and translation-invariant fields in the sense of [1, 4, 5].

In the present paper we investigate the simplest of the mentioned types – potentials satisfying the Yang–Mills equations and having constant components with respect to some gauge and some coordinate system. The partial differential equation problem collapses then to a purely algebraic problem. The Lie algebra valued fields do no longer live on a curved manifold but more modestly on a vector space. The gauge group and the group of coordinate transformations shrink to small residues.

Two main types of results are expected and will in fact come out here: In certain situations each constant Yang–Mills field turns out to be flat, i.e. to have vanishing field strength. In certain other situations non-flat constant Yang–Mills fields exist and can be described explicitly. We show that, on the one hand, each constant Yang–Mills field on a Euclidean or Lorentzian vector space with values in some compact Lie algebra is flat. On the other hand, nilpotent Lie algebras produce examples of nonflat constant Yang–Mills fields. The problem in general remains open; our paper is thought as a first step towards its solution.

THE PROBLEM

For a finite-dimensional vector space E we write

$\otimes^p E := E \otimes E \otimes \dots \otimes E =$ p-fold tensor product space,

$\wedge^p E :=$ subspace of $\otimes^p E$ consisting of alternating tensors,

$\prod^p E :=$ subspace of $\otimes^p E$ consisting of symmetric tensors,

$E^* :=$ dual vector space to E.

Definition 1. Let E be a finite-dimensional vector space, g a definite or indefinite scalar product on E and L a finite-dimensional Lie algebra.

1. An element $A \in E^* \otimes L$ is called a one-form on E with values in L or, for shortness, a potential.

2. A times A in the sense of the tensor product $\otimes$ with respect to E and the commutator product [,] with respect to L is called the field strength F of A; we write $F = [A, \otimes A] \in (\otimes^2 E) \otimes L$.

3. F times A in the sense of the inner product $\llcorner$ with respect to (E, g) and the commutator product [,] with respect to L is called the current J of A; we write $J = [F, \llcorner A] \in E^* \otimes L$.

4. A is called flat if $F = 0$.

5. A is called a constant Yang–Mills field on (E, g) with values in L if $J = 0$.

It is easy to show that F is an L-valued alternating tensor on E; thus we may write $F = [A, \wedge A] \in (\wedge^2 E^*) \otimes L$. The problem we are concerned with is to solve the Yang–Mills equation for A

$$\text{(YM)} \qquad [[A, \wedge A], \llcorner A] = 0.$$

Since both $\dim E =: n$ and $\dim L =: N$ are finite it is possible to introduce

– a base $e_\alpha = e_1, e_2, \dots, e_n$ of E,

– a base $e^\alpha = e^1, e^2, \dots, e^n$ of E^* dual to the base e_α,

– the components $g_{\alpha\beta} := g(e_\alpha, e_\beta)$ of g and the inverse $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$,

– a base $X_i = X_1, X_2, \dots, X_N$ of L, and

– the structure constants c^k_{ij} of L with respect to the base X_i.

Accordingly, there are numbers a^i_α ($\alpha = 1, 2, \dots, n$; $i = 1, 2, \dots, N$), called the components of A, such that $A = a^i_\alpha e^\alpha \otimes X_i$. We have a representation with Greek indices $\alpha, \beta, \ldots = 1, 2, \ldots, n$

$$A = e^\alpha \otimes A_\alpha, \quad \text{where} \quad A_\alpha := a^i_\alpha X_i \in L,$$
$$F = (e^\alpha \wedge e^\beta) \otimes F_{\alpha\beta}, \quad \text{where} \quad F_{\alpha\beta} := [A_\alpha, A_\beta],$$
$$J = e^\alpha \otimes J_\alpha, \quad \text{where} \quad J_\alpha := [F_{\alpha\beta}, A^\beta],$$

and alternatively a representation with Latin indices $i, j, \ldots = 1, 2, \ldots, N$

$$A = a^i \otimes X_i, \quad \text{where} \quad a^i := a^i_\alpha e^\alpha \in E^*,$$
$$F = f^k \otimes X_k, \quad \text{where} \quad f^k := c^k_{ij}(a^i \wedge a^j).$$
$$J = j^m \otimes X_m, \quad \text{where} \quad j^m := c^k_{ij} c^m_{kl}(a^i \wedge a^j) \lfloor a^l.$$

Raising and lowering of Greek indices is done by means of $(g^{\alpha\beta})$ and $(g_{\alpha\beta})$ respectively. In particular,

$$A^\alpha := g^{\alpha\beta} A_\beta, \qquad F^\alpha_{\cdot\beta} := g^{\alpha\mu} F_{\mu\beta}, \qquad F^{\alpha\beta} := g^{\beta\mu} F^\alpha_{\cdot\mu}.$$

THE RESULTS AND THEIR PROOFS

Proposition 1. *Let $\langle , \rangle$ be an ad-invariant definite or indefinite scalar product on L. Then*

(1) $$\langle J_\alpha, A_\beta \rangle = \langle F_{\alpha\mu}, F^\mu_{\cdot\beta} \rangle$$

holds as an identity and

(2) $$\langle F_{\alpha\mu}, F^\mu_{\cdot\beta} \rangle = 0$$

holds as a consequence of the Yang–Mills equation (YM).

Proof. The ad-invariance of $\langle , \rangle$ implies

$$\langle J_\alpha, A_\beta \rangle = \langle [F_{\alpha\mu}, A^\mu], A_\beta \rangle = \langle F_{\alpha\mu}, [A^\mu, A_\beta] \rangle = \langle F_{\alpha\mu}, F^\mu{}_\beta \rangle.$$

Theorem 1. *Every constant Yang–Mills field on a Euclidean or Lorentzian vector space (E, g) with values in a compact Lie algebra L is flat.*

Proof. By definition, a compact Lie algebra L admits a definite ad-invariant scalar product $\langle , \rangle$; we will work with this.

1. The trace part of (2) reads

(3) $$\langle F_{\alpha\beta}, F^{\alpha\beta} \rangle = 0.$$

If (E, g) is Euclidean then (3) and the definiteness of g and of $\langle , \rangle$ together imply $F_{\alpha\beta} = 0$ $(\alpha, \beta = 1, 2, \ldots, n)$.

2. The trace-free part of (2) reads

(4) $$T_{\alpha\beta} := \langle F_{\alpha\mu}, F^\mu_\beta \rangle - \frac{1}{n} g_{\alpha\beta} \langle F_{\mu\nu}, F^{\mu\nu} \rangle = 0.$$

Let (E, g) be Lorentzian and $n \geqq 3$ and choose a base $e^\alpha = e^1, e^a$ $(a = 2, 3, \ldots, n)$ of E^* such that $g = -e^1e^1 + e^ae^a$. Then (4) implies

$$nT_{11} = (n-2)\langle F_{1a}, F_{1a}\rangle + \langle F_{ab}, F_{ab}\rangle = 0,$$

and again definiteness arguments give $F_{1a} = 0$, $F_{ab} = 0$ $(a, b = 2, \ldots, n)$. Note that $T := T_{\alpha\beta}e^\alpha e^\beta \in \prod^2 E^*$ is physically interpreted as the energy-momentum tensor of the potential A with respect to $\langle , \rangle$ and $T_{11} = T(e_1, e_1)$ is interpreted as the energy density of A in the direction e_1.

3. For $n = 1$ every Yang–Mills field is flat. For $n = 2$ the condition (3) reduces to $\langle F_{12}, F_{12}\rangle = 0$, and the definiteness of $\langle , \rangle$ implies $F_{12} = 0$.

Example 1. Theorem 1 applies to the Cartan series of simple compact Lie algebras

$$\begin{aligned} A_l &= su(l+1) && \text{for } l \geqq 1, \\ B_l &= so(2l+1) && \text{for } l \geqq 2, \\ C_l &= sp(2l) && \text{for } l \geqq 3, \\ D_l &= so(2l) && \text{for } l \geqq 4, \end{aligned}$$

as well as to the exceptional simple compact Lie algebras

$$E_6, E_7, E_8, F_4, G_2.$$

A compact Lie algebra L is the direct sum of some Abelian Lie algebra, namely the centre of L, and of some semisimple Lie algebra. As a consequence, theorem 1 applies also to the classical matrix Lie algebras

$$u(l), o(l+1) \qquad \text{for } l \geqq 1.$$

The question arises whether there exist non-flat constant Yang–Mills fields at all. The following gives a positive answer.

Theorem 2. *Every one-form on (E, g) with values in some three-nilpotent Lie algebra L represents a constant Yang–Mills field.*

Proof. The three-nilpotency $[[L, L], L] = 0$ trivially implies (YM).

Example 2. Theorem 2 applies to the m-th Heisenberg algebra, i.e. the Lie algebra of all $m \times m$ matrices of the form

$$\begin{pmatrix} 0 & a_1 & a_2 & \ldots & a_{m-2} & c \\ \cdot & & & & & b_1 \\ \cdot & & & & & b_2 \\ & & & & & \vdots \\ \cdot & & & & & b_{m-2} \\ 0 & & & \ldots & & 0 \end{pmatrix},$$

where all unspecified matrix elements are defined to be zero.

Example 3. Let V be some finite-dimensional vector space and

$$\wedge^+ V := \sum_{p \geqq 0} \wedge^{2p} V, \qquad \wedge^- V := \sum_{p \geqq 0} \wedge^{2p+1} V.$$

Make the associative algebra $\wedge V = \wedge^+ V + \wedge^- V$ to a Lie algebra in the usual way $[A, B] := A \wedge B - B \wedge A$. Here Σ and $+$ denote the direct sum of vector spaces and $\wedge^0 V :=$ number field belonging to V, $\wedge^1 V := V$. The Lie algebra $\wedge V$ is three-nilpotent because of

$$[\wedge^+ V, \wedge V] = \{0\}, \qquad [\wedge^- V, \wedge^- V] \subset \wedge^+ V.$$

Hence, every potential $A \in E^* \otimes \wedge V$ identically satisfies the Yang–Mills equation (YM).

Let us keep to the nilpotent Lie algebras for a moment. A version of a well-known theorem of Engel says that every finite-dimensional nilpotent Lie algebra may be realized as a Lie algebra of upper triangular $m \times m$ matrices for a suitable m, i.e. of matrices $(a_{ij})_{i,j=1,2,\ldots,m}$ with $a_{ij} = 0$ for $i \geqq j$.

Theorem 3. *Consider a constant Yang–Mills field A on some Euclidean space (E, g) with values in the Lie algebra of upper triangular $m \times m$ matrices, where $m \geqq 4$, and denote by a_k the element of A in the k-th row and the $(k + 1)$-th column. Then for each $k = 1, 2, \ldots, m - 3$ the tripel (a_k, a_{k+1}, a_{k+2}) obeys:*

(i) (at least) *one of the elements vanishes,* or
(ii) *the three elements are pairwise linearly dependent,* or
(iii) *the three elements are pairwise orthogonal.*

Proof. From the assumed form of A and (YM) there follows for $k = 1, 2, \ldots, m - 3$

$$(a_{k+1} \llcorner a_{k+2})\, a_k + (a_k \llcorner a_{k+1})\, a_{k+2} = 2(a_k \llcorner a_{k+2})\, a_{k+1}. \tag{5}$$

Imbed span (a_k, a_{k+1}, a_{k+2}) in a three-dimensional subspace of E^* and equip this subspace with the vector cross product $\times$ compatible with the inner product $\llcorner$ Then (5) may be rewritten as

$$(a_k \times a_{k+1}) \times a_{k+2} = a_k \times (a_{k+2} \times a_{k+1}).$$

It is an exercise of elementary vector algebra to derive herefrom (i) or (ii) or (iii).

Example 4. Let $n \geqq m - 1 \geqq 2$ and $e^\alpha = e^1, e^2, \ldots, e^n$ be an orthonormal, with respect to a Euclidean metric g of E, base of E^*. Then

$$A = \begin{pmatrix} 0 & e^1 & 0 & \dots & 0 \\ \cdot & & e^2 & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot\cdot \\ & & & & e^{m-1} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

represents a non-flat constant Yang–Mills field on E with values in the Lie algebra of upper triangular $m \times m$ matrices.

Special examples of the type of example 4 are given by the author in [9].

Theorem 4. *Let A be a constant Yang–Mills field on a Euclidean space (E, g) with values in some Lie algebra L of dimension $\geqq 3$. Then A is flat or L is the three-dimensional Heisenberg algebra.*

Proof. An Abelian Lie algebra L fulfils the theorem by A being flat. Every non-Abelian two-dimensional Lie algebra L admits a base $X_i = X_1, X_2$ such that $[X_1, X_2] = X_2$. The ansatz $A = a_1X_1 + a_2X_2$ (We use here lower Latin indices for the components of A with respect to the X_i!) gives then $F = 2(a_1 \wedge a_2) X_2$ and (YM) reduces to $(a_1 \wedge a_2) \llcorner a_1 = 0$. The latter equation implies that a_1, a_2 are linearly dependent, hence $F = 0$. We proceed with the non-Abelian three-dimensional Lie algebras. These are listed up in the book [2] with the use of the abbreviations

$$X^1 := [X_2, X_3], \qquad X^2 := [X_3, X_1], \qquad X^3 := [X_1, X_2]$$

as follows:

(1a) $X^1 = X^2 = 0,\ X^3 = X_3;$

(1b) $X^1 = X^2 = 0,\ X^3 = X_2;$

(2a) $X^1 = 0,\ X^2 = -\alpha X_3,\ X^3 = X_2,$ where $\alpha \neq 0;$

(2b) $X^1 = 0,\ X^2 = -X_3,\ X^3 = X_2 + \beta X_3$ where $\beta \neq 0;$

(3a) $X^1 = X_1,\ X^2 = X_2,\ X^3 = X_3;$

(3b) $X^1 = X_1,\ X^2 = X_2,\ X^3 = -X_3.$

The inspection shows: (1a) describes the Heisenberg algebra, (1b) completes the two-dimensional non-Abelian Lie algebra by a central element X_3, and (3a) describes the compact Lie algebra so (3). We calculate for the remaining cases (2a), (2b), (3b) the field strength F and the necessary and sufficient conditions for $A = a_1X_1 + a_2X_2 + a_3X_3$ to satisfy (YM):

(2a) $F = 2(a_1 \wedge a_2) X_2 + 2\alpha(a_1 \wedge a_3) X_3,$
$(a_1 \wedge a_2) \llcorner a_1 = 0,\ (a_1 \wedge a_3) \llcorner a_1 = 0;$

(2b) $$F = 2(a_1 \wedge a_2)(X_2 + \beta X_3) + 2a_1 \wedge a_3 X_3,$$
$$(a_1 \wedge a_2) \llcorner a_1 = 0, (a_1 \wedge a_3) \llcorner a_1 = 0;$$

(3b) $$F = -2(a_1 \wedge a_2) X_3 - 2(a_1 \wedge a_3) X_2 + 2(a_2 \wedge a_3) X_1,$$
$$(a_1 \wedge a_2) \llcorner a_2 = (a_1 \wedge a_3) \llcorner a_3, (a_1 \wedge a_2) \llcorner a_1 = (a_2 \wedge a_3) \llcorner a_3,$$
$$(a_1 \wedge a_3) \llcorner a_1 = (a_2 \wedge a_3) \llcorner a_2.$$

Elementary calculations give the result $F = 0$ in each case.

We note that (3b) describes the matrix Lie algebra $sl(2, R)$. Adding to $sl(2, R)$ a central element we obtain $gl(2, R)$. These remarks establish

Example 5. A constant Yang–Mills field on a Euclidean space with real 2×2 matrices as the values is flat.

The author stated the result of example 5 in another fashion already in [10].

DISCUSSION AND APPLICATIONS

Let us connect the situation here with the general Yang–Mills theory. If L is the Lie algebra to some Lie group G then $A \in E^* \otimes L$ may be interpreted mathematically as a connection and physically as a gauge field on the trivial fibre bundle $E \times G$ with the basis E and the fibre G. Through this interpretation the terms used by us – potential, field strength, Yang–Mills equation etc. – have their usual meaning.

There is a series of physical papers [7, 1, 4, 6, 5, 8, 12] dealing with gauge fields which satisfy certain constancy conditions and have values in the Lie algebra $su(N)$ for some N. Constant potentials in our sense appear there as a special case, but are not meant to solve the Yang–Mills equations. The behavior of variable linearized gauge fields around "constant fields" and the behavior of Yang–Mills charged test matter are the main topics which are discussed.

The following problems concerning constant Yang–Mills fields are actual ones in our opinion:

– Is there a gauge- and coordinate-invariant characterization of those Yang–Mills fields which admit constant potentials with respect to some gauge and some coordinate system?

– Find as many as possible (in the ideal case: all) constant Yang–Mills fields and classify them!

– What will happen if the finite-dimensional Lie algebra L is replaced by an infinite-dimensional one?

We will not end the paper without presenting analytical applications of our algebraic considerations.

Theorem 5. *Consider four independent variables* $x^\alpha = x^1, x^2, x^3, x^4$, *m dependent variables* $u^i = u^1, u^2, \ldots, u^m$ *collected to a column* $u = (u^i)$, *a real nondegenerated constand symmetric* 4×4 *matrix* $(g^{\alpha\beta})$, *and four constant* $m \times m$ *matrices* $A_\alpha = (A^i_{\alpha j})$. *If the partial defferenttal eguation*

$$g^{\alpha\beta}\left(\frac{\partial}{\partial x^\alpha} + A_\alpha\right)\left(\frac{\partial}{\partial x^\beta} + A_\beta\right)u = 0 \tag{6}$$

admits a logarithm-free elementary solution in the sense of Hadamard then the Yang–Mills equation (YM) holds true in the following interpretation:

$$(g_{\alpha\beta}) := (g^{\alpha\beta})^{-1},$$

L = Lie subalgebra of $gl(m, R)$ *which is generated by the matrices* $A_\alpha = A_1, A_2, A_3, A_4$.

E = vector space which is generated by the $e^\alpha := \dfrac{\partial}{\partial x^\alpha}$,

E^* *= vector space which is generated by the* $e_\alpha := \mathrm{d}x^\alpha$.

$A = e^\alpha \otimes A_\alpha \in E^* \otimes L$.

A proof and background information are given by the author in [9, 10] for Lorentzian signature of g and in [11] for arbitrary signature of g. Note that the potential A is flat if and only if (6) is gauge-equivalent to

$$g^{\alpha\beta}\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} = 0. \tag{7}$$

Example 6. If the equation (6) admits a logarithm-free elementary solution and if the matrices $A_\alpha = A_1, A_2, A_3, A_4$ are either real and antisymmetric or complex and antihermitean then (6) is gauge-equivalent to (7).

Theorem 6. *If a variable gauge potential* $A = A(x)$ *on an n-dimensional Euclidean space* (E, g) *solves the full Yang–Mills (partial differential) equations and if it is analytic at* ∞ *then the constant asymptotic value*

$$A(\infty) := \lim_{|x| \to \infty} A(x)$$

solves the reduced Yang–Mills equations (YM).

Sketch of the proof. By definition, a field on (E, g) is analytic at ∞ if it admits a convergent Taylor expansion in the reciprocal values $(x^\alpha)^{-1}$ of the Cartesian coordinates x^α of $x \in E$ for sufficiently large $|x|^2 := g_{\alpha\beta}x^\alpha x^\beta$. Working with components, $A_\alpha(\infty)$ is the absolute term in the assumed Taylor expansion of $A_\alpha(x)$ and $[[A_\alpha(\infty), A_\beta(\infty)], A^\beta(\infty)]$ is just the absolute term in the Taylor expansion of the left-hand side of the full Yang–Mills equations.

Example 7. If $A = A(x)$ in theorem 6 has values in some compact Lie algebra then $A(\infty)$ is a flat Yang–Mills field.

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