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REMARKS ON INJECTIVITY

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Abstract. Various properties of injective modules and generalizations are studied. Quasi-Frobeniusean and pseudo-Frobeniusean rings are characterized.

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INTRODUCTION

In this sequel to [10], certain properties of injectivity and generalizations are considered. The concept of injectivity is one of the fundamental concepts in the theory of rings and modules (cf. [3], [4], [5]) and has been extensively studied since several years. CE-injective modules, introduced in [10], are here further developed. This note contains the following results: (1) If A is a prime left self-injective regular ring, then for any left ideals B, D with an isomorphism $g : B \approx D$, there exist left ideals U, V containing B, D respectively and an isomorphism $f: U \approx V$ extending g such that either $U = A$ or $V = A$; (2) If M is a CE-injective left A-module such that any left submodule isomorphic to a complement submodule is a complement submodule, $B = \text{End}\,({}_AM)$, the following are then equivalent: (a) B is semi-perfect; (b) Every simple left B-module has a projective cover; (c) B contains no infinite set of orthogonal idempotents; (3) A is left and right pseudo-Frobeniusean iff the injective hull of every simple left A-module and the injective hull of every cyclic projective right A-module are projective; (4) A is quasi-Frobeniusean iff every left A-module has an injective projective left cover; (5) The following conditions are equivalent: (a) Every factor ring of A is quasi-Frobeniusean; (b) A is a left GFC ring such that the injective hull of every cyclic left A-module is cyclic projective; (c) The injective hull of every cyclic left A-module is cyclic projective and every simple left A-module has a projective cover; (6) A is semi-simple Artinian iff A is a left p.p. ring such that every simple left A-module has a p-injective projective cover.

Throughout, A denotes an associative ring with identity and A-modules are unital. Z, J will stand respectively for the left singular ideal and the Jacobson radical of A.

An ideal of A will always mean a two-sided ideal and A is called left duo if every left ideal of A is an ideal. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. A left A-module M is called p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. A is von Neumann regular iff every left (right) A-module is flat if every left (right) A module is p-injective. In general, there is no inclusion relation between the classes of flat modules and p-injective modules. However, if K is a maximal left ideal of A which is an ideal, then ${}_AA/K$ is flat iff A/K_A is injective iff A/K_A is p-injective. For any left A-module M, $Z(M) = \{y \in M \backslash l(y)$ is essential in ${}_AA\}$ is the singular submodule of M. M is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$). A is called semi-local if A/J is Artinian.

We start by considering non-singular left ideals in left self-injective rings.

Lemma 1. *Let A be a left self-injective ring. If I is a non-singular left ideal of A, for any $b \in I$, Ab is generated by an idempotent.*

Proof. Let $0 \neq b \in I$, K a non-zero complement left ideal of A such that $L = l(b) \oplus K$ is an essential left ideal. If $f : Kb \to A$ is the map $kb \to k (k \in K)$, since ${}_AA$ is injective, there exists $c \in A$ such that $f(kb) = kbc$ for all $k \in K$. Therefore $K \subseteqq l(b - bcb)$ which implies $L \subseteqq l(b - bcb)$, whence $b - bcb \in Z(I) = 0$. Thus $Ab = Ae$, where $e = cb$ is idempotent.

Proposition 2. *Let A be a left self-injective ring containing a non-singular left ideal I. If B, D are left ideals of A contained in I with an isomorphism $g : B \approx D$, there exist injective non-singular left ideals U_0, V_0 containing B, D respectively with an isomorphism $f_0 : U_0 \approx V_0$ extending g, and injective non-singular left ideals P, Q which do not contain any non-zero mutually isomorphic left ideals of A such that $U_0 \oplus P = V_0 \oplus Q$ is the injective hull of I and $PQ = QP = 0$. If, further, A is semi-prime, then there exist central idempotents u_1, v_1 of A such that $P \subseteqq Au_1$, $Q \subseteqq Av_1$, $Pv_1 = Qu_1 = 0$.*

Proof. The set of essential extensions of ${}_AI$ in ${}_AA$ has, by Zorn's Lemma, a maximal member C which is a complement left ideal of A. Then ${}_AC$ is the injective hull of ${}_AI$. Also ${}_AC$ is non-singular by [8, Lemma 2]. Consider the set E of elements (U, V, f), where U, V are left ideals of A in C containing B, D respectively and $f : U \approx V$ extending g, ordered by the following: $(U, V, f) \subseteqq (U', V', f')$ iff $U \subseteqq U'$, $V \subseteqq V'$ and f' extends f. Then, by Zorn's Lemma, E has a maximal member (U_0, V_0, f_0). If $\bar{U}$, $\bar{V}$ are the injective hulls of U_0, V_0 respectively in ${}_AC$, then f_0 extends to an isomorphism of $\bar{U}$ into $\bar{V}$. By the maximality of (U_0, V_0, f_0), we have $\bar{U} = U_0$, $\bar{V} = V_0$, whence $C = U_0 \oplus P = V_0 \oplus Q$, where $P = Au$, $Q = Av$, u, v being idempotents in C, and P, Q do not contain any mutually isomorphic left ideals. We claim that $PQ = 0$. Suppose the contrary: if $b \in A$ such that $ubv \neq 0$, $h : Au \to Av$ the map defined by $h(au) = aubv$ for all $a \in A$, then $h(Au) = Aw$, $0 \neq w = w^2 \in Av$ by Lemma 1, whence $\ker h$ is a direct summand of ${}_AAu$. Therefore $Au = \ker h \oplus Az$,

$0 \neq z = z^2 \in Au$ and $Az \approx Aw$, which is a contradiction! This proves that $PQ = 0$. Similarly $QP = 0$. Now suppose that A is semi-prime. Then $P \subseteq l(r(PA)) = Au_1$ and $Q \subseteq r(l(Q)) = Av_1$, where u_1, v_1 are central idempotents. Since $PQ = 0$, then $v \in r(PA)$ implies that $Au_1 \subseteq l(v)$, whence $Qu_1 = 0$. Similarly, $Pv_1 = 0$.

Corollary 2.1. *If A is prime left self-injective regular, then for any left ideals B, D with $g : B \approx D$, there exist left ideals U, V containing B, D respectively and $f : U \approx V$ extending g such that either $U = A$ or $V = A$.*

Left p-injective rings whose complement left ideals are principal generalize left self-injective rings and left continuous regular rings. The next proposition may be similarly proved.

Proposition 3. *Let A be a left p-injective ring whose complement left ideals are principal and K and injective non-singular left ideal. If B, D are left ideals contained in K with an isomorphism $g : B \approx D$, there exist left ideals U, V containing B, D respectively with an isomorphism $f : U \approx V$ extending g such that $K = U \oplus P = V \oplus Q$, where P, Q do not contain any non-zero mutually isomorphic left ideals and $PQ = QP = 0$. Consequently, if A is prime, then either $K = U$ or $K = V$.*

Remark 1. Let A be a left p-injective ring containing a reduced injective left ideal K. If B, D are isomorphic left ideals contained in K, then the conclusion of Proposition 3 holds.

As usual, (1) a left A-module M is said to have a projective cover if there exist a projective left A-module P and an epimorphism $g : P \to M$ such that ker g is superfluous in P. H. BASS [1] called A left perfect if every left A-module has a projective cover. (2) ${}_AM$ is a generator if, for any left A-module N, there exists an epimorphism from a direct sum of copies of M onto N. (3) ${}_AM$ is a cogenerator if, for any left A-module N, there exists a monomorphism of N into a direct product of copies of M. A is called left pseudo-Frobeniusean (resp. FPF) if every faithful (resp. finitely generated faithful) left A-module generates the category of left A-modules (cf. [3], [5]). The following conditions are equivalent: (1) A is left pseudo-Frobeniusean; (2) A is an injective cogenerator; (3) A is a semi-local left cogenerator; (4) A is a left cogenerating right Kasch ring. (A is right Kasch if every maximal right ideal of A is a right annihilator ideal.) Also, A is left cogenerating iff the injective hull of every simple left A-module is projective. Recall that A is a left p.p. ring if every principal left ideal of A is a projective left A-module.

Remark 2. A is von Neumann regular iff A is a left p.p. ring such that there exists a p-injective left generator.

Following [10], a left A-module M is CE-injective if, for any left submodule N containing a non-zero complement left submodule of M, every left A-homomorphism of N into M extends to an endomorphism of ${}_AM$. We now consider the ring of endomorphisms of a generalization of quasi-injective modules.

Proposition 4. *Let M be a* CE*-injective left A-module such that any left submodule isomorphic to a complement left submodule is a complement submodule. If $B =$ $=$* End $({}_AM)$, *the following conditions are equivalent:*

(1) *B is semi-perfect;*

(2) *Every simple left B-module has a projective cover;*

(3) *B contains no infinite set of orthogonal idempotents.*

Proof. Since B is semi-perfect iff every finitely generated left B-module has a projective cover [1, Theorem 2.1], then (1) implies (2).

Assume (2). Let W denote the Jacobson radical of B and $\bar{K} = K + W$ a maximal left ideal of $\bar{B} = B/W$, where K is a maximal left ideal of B. Since B/K has a projective cover, let $g : P \to B/K$ be an epimorphism, where ${}_BP$ is projective and $\ker g$ is superfluous in P. If $p : B \to B/K$ is the natural projection, there exists a left B-homomorphism $h : B \to P$ such that $gh = p$ and for any $c \in P$, there exists $y \in B$ such that $g(c) = p(y) = gh(y)$ which yields $P = \ker g + h(B)$, whence $h(B) = P$. If $h(1) = d$, then $P = Bd$ and $h(B) = Bd$. Since $B/\ker h \approx P$, then $\ker h$ is a direct summand of ${}_BB$ (because ${}_BP$ is projective). If $h(K) = 0$, then $K = \ker h$ and $B = K \oplus Be$, $0 \neq e = e^2 \in B$, whence $Ke = 0$. In that case, $\bar{K} = l(\bar{e})$ (since $e \notin W$). If $h(K) \neq 0$, since $gh(K) = 0$, then $h(K)$ is superfluous in P. Since $h(B) = P$ is projective, there exists a left B-homomorphism $t : h(B) \to B$ such that $ht = i$, the identity map on $h(B)$. Since $h(K)$ is superfluous in $h(B)$, then $th(K)$ is superfluous in B. Now let $t(d) = b \in B$. Then $d = i(d) = ht(d) = h(b) = bh(1) = bd$ implies $0 \neq b = t(d) =$ $= t(bd) = bt(d) = b^2$ and $Kb = Kt(d) = t(Kd) = th(K)$ is superfluous in B. Thus in case $h(K) \neq 0$, there exists also a non-zero idempotent b such that Kb must be contained in every maximal left ideal of B, whence $Kb \subseteqq W$. Therefore $\bar{K} = l_{\bar{B}}(\bar{b})$ (in as much as the Jacobson radical W contains no non-zero idempotent of B). The fact that $\bar{b}$ is an idempotent in $\bar{B}$ implies that $\bar{K}$ is a direct summand of ${}_{\bar{B}}\bar{B}$. Therefore, whether $k(K) = 0$ or not, $\bar{K}$ must be a direct summand of ${}_{\bar{B}}\bar{B}$ which proves that $\bar{B}$ is semi-simple Artinian. B is therefore a semi-local ring whose idempotents can be lifted [10, Proposition 4 and Remark 6], whence (2) implies (1).

(1) and (3) are equivalent by [5, P. 305 ex. 8] and [10, Proposition 4].

Applying [4, Corollary 2.22], we get

Corollary 4.1. *If ${}_AM$ is non-singular quasi-injective, $B =$* End $({}_AM)$, *then B is semi-simple Artinian if every simple left B-module has a projective cover.*

It is well-known that if A is left self-injective, then idempotents of A/J can be lifted. Using [1, Theorem 2.1], one can similarly prove the next result.

Theorem 5. *The following conditions are equivalent:*

(1) *A is left pseudo-Frobeniusean;*

(2) *For any simple left A-module U, U has a projective cover and the injective hull of ${}_AU$ is projective;*

(3) *Every simple left A-module has a projective cover and there exists a projective left cogenerator;*

(4) *A is left cogenerating such that every simple left A-module has a projective cover.*

Remark 3. If A is left pseudo-Frobeniusean, then (a) the injective hull of every simple left A-module is cyclic; (b) a simple left A-module is projective iff it is injective.

Remark 4. If A is left f-injective with an injective maximal left ideal such that the injective hull of every simple left A-module is projective, then A is left pseudo-Frobeniusean. (A is called left f-injective if, for any finitely generated left ideal F of A, every left A-homomorphism of F into A extends to an endomorphism of ${}_AA$).

Proposition 6. *The following conditions are equivalent:*

(1) *A is left and right pseudo-Frobeniusean;*

(2) *The injective hull of every simple left A-module and the injective hull of every cyclic faithful projective right A-module are projective.*

Proof. Assume (1). Since A is a left cogenerator, then the injective hull of every simple left A-module is projective. Let C be a cyclic faithful projective right A-module. If $C = cA$, then $r(c)$ is a direct summand of A_A which implies that $C_A(\approx A/r(c))$ is injective. Consequently, (1) implies (2).

Assume (2). Since A is a left cogenerator and hence left Kasch, then any proper finitely generated left ideal of A has non-zero right annihilator. If E_A is the injective hull of A_A, by hypothesis, E_A is projective and by [1, Theorem 5.4], A_A is a direct summand of E_A which implies $A = E$. Then (2) implies (1) by [5, Theorem 12.1.1].

We say that a left A-module M has an injective (resp. p-injective) projective cover if there exist an injective (resp. p-injective) projective left A-module P with an epimorphism $g : P \to M$ such that ker g is superfluous in P.

Theorem 7. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean;*

(2) *A is left Noetherian with an injective left generator;*

(3) *Every left A-module has an injective projective left cover.*

Proof. Since ${}_AA$ is a generator, then (1) implies (2).

Assume (2). Let G be an injective left generator. For any projective left A-module P, there exists an epimorphism $g : D \to P$, where D is a direct sum of copies of G. Since A is left Noetherian, then ${}_AD$ is injective. Therefore $D/\ker g \approx P$ implies that $\ker g$ is a direct summand of ${}_AD$, whence ${}_AP$ is injective. Since a left Artinian ring is left (and right) perfect, then by [3, Theorem 24.20], (2) implies (3).

Assume (3). For any projective left A-module P, there exists an injective projective left A-module Q with an epimorphism $g : Q \to P$ such that ker g is superfluous in Q. Then ${}_AP(\approx Q/\ker g)$ is injective and (3) implies (1) by [3, Theorem 24.20].

Remark 5. The following conditions are equivalent:

(1) A is left p-injective left perfect;

(2) A is a left p-injective ring whose simple left modules have projective covers such that Z is left T-nilpotent;

(3) Every flat left A-module is p-injective projective.

We now turn to sufficient conditions for right Kasch rings to be left pseudo-Frobeniusean.

Proposition 8. *Let A be a right Kasch ring whose indecomposable injective left modules are projective. If A is of left finite Goldie dimension, then A is left pseudo-Frobeniusean.*

Proof. A contains an essential left ideal L which is a finite direct sum of non-zero uniform left ideals. If E is the injective hull of ${}_AA$, since the injective hull of any uniform left ideal in ${}_AE$ is an indecomposable left A-module, then E contains an essential left submodule F which is a finite direct sum of indecomposable injective left submodules. By hypothesis, F is an injective projective left A-module which yields $E = F$. Since A is right Kasch, then any proper finitely generated right ideal has non-zero left annihilator which implies that ${}_AA$ is a direct summand of ${}_AE$, whence $A = E$ is injective. Now the injective hull of any simple left A-module is indecomposable and therefore projective which implies that ${}_AA$ is a cogenerator. This proves that A is left pseudo-Frobeniusean.

Let us now characterize rings which are fully quasi-Frobeniusean. Following BIRKENMEIER [2], A is called a left GFC ring if every cyclic faithful left A-module is a generator. Left GFC rings generalize left pseudo-Frobeniuseaun and left FPF rings. Also, if every non-zero left ideal of A contains a non-zero ideal, then A is left GFC.

Theorem 9. *The following conditions are equivalent:*

(1) *Every factor ring of A is quasi-Frobeniusean;*

(2) *The injective hull of every cyclic left A-module is cyclic projective and every simple left A-module has a projective cover;*

(3) *A is a left* GFC *ring such that the injective hull of every cyclic left A-module is cyclic projective;*

(4) *A is left* GFC *satisfying the maximum condition on left annihilators such that the injective hulls of cyclic left A-modules are cyclic.*

Proof. It is well-known that (1) implies (2).

Assume (2). Suppose there exists an injective left A-module Q which is not a direct sum of indecomposable submodules. Then ${}_AQ$ is not uniform. Therefore, there exist non-zero left submodules Q_1, M_2 such that $Q = Q_1, \oplus M_2$. We may suppose that M_2 is not uniform (by changing the notation, if necessary). Then $M_2 = Q_2 \oplus M_3$, where M_3 is again supposed not uniform (by changing the notation again, if necessary). This decomposition may be continued such that we obtain, for each positive integer n, $Q = Q_1 \oplus Q_2 \oplus \dots \oplus Q_n \oplus M_{n+1}$ where, M_{n+1} is supposed not uniform. Since each Q_i ($1 \leqq i \leqq n$) contains a cyclic projective submodule P_i, then for any positive integer n, Q contains a direct sum of cyclic projective submodules $P_1, \ldots, P_n$. Each P_i is isomorphic to a left ideal K_i. Now since the injective hull of every simple left A-module is projective, then ${}_AA$ is a cogenerator and since every simple left A-module has a projective cover, then A is semi-local which yields A left pseudo-Frobeniusean. Then $F_n = K_1 \oplus \ldots \oplus K_n$ is a finitely generated projective submodule which is a direct summand of ${}_AA$ (in as much as ${}_AA$ is injective). We thus produce an infinite ascending chain of direct summands $F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$ which contradicts A left pseudo-Frobeniusean. This proves that every injective left A-module is a direct sum of indecomposable submodules, whence A is left Noetherian and therefore (2) implies (1) by [3, Proposition 25.4.6 B].

It is evident that (1) implies (3).

Assume (3). If E denotes the injective hull of ${}_AA$, then ${}_AE$ is a generator and there exists an epimorphism $g : F \to A$, where F is a finite direct sum of copies of E. Then ${}_AF$ is injective which implies that ${}_AA$ is injective. Since ${}_AA$ is a cogenerator, then A is left pseudo-Frobeniusean and the proof of "(2) implies (1)" shows that (3) implies (1).

Similarly, (1) and (4) are equivalent by [3, Theorem 24.20].

Corollary 9.1. *If A is left duo, the following are equivalent:* (a) *Every factor ring of A is quasi-Frobeniusean;* (b) *Every cyclic left A-module has a cyclic projective injective hull.*

Following [6], a left A-module M is called semi-simple if the intersection of all maximal left submodules is zero.

Theorem 10. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*

(2) *A is a left* p.p. *ring such that every simple left A-module has a p-injective projective cover;*

(3) *Every cyclic semi-simple left A-module is flat and has a projective cover;*

(4) *Every essential left ideal of A is a left annihilator and Z contains no non-zero nilpotent right ideal.*

Proof. (1) implies (2) evidently.

Assume (2). Let U be a simple left A-module. There exist a p-injective projective left A-module P and an epimorphism $g : P \to U$ such that $\ker g$ is superfluous in P. Then $P/\ker g \approx U$ and since A is left p.p., by [9, Remark 2], ${}_AU$ is p-injective which implies that $J = 0$ [9, Lemma 1]. The proof of Proposition 4 then shows that A is semi-simple Artinian and therefore (2) implies (3).

Assume (3). Then ${}_AA/J$ is semi-simple and hence flat which yields $J = 0$. Since every simple left A-module has a projective cover, then A is semi-simple Artinian and (3) implies (4).

Assume (4). Suppose there exists a maximal left ideal M which is not a direct summand of ${}_AA$. Then M is an essential left ideal which implies that $M = l(b)$,

$0 \neq b \in A$. For any non-zero elements u, v in $r(M)$ such that $uv \neq 0$, there exists $d \in A$ such that $0 \neq du \in M$ and $duv = 0$. Now $M = l(uv)$ implies that $d \in M$, whence $du = 0$ which is a contradiction! Therefore $(r(M))^2 = 0$ and since $r(M) \subseteq Z$, by hypothesis, $r(M) = 0$ which contradicts $b \neq 0$. This proves that every maximal left ideal of A is a direct summand of ${}_A A$ which yields A semi-simple Artinian. Thus (4) implies (1).

We conclude with two more remarks.

Remark 6. If every cyclic left A-module has a cyclic injective hull, the following are then equivalent: (a) A is left pseudo-Frobeniusean; (b) A is left GFC such that A/J satisfies the ascending chain condition on direct summands; (c) Every simple left A-module has a projective cover. In that case, A is local iff the left ideals of A are linearly ordered. (cf. [7, Corollary 1.11] and [10, Lemma 12].

Remark 7. (1) If A is left GFC, then A is left self-injective iff the injective hull of every cyclic projective faithful left A-module is cyclic; consequently, the following are equivalent : (a) A is left and right self-injective strongly regular; (b) A is semi-prime left duo such that the injective hull of every cyclic projective faithful left A-module is cyclic.

(2) If A is left FPF, then A is left self-injective iff the injective hull of every cyclic projective faithful left A-module is projective.

(3) If every non-zero left ideal of A contains a non-zero ideal, then A is left pseudo-Frobeniusean iff the injective hull of every cyclic faithful projective left A-module is a cyclic left cogenerator.

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