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Jan Osička

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# ON THE ASYMPTOTIC STABILITY OF TWO-DIMENSIONAL LINEAR SYSTEMS

JAN OSIČKA

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**Abstract.** In this paper, the differential system of second-order with variable coefficients is studied, and some criteria of the asymptotic stability for solutions are given.

**Key words.** Oscillation, nonoscillation, convergence of the solution, asymptotic stability.

**MS Classification.** 34 D 05

## 1. INTRODUCTION

In the present paper we consider a system of differential equations

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2, \qquad \frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2, \tag{1.1}$$

where $a_{ik} : R_+ \to R$ $(i, k = 1, 2)$ are functions summable on every compact interval.

It will be assumed throughout that

$$\sigma a_{12}(t) > 0 \qquad \text{if } t \in R_+, \tag{1.2}$$

where $\sigma \in \{-1, 1\}$ and the function $\frac{a_{21}}{a_{12}}$ is integrable on every compact interval.

Let

$$a(t) = -\frac{a_{21}(t)}{a_{12}(t)} \exp\left[2\int_0^t (a_{11}(\tau) - a_{22}(\tau))\,d\tau\right], \tag{1.3}$$

$$\varphi(t) = \int_0^t |a_{12}(\tau)| \exp\left[\int_0^\tau (a_{22}(\xi) - a_{11}(\xi))\,d\xi\right] d\tau. \tag{1.4}$$

**Lemma 1.** *By means of the transformation*

$$x_i(t) = \exp\left[\int_0^t a_{ii}(\tau)\,d\tau\right] y_i(s) \qquad (i = 1, 2),\ s = \varphi(t), \tag{1.5}$$

*the system* (1.1) *will take the form*

$$\frac{dy_1}{ds} = \sigma y_2,$$
(1.6)
$$\frac{dy_2}{ds} = -\sigma p(s)\, y_1,$$

*where*

(1.7) $$p(s) = a(\varphi^{-1}(s)) \quad \text{if } 0 \leqq s < s_0,\ s_0 = \lim_{t\to\infty} \varphi(t)$$

*and* $\varphi^{-1}$ *is the inverse to* $\varphi$.

Proof. Let $(x_1, x_2)$ be an arbitrary solution of the system (1.1). In view of (1.2) and (1.5)

(1.8) $$x_1'(t) = a_{11}(t)\, x_1(t) + \sigma a_{12}(t) \exp\left[\int_0^t a_{22}(\tau)\, d\tau\right] y_1'(s),$$

$$x_2'(t) = a_{22}(t)\, x_2(t) + \sigma a_{12}(t) \exp\left[\int_0^t (2a_{22}(\tau) - a_{11}(\tau))\, d\tau\right] y_2'(s).$$

By substituting (1.5) and (1.8) into (1.1) we obtain (1.6). The lemma is proved.

**Lemma 2.** *Let*

(1.9) $$a(t) > 0 \quad \text{if } t \in R_+, \qquad \limsup_{t\to\infty} \int_0^t a_{11}(\tau)\, d\tau < \infty$$

and

$$\limsup_{t\to\infty} \left| \frac{a_{21}(t)}{a_{12}(t)} \right| < \infty$$

*and every solution of the differential equation*

(1.10) $$u'' + p(s)\, u = 0, \qquad (0 \leqq s < s_0),$$

*where* $s_0$ *and* $p$ *are defined by* (1.7), *satisfies the condition*

(1.11) $$\lim_{s\to s_0} \left[ \frac{u'^2(s)}{p(s)} + u^2(s) \right] = 0.$$

*Then the system* (1.1) *is asymptotically stable.*

Proof. Let $(x_1, x_2)$ be an arbitrary solution of the system (1.1) and let $(y_1, y_2)$ be the vector function given by (1.5). According to Lemma 1, $(y_1, y_2)$ is a solution of the system (1.6). In view of (1.5) and (1.6), the function $u(s) = y_1(s)$ is a solution of (1.10) and

$$x_1(t) = \exp\left[\int_0^t a_{11}(\tau)\, d\tau\right] u(s),$$
(1.12)
$$x_2(t) = \sigma \exp\left[\int_0^t a_{22}(\tau)\, d\tau\right] u'(s).$$

Therefore,

$$(1.13)\qquad x_1^2(t) + x_2^2(t) = \exp\left[2\int_0^t a_{11}(\tau)\,d\tau\right]\left[u^2(s) + \left|\frac{a_{21}(t)}{a_{12}(t)}\right| \frac{u'^2(s)}{p(s)}\right] \leqq$$

$$\leqq \exp\left[2\int_0^t a_{11}(\tau)\,d\tau\right]\left[1 + \left|\frac{a_{21}(t)}{a_{12}(t)}\right|\right]\left[u^2(s) + \frac{u'^2(s)}{p(s)}\right]$$

for $t \in R_+$. From (1.9), (1.11) and (1.13), it holds that

$$\lim_{t\to\infty} (x_1^2(t) + x_2^2(t)) = 0,$$

i.e. system (1.1) is asymptotically stable.

## 2. LEMMAS ON THE SOLUTION OF EQUATION (1.10)

Consider the equation (1.10), where $s_0 < \infty$ and the function $p : [0, s_0) \to [0, \infty)$ is integrable on $[0, s_0 - \varepsilon]$ for any arbitrarily small $\varepsilon > 0$.

**Lemma 3.** *Let equation* (1.10) *be nonoscillatory. Then the condition*

$$(2.1)\qquad \int_0^{s_0} (s_0 - s)\,p(s)\,ds = \infty$$

*is necessary and sufficient for every solution of equation* (1.10) *to tend to zero as* $s \to s_0$.

Proof. We prove first the sufficiency. Let $u$ be an arbitrary solution of (1.10). Since (1.10) is nonoscillatory and $p$ is of constant sign, there exists a point $s_1 \in [0, s_0)$ such that $u(s) \neq 0$ and $u'(s) \neq 0$ if $s_1 \leqq s < s_0$. Consequently, there exists a finite or infinite limit

$$c_0 = \lim_{s\to s_0} u(s).$$

Our aim is to show that $c_0 = 0$. We assume the contrary, $c_0 \neq 0$. Then without of generality it will be assumed that

$$(2.2)\qquad u(s) \geqq \delta \qquad \text{if } s_1 \leqq s < s_0,$$

where $\delta$ is a positive number.

From (1.10) we have

$$\int_{s_1}^{s} (s_0 - t)\,u''(t)\,dt + \int_{s_1}^{s} (s_0 - t)\,p(t)\,u(t)\,dt = 0 \qquad (s_1 \leqq s < s_0).$$

This and (2.2) imply

$$(2.3)\qquad (s_0 - s)\,u'(s) + u(s) \leqq c_1 - \delta \int_{s_1}^{s} (s_0 - t)\,p(t)\,dt$$

for $s \in [s_1, s_0)$, where $c_1 = (s_0 - s_1) u'(s_1) + u(s_1)$. In view of (2.1), it follows from (2.3) that the inequality

$$(s_0 - s) u'(s) + u(s) < 0$$

holds for some $s_2 \in (s_1, s_0)$. Therefore

$$\frac{u'(s)}{u(s)} < -\frac{1}{s_0 - s} \qquad \text{for } s_2 < s < s_0 .$$

Integrating this inequality yields

$$u(s) \leqq \frac{u(s_2)}{s_0 - s_2} (s_0 - s)$$

for $s_2 < s < s_0$, which contradicts (2.2). This proves that $c_0 = 0$.

We now turn to the proof of necessity, i.e. we prove that if condition (2.1) is violated, then equation (1.10) has a solution, not tending to zero as $s \to s_0$.

We choose $s_1 \in (0, s_0)$ so that

$$\int_{s_1}^{s_0} (s_0 - s) p(s) \, ds < \ln \frac{3}{2} . \tag{2.4}$$

Let $u$ be a solution of (1.10) satisfying the initial conditions $u(s_1) = 1$, $u'(s_1) = 0$. Then

$$u(s) = 1 - \int_{s_1}^{s} (s - t) p(t) u(t) \, dt \tag{2.5}$$

and

$$| u(s) | \leqq 1 + \int_{s_1}^{s} (s_0 - t) p(t) | u(t) | \, dt,$$

for $s_1 \leqq s < s_0$. Hence according to Gronwall–Bellman's lemma, it holds that

$$| u(s) | \leqq \exp \left[ \int_{s_1}^{s} (s_0 - t) p(t) \, dt \right]$$

for $s_1 \leqq s < s_0$. This and (2.5) imply that

$$u(s) \geqq 1 - \int_{s_1}^{s} (s_0 - t) p(t) \exp \left[ \int_{s_1}^{t} (s_0 - \tau) p(\tau) \, d\tau \right] dt =$$

$$= 2 - \exp \left[ \int_{s_1}^{s} (s_0 - \tau) p(\tau) \, d\tau \right]$$

for $s_1 \leqq s < s_0$. From this in view of (2.4) we get

$$u(s) \geqq 2 - \frac{3}{2} = \frac{1}{2}$$

for $s_1 \leqq s < s_0$. The lemma is proved.

**Lemma 4.** *Let the function $p$ be absolutely continous on $[0, s_0 - \varepsilon]$ for any $\varepsilon$ $(0 < \varepsilon < s_0)$ and*

$$p(s) > 0, \qquad p'(s) \geqq 0 \tag{2.6}$$

*if $0 \leqq s < s_0$. Moreover, let (1.10) be either oscillatory or nonoscillatory and*

$$\int_{s_1}^{s_0} \sqrt{p(t)}\,\mathrm{d}t = \infty. \tag{2.7}$$

*Then every solution $u$ of (1.10) that tends to zero as $s \to s_0$, satisfies condition (1.11).*

Proof. Introduce the function

$$\varrho(s) = \frac{u'^2(s)}{p(s)} + u^2(s).$$

In view of (1.10)

$$\varrho'(s) = -\frac{p'(s)}{p^2(s)}\,u'^2(s) \leqq 0$$

if $0 \leqq s < s_0$. Consequently, $\varrho$ as a monotone function, has a limit

$$\varrho_0 = \lim_{s \to s_0} \varrho(s).$$

Our aim is to prove that $\varrho_0 = 0$. If (1.10) is oscillatory, then there exists the sequence $s_k \to s_0$ as $k \to \infty$ such that

$$u'(s_k) = 0 \qquad (k = 1, 2, \ldots).$$

Hence,

$$\varrho(s_k) = u^2(s_k) \qquad (k = 1, 2, \ldots)$$

and

$$\varrho_0 = \lim_{k \to \infty} u^2(s_k) = 0.$$

Now assume that (1.10) is nonoscillatory and $\varrho_0 > 0$. Since

$$\lim_{s \to s_0} u^2(s) = 0, \tag{2.8}$$

one can find a number $s^*$ such that

$$\frac{u'^2(s)}{p(s)} > \frac{\varrho_0}{2},$$

i.e.

$$u'(s) < -\sqrt{\frac{\varrho_0}{2}}\sqrt{p(s)} \qquad (s^* \leqq s < s_0).$$

By integrating this inequality, we get

$$u(s) \leqq u(s^*) - \sqrt{\frac{\varrho_0}{2}} \int_{s^*}^{s} \sqrt{p(t)}\,\mathrm{d}t.$$

Hence, in view of (2.7) we find that

$$\lim_{s \to s_0} u(s) = -\infty,$$

which contradicts (2.8). This proves that $\varrho_0 = 0$. The lemma is proved.

**Lemma 5.** *Let conditions* (2.1), (2.6) *and* (2.7) *be fulfilled. Besides, in some left-hand neighbourhood of the point* $s_0$ *the inequality*

$$p(s) \leqq \frac{1}{4(s_0 - s)^2} \tag{2.9}$$

*holds. Then every solution of equation* (1.10) *satisfies* (1.11).

Proof. Since equation

$$v'' + \frac{1}{4(s_0 - s)^2} v = 0$$

has a nonoscillatory solution $v(s) = (s_0 - s)^{-1/2}$, in view of Sturm's lemma and inequality (2.9) it is evident that (1.10) is nonoscillatory. By Lemmas 3 and 4, every solution of (1.10) tends to zero as $s \to s_0$ and satisfies (1.11). The lemma is proved.

**Lemma 6.** *Let the function* $p$ *be absolutely continous on* $[0, s_0 - \varepsilon]$ *for any* $\varepsilon$ $(0 < \varepsilon < s_0)$ *and*

$$(s_0 - s)^2 p(s) > \frac{1}{4}, \qquad \frac{d}{ds}(p(s)(s_0 - s)^2) \geqq 0 \tag{2.10}$$

*if* $0 \leqq s < s_0$. *Then for any solution* $u$ *of* (1.10), *one can find a positive number* $\eta$ *such that*

$$\frac{u'^2(s)}{p(s)} + u^2(s) \leqq \eta(s_0 - s) \qquad (0 \leqq s < s_0). \tag{2.11}$$

Proof. Let $u$ be an arbitrary solution of (1.10). We set

$$u(s) = (s_0 - s)^{1/2} w(t), \qquad t = \ln \frac{s_0}{s_0 - s}.$$

Then

$$u'(s) = -\frac{1}{2}(s_0 - s)^{-1/2} w(t) + (s_0 - s)^{-1/2} w'(t)$$

and

$$w'' + \hat{p}(t) w = 0,$$

where

$$\hat{p}(t) = (s_0 - s)^2 p(s) - \frac{1}{4}.$$

In view of (2.10) $\hat{p}(t) > 0, \hat{p}'(t) \geqq 0$ if $0 \leqq t < \infty$. Therefore the function

$$\frac{w'^2(t)}{\hat{p}(t)} + w^2(t)$$

does not increase. Hence

$$\frac{w'^2(t)}{\hat{p}(t)} + w^2(t) \leqq \delta^2,$$

where

$$\delta^2 = \frac{w'^2(0)}{\hat{p}(0)} + w^2(0).$$

Thus we have

$$|w'(t)| \leqq \delta(s_0 - s)\sqrt{p(s)}, \qquad |w(t)| \leqq \delta$$

if $0 \leqq t < \infty$. Therefore,

$$|u(s)| \leqq \delta(s_0 - s)^{1/2},$$

$$|u'^2(s)| \leqq \frac{\delta^2}{4(s_0 - s)} + \delta^2\sqrt{p(s)} + \delta^2(s_0 - s)\, p(s)$$

if $0 \leqq s < s_0$. This and (2.10) imply that (2.11) holds with $\eta = 5\delta^2$. The lemma is proved.

## 3. CRITERIA OF THE ASYMPTOTIC STABILITY FOR THE SYSTEM (1.1)

**Theorem 1.** *Let* $a_{12}(t) \neq 0$ *for* $t \in R_+$,

$$\lim_{t\to\infty} \int_0^t a_{11}(\tau)\, d\tau = -\infty \qquad \text{and} \qquad \limsup_{t\to\infty} \left| \frac{a_{21}(t)}{a_{12}(t)} \right| < \infty. \tag{3.1}$$

*The function a is absolutely continous on every compact interval and*

$$a(t) > 0, \qquad a'(t) \geqq 0 \tag{3.2}$$

*when* $t \in R_+$. *Then the system* (1.1) *is asymptotically stable.*

Proof. As shown above, an arbitrary solution $(x_1, x_2)$ of the system (1.1) satisfies (1.13), where $u$ is a solution of (1.10). In view of (3.2), $p(s) > 0, p'(s) \geqq 0$ if $0 \leqq s < s_0$. Therefore, the function

$$\varrho(s) = \frac{u'^2(s)}{p(s)} + u^2(s),$$

satisfies the condition

$$\varrho'(s) = -\frac{p'(s)}{p^2(s)} u'^2(s) \leqq 0$$

for $0 \leqq s < s_0$ and hence,

$$\varrho(s) \leqq \varrho(0) \tag{3.3}$$

for $0 \leqq s < s_0$. From (1.13), (3.1) and (3.3), it holds that

$$\lim_{t\to\infty} x_i(t) = 0 \qquad (i = 1, 2).$$

The theorem is proved.

**Theorem 2.** *Let $a_{12}(t) \neq 0$ for $t \in R_+$. Let be an absolutely continous function a on every compact interval and let the condition* (3.2) *be satisfied. Let, in addition*

$$\limsup_{t\to\infty} \int_0^t a_{11}(\tau)\,\mathrm{d}\tau < \infty, \qquad \limsup_{t\to\infty} \left|\frac{a_{21}(t)}{a_{12}(t)}\right| < \infty \tag{3.4}$$

*and*

$$b(t) = \sqrt{\left|\frac{a_{21}(t)}{a_{12}(t)}\right|} \int_t^\infty |a_{12}(\tau)| \exp\left[\int_t^\tau (a_{22}(\xi) - a_{11}(\xi))\,\mathrm{d}\xi\right] \mathrm{d}\tau \leqq \frac{1}{2} \tag{3.5}$$

*for $t \in R_+$. Then* (1.1) *is asymptotically stable provided that*

$$\int_0^\infty \sqrt{|a_{12}(t)\,a_{21}(t)|}\, b(t)\,\mathrm{d}t = \infty. \tag{3.6}$$

Proof. In view of (3.2) and (3.5), equalities (1.3), (1.4) and (1.7) imply (2.6) and (2.9). On the other hand, (3.6) implies (2.1).

According to (3.5) and (3.6), we can write

$$\int_0^{s_0} \sqrt{p(s)}\,\mathrm{d}s =$$

$$= \int_0^\infty \sqrt{\left|\frac{a_{21}(t)}{a_{12}(t)}\right|} \exp\left[\int_0^t (a_{11}(\tau) - a_{22}(\tau))\,\mathrm{d}\tau\right] |a_{12}(t)| \exp\left[\int_0^t (a_{22}(\tau) - a_{11}(\tau))\,\mathrm{d}\tau\right] \mathrm{d}t =$$

$$= \int_0^\infty \sqrt{|a_{21}(t)a_{12}(t)|}\,\mathrm{d}t \geqq 2\int_0^\infty \sqrt{|a_{21}(t)a_{12}(t)|}\, b(t)\,\mathrm{d}t = \infty.$$

Consequently, all the conditions of Lemmas 2 and 5 are satisfied. Therefore, (1.1) is asymptotically stable. This completes the proof of Theorem 2.

**Theorem 3.** *Let the conditions* (3.2), (3.4), (3.5) *be fulfilled and*

$$\liminf_{t\to\infty} \int_0^t a_{11}(\tau)\,\mathrm{d}\tau > -\infty. \tag{3.7}$$

*Then* (3.6) *is necessary and sufficient for the asymptotic stability of* (1.1).

Proof. The sufficiency follows from Theorem 2. We now prove the necessity.

Let (1.1) be asymptotically stable. Then (1.13) and (3.7) imply that every solution of (1.10) tends to zero as $s \to s_0$. By Lemma 3, (2.1) is satisfied which implies also (3.6).

**Corollary 1.** (A. G. Surkov [1]) *Let* $a_{11}(t) = 0$, $a_{12}(t) = -a_{21}(t) > 0$, $a_{22}(t) \leqq -2a_{12}(t)$ for $t \in R_+$. *Then the condition*

$$\int_0^\infty a_{12}(t)\left(\int_t^\infty a_{12}(\tau)\exp\left[\int_t^\tau a_{22}(\xi)\,d\xi\right]d\tau\right)dt = \infty \tag{3.8}$$

*is necessary and sufficient for the asymptotic stability of* (1.1).

Proof. We have

$$a(t) = \exp\left[-2\int_0^t a_{22}(\tau)\,d\tau\right],$$

$$b(t) = \int_t^\infty a_{12}(\tau)\exp\left[\int_t^\tau a_{22}(\xi)\,d\xi\right]d\tau \leqq$$

$$\leqq \int_t^\infty a_{12}(\tau)\exp\left[-2\int_t^\tau a_{12}(\xi)\,d\xi\right]d\tau \leqq \frac{1}{2}.$$

Hence, conditions (3.2), (3.4), (3.5) and (3.7) are satisfied. Therefore, in view of Theorem 2, condition (3.8) is necessary and sufficient.

**Theorem 4.** *Let* $a_{12}(t) \neq 0$ *for* $t \in R_+$, *the function* $a$ *is absolutely continous on every finite segment and* $a(t) > 0$ *for* $t \in R_+$. *Assume also that*

$$\lim_{t\to\infty}\left(\exp\left[2\int_0^t a_{11}(\tau)\,d\tau\right]\left(1 + \left|\frac{a_{21}(t)}{a_{12}(t)}\right|\right)\times \int_t^\infty |a_{12}(\tau)|\exp\left[\int_0^\tau (a_{22}(\xi) - a_{11}(\xi))\,d\xi\right]d\tau\right) = 0 \tag{3.9}$$

*and*

$$b(t) = \sqrt{\left|\frac{a_{21}(t)}{a_{12}(t)}\right|}\int_t^\infty |a_{12}(\tau)|\exp\left[\int_t^\tau (a_{22}(\xi) - a_{11}(\xi))\,d\xi\right]d\tau > \frac{1}{2}, \tag{3.10}$$

$$b'(t) \geqq 0 \tag{3.11}$$

*hold for* $t \in R_+$. *Then the system* (1.1) *is asymptotically stable.*

Proof. In view of (3.10) and (3.11), equalities (1.3), (1.4) and (1.7) imply (1.13). Therefore, according to Lemma 2, the system (1.1) is asymptotically stable. The theorem is proved.

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*J. Osička*
*Department of Mathematics*
*J. E. Purkyně University*
*662 95 Brno, Janáčkovo nám. 2a*
*Czechoslovakia*