Demeter Krupka

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Archivum Mathematicum, Vol. 22 (1986), No. 4, 211--228

Persistent URL: http://dml.cz/dmlcz/107267

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 22, No. 4 (1986), 211–228

GEOMETRY OF LAGRANGEAN STRUCTURES. 2.*)

DEMETER KRUPKA

(Received January 14, 1986)

Abstract. Underlying notions of the global calculus of variations in fibered spaces, such as the r-jet prolongation of a fibered manifold, horizontal and contact differential forms and odd base forms are introduced, and their basic properties are discussed.

Key words. Fibered manifold, homomorphism of fibered manifolds, r-jet prolongation of a fibered manifold, horizontalization, horizontal forms, contact forms.

MS Classification. 58 E 99, 58 A 20.

2. DIFFERENTIAL FORMS ON JET PROLONGATIONS OF FIBERED MANIFOLDS

This paper is devoted to the theory of horizontal and contact differential forms, and differential odd base forms on (finite) jet prolongations of fibered manifolds. The subject had been developed in the period 1970–1980, and reflects the effort to achieve a deeper understanding of the geometrical, and conceptual structure of the global calculus of variations. It is very closely related to the basic variational notions: the horizontal and contact forms allow to introduce the global concepts such as, for example, the lagrangian, the Lepagean form, the Euler-Lagrange form, and the first variation formula.

In this paper, as well as throughout this work, a uniform numbering of sections, formulas, and references is used, beginning in Section 1, and continuing in the next sections.

The theory of horizontal and contact forms for higher order variational problems starts with the notion of horizontalization [9]; in [7], [8], [10], [11] and [12] this theory gets a relatively closed form. It should be pointed out, however, that the theory of Takens, Tulczyjew, and Kupershmidt involves, in addition, infinite constructions (infinite jets, direct and indirect limits), which are not needed in our approach.

For basic facts on the Ehresmann's theory of jets the reader is referred to [1].

* Part 1 of this work has been published in Arch. Math. (Brno) 22, No. 3 (1986).

2.1. Jet prolongations of fibered manifolds. Let Y be a *fibered manifold* with *base* X and *projection* π, let $n = \dim X$, $m = \dim Y - n$. By definition, π is a surjective submersion; in particular, π is an open mapping. Thus to each point $y \in Y$ there exists a chart (V, ψ), $\psi = (u^1, \ldots, u^n, y^1, \ldots, y^m)$, at y and a unique chart (U, φ), $\varphi = (x^1, \ldots, x^n)$, at $x = \pi(y)$ such that $U = \pi(V)$ and $u^i = x^i \circ \pi$ for all i, $1 \leqq i \leqq n$, or, which is the same, such that $\varphi \circ \pi = pr_1 \circ \psi$, where $pr_1 = R^n \times R^m \to R^n$ is the first canonical projection. (V, ψ) is called a *fiber chart* on Y, and (U, φ) is called *associated* with (V, ψ).

For simplicity, a fiber chart on Y is usually denoted by (V, ψ), $\psi = (x^i, y^\sigma)$, where $1 \leqq i \leqq n$, $1 \leqq \sigma \leqq m$, and the associated chart on X is denoted by (U, φ), $\varphi = (x^i)$.

The *r-jet* of a mapping f at a point x is denoted by $J^r_x f$. The manifold of r-jets $J^r_x\gamma$ of (local) sections γ of Y is called the r-jet *prolongation* of Y, and is denoted by J^rY. J^rY has the structure of a fibered manifold with base X (resp. J^sY, $0 \leqq s \leqq r$) and projection π^s (resp. $\pi_{r,s}$) defined by $\pi_r(J^r_x\gamma) = x$ (resp. $\pi_{r,s}(J^r_x\gamma) = J^s_x\gamma$). If γ is a section of Y over an open set $U \subset X$, then the mapping $x \to J^r_x\gamma$ is a section of J^rY over U; this section is called the *r-jet prolongation* of γ, and is denoted by $J^r\gamma$.

Any fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y defines the *associated* fiber chart (V_r, ψ_r), $\psi_r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_r})$, on J^rY, where $1 \leqq i \leqq n$, $1 \leqq \sigma \leqq m$, $1 \leqq j_1 \leqq \ldots \leqq j_k \leqq n$, $1 \leqq k \leqq r$, by the formula

$$y^\sigma_{j_1 \ldots j_k}(J^r_x\gamma) = D_{j_1} \ldots D_{j_k}(y^\sigma\gamma\varphi^{-1})(\varphi(x)), \tag{2.1.1}$$

where (U, φ) is the chart on X associated with (V, ψ), and D_i denotes the i-th partial derivative operator. Let (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\bar V, \bar\psi)$, $\bar\psi = (\bar x^i, \bar y^\sigma)$, be two fiber charts such that $V \cap \bar V \neq \emptyset$. Then $V_s \cap \bar V_s \neq \emptyset$, and we have for any $J^r_x\gamma \in V_r \cap \bar V_r$ and any k, $1 \leqq k \leqq r$, $p_1, \ldots, p_k = 1, 2, \ldots, n$, and ν, $1 \leqq \nu \leqq m$,

$$\begin{aligned} \bar y^\nu_{p_1 \ldots p_k}(J^r_x\gamma) &= (\bar y^\nu_{p_1 \ldots p_k} \circ J^r\gamma \circ \bar\varphi^{-1})(\bar\varphi(x)) = \\ &= D_{p_k}(\bar y^\nu_{p_1 \ldots p_{k-1}} \circ J^r\gamma \circ \bar\varphi^{-1})(\bar\varphi(x)) = \\ &= D_{p_k}(\bar y^\nu_{p_1 \ldots p_{k-1}}\psi_r^{-1} \circ \psi_r \circ J^r\gamma \circ \varphi^{-1} \circ \varphi\bar\varphi^{-1})(\bar\varphi(x)). \end{aligned} \tag{2.1.2}$$

Thus we obtain, using the chain rule, the *transformation formula* in a recurrent form,

$$\bar y^\nu_{p_1 \ldots p_k} = \left(\frac{\partial \bar y^\nu_{p_1 \ldots p_{k-1}}}{\partial x^s} + \sum_{q=0}^{k-1} \sum \frac{\partial \bar y^\nu_{p_1 \ldots p_{k-1}}}{\partial y^\sigma_{j_1 \ldots j_q}} y^\sigma_{j_1 \ldots j_q s}\right) \frac{\partial x^s}{\partial \bar x^{p_k}}, \tag{2.1.3}$$

where the second summation sign denotes the summation over all q-tuples tuples $(j_1, \ldots, j_q)$ such that $1 \leqq j_1 \leqq \ldots \leqq j_q \leqq n$.

Remark 2.1. We note that (2.1.1) defines the functions $y^\sigma_{j_1 \ldots j_k} : V_r \to R$ for *all* k-tuples $(j_1, \ldots, j_k)$, not only for non-decreasing ones; however, the coordinates of the chart (V_r, ψ_r) are only those of them whose subscripts form a non-decreasing k-tuples.

Remark 2.2. One could suggest to use *multi-indices* instead of the non-decreasing k-tuples of indices $(j_1, \dots, j_k)$ in the expressions like (2.1.1). It will be seen later, however, that some operations over the indices, as symmetrization in a *part* of them, cannot be effectively described by multi-indices. For this reason we prefer the use of non-decreasing k-tuples.

Let $\pi_1 : Y_1 \to X_1$, $\pi_2 : Y_2 \to X_2$ be two fibered manifolds, $V \subset Y_1$ an open set, and $\alpha : V \to Y_2$ a homomorphism of fibered manifolds. Since π_1 is an open mapping, $U = \pi_1(V)$ is an open subset of X. Recall that α is said to be a *homomorphism* of fibered manifolds if there exists a mapping $\alpha_0 : U \to X_2$ such that

$$\pi_2 \circ \alpha = \alpha_0 \circ \pi_1 \tag{2.1.4}$$

on V. If α_0 exists, it is unique, and is called the *projection* of α. We write for simplicity $\alpha_0 = pr\,\alpha$.

It is clear that if for two homomorphisms α, β of fibered manifolds the composition $\beta \circ \alpha$ is defined, then it is again a homomorphism of fibered manifolds, and $pr(\beta \circ \alpha) = (pr\,\beta) \circ (pr\,\alpha)$.

Let $\alpha : V \to Y_2$ be a homomorphism of fibered manifolds. Suppose that $\dim X_1 = \dim X_2$, and that $pr\,\alpha : U \to pr\,\alpha(U) \subset X_2$ is a diffeomorphism. Let γ be a section of Y_1, mapping its domain of definition into V. Then by (2.1.4), $\alpha\gamma(pr\,\alpha)^{-1}$ is a section of π_2, defined on the open set $pr\,\alpha(U) \subset X_2$. Thus the r-jet $J^r_{pr\alpha(x)}(\alpha\gamma(pr\,\alpha)^{-1})$ is defined for each x from the domain of definition of $pr\,\alpha$, and the formula

$$J^r\alpha(J^r_x\gamma) = J^r_{pr\alpha(x)}(\alpha\gamma(pr\,\alpha)^{-1}) \tag{2.1.5}$$

defines a mapping $J^r\alpha : \pi_{r,0}^{-1}(V) \to J^rY_2$. $J^r\alpha$ is a smooth mapping such that

$$(\pi_1)_r \circ J^r\alpha = pr\,\alpha \circ (\pi_2)_r, \qquad (\pi_1)_{r,s} \circ J^r\alpha = J^s\alpha \circ (\pi_2)_{r,s}. \tag{2.1.6}$$

Thus $J^r\alpha$ is a homomorphism of the fibered manifold J^rY_1 with base X_1 (resp. J^sY_1) and projection $(\pi_1)_s$ (resp. $(\pi_1)_{r,s}$) into the fibered manifold J^rY_2 with base X_2 (resp. J^sY_2) and projection $(\pi_2)_r$ (resp. $(\pi_2)_{r,s}$). We call $J^r\alpha$ the ***r*-jet prolongation** of α.

Notice that (2.1.5) can be written in the form

$$J^r\alpha \circ J^r\gamma \circ (pr\,\alpha)^{-1} = J^r(\alpha\gamma(pr\,\alpha)^{-1}) \tag{2.1.7}$$

for every section γ of π.

If α, β are two homomorphisms of fibered manifolds such that $\beta \circ \alpha$ is defined and $pr\,\alpha$, $pr\,\beta$ are diffeomorphisms, then the r-jet prolongation $J^r(\beta \circ \alpha)$ is defined, and

$$J^r(\beta \circ \alpha) = J^r\beta \circ J^r\alpha. \tag{2.1.8}$$

The definition of the r-jet prolongation of a fibered manifold can be applied to the s-jet prolongation of this fibered manifold, where r and s are any non-negative integers. Let Y be a fibered manifold with base X and projection π. We obtain in this way a fibered manifold J^rJ^sY with base X (resp. J^pJ^sY, $0 \leqq p \leqq r$) and projection $(\pi_s)_r$ (resp. $(\pi_s)_{r,p}$). Jet prolongations of Y of this type are usually referred to as the *non-holonomic prolongations*, or the *(s, r)-jet prolongations* of Y; the elements of J^rJ^sY are usually called *non-holonomic jets*.

Let s and r be non-negative integers, $J_x^{r+s}\gamma \in J^{r+s}Y$ a point. For a representative γ of the r-jet $J_x^{r+s}\gamma$, $J^s\gamma$ is a section of J^sY, and the r-jet of this section $J_x^rJ^s\gamma$, is a well-defined element of J^rJ^sY; obviously, chosing a fiber chart (V, ψ) on Y such that $J_x^rJ^s\gamma \in (V_s)_r$, and expressing $J_x^rJ^s\gamma$ with respect to this fiber chart we can see at once that this r-jet depends only on the $(r + s)$-jet $J_x^{r+s}\gamma$. Thus, putting

$$\iota(J_x^{r+s}\gamma) = J_x^rJ^s\gamma \tag{2.1.9}$$

we obtain a mapping $\iota : J^{r+s}Y \to J^rJ^sY$. It is easily verified in terms of charts that ι is an embedding. We call it the *canonical embedding* of $J^{r+s}Y$ into J^rJ^sY.

2.2. Horizontalization and horizontal forms. In this section, Y is a fixed fibered manifold with base X and projection π.

Let ϱ be a form, or an odd base form on Y. ϱ is called *π-horizontal* if $i_\xi\varrho = 0$ for every π-vertical vector $\xi \in TY$. ϱ is called *π-projectable* if there exists a form, or an odd form, ϱ_0 on X such that $\pi^*\varrho_0 = \varrho$. If ϱ_0 exists it is unique; we call it the *π-projection* of ϱ.

The module of p-forms (resp. the module of odd base p-forms) over the ring of functions, defined on an open set $W \subset Y$, is denoted by $\Omega^p(W)$ (resp. $\hat{\Omega}^p(W)$). π-horizontal p-forms (resp. odd base p-forms) form a sub-module of this module, denoted by $\Omega_X^p(W)$ (resp. $\hat{\Omega}_X^p(W)$). We put

$$\begin{aligned} \Omega(W) &= \Sigma\,\Omega^p(W), & \Omega_X(W) &= \Sigma\,\Omega_X^p(W), \\ \hat{\Omega}(W) &= \Sigma\hat{\Omega}^p(W), & \hat{\Omega}_X(W) &= \Sigma\hat{\Omega}_X^p(W) \end{aligned} \tag{2.2.1}$$

(the direct sum of modules, summation over $p = 0, 1, \ldots, n + m$). $\Omega(W)$ (resp. $\hat{\Omega}(W)$ is a graded module, and $\Omega_X(W)$ (resp. $\hat{\Omega}_X(W)$) is its graded submodule. The exterior product of forms defines on $\Omega(W)$ the structure of the exterior algebra; $\Omega_X(W) \subset \Omega(W)$ is its subalgebra. The exterior product of forms and odd base forms defines on $\hat{\Omega}(W)$ the structure of a left module over the algebra $\hat{\Omega}(Y)$.

Let now $W \subset J^rY$ be an open set, $f \in \Omega^0(W)$ and $\varrho \in \Omega^p(W)$, where $p \geqq 1$. Denote $W' = \pi_{r+1,r}^{-1}(W)$. We set for each $J_x^{r+1}\gamma \in W'$ and any vectors $\xi_1, \ldots, \xi_p \in TJ^{r+1}Y$ at the point $J_x^{r+1}\gamma$

$$\begin{gathered} h(f)\,(J_x^{r+1}\gamma) = f(J_x^r\gamma), \\ h(\varrho)\,(J_x^{r+1}\gamma)\,(\xi_1, \ldots, \xi_p) = \varrho(J_x^r\gamma)\,(T_xJ^r\gamma\,.\,T\pi_{r+1}\,.\,\xi_1, \ldots, T_xJ^r\gamma\,.\,T\pi_{r+1}\,.\,\xi_p). \end{gathered} \tag{2.2.2}$$

Then $h(f) \in \Omega^0(W')$ and $h(\varrho) \in \Omega^p_X(W')$, i.e., $h(\varrho)$ is π_{r+1}-horizontal. The mapping $\varrho \to h(\varrho)$ of $\Omega(W)$ into $\Omega_X(W')$, defined by (2.2.2), is called the *π-horizontalization* (of forms).

We shall now transfer the notion of π-horizontalization to odd base forms. Let us consider J^rY as a fibered manifold with base X and projection π_r. Notice that we have a commutative diagram

$$\begin{array}{ccc} \pi^*_{r+1,r}\pi^*_r\hat{R}X & \xrightarrow{\nu_{r+1,r}} & \pi^*_r\hat{R}X \\ \downarrow & & \downarrow \\ J^{r+1}Y & \xrightarrow{\pi_{r+1,r}} & J^rY \end{array} \tag{2.2.3}$$

where $\nu_{r+1,r}$ is the canonical homomorphism of vector bundles (see the beginning of Sec. 1.2). Identifying $\pi^*_{r+1,r}\pi^*_r\hat{R}X$ with $\pi^*_{r+1}\hat{R}X$ we can interpret $\nu_{r+1,r}$ as a homomorphism of $\pi^*_{r+1}\hat{R}X$ into $\pi^*_r\hat{R}X$. $\nu_{r+1,r}$ is a linear isomorphism on each fiber. Let $W \subset J^rY$ be an open set, $\varrho \in \hat{\Omega}^p(W)$, where $p \geqq 0$, and $W' = \pi^{-1}_{r+1,r}(W)$. Let $J^{r+1}_x\gamma \in W'$ be any point, $\xi_1, \ldots, \xi_p \in TJ^{r+1}Y$ any vectors at this point. There exists one and only one odd base scalar $h(\varrho)(J^{r+1}_x\gamma)(\xi_1, \ldots, \xi_p)$ at the point $J^{r+1}_x\gamma$ such that

$$\begin{aligned} &\nu_{r+1,r}(h(\varrho)(J^{r+1}_x\gamma)(\xi_1, \ldots, \xi_p)) = \\ &= \varrho(J^r_x\gamma)(T_xJ^r\gamma \,.\, T\pi_{r+1} \cdot \xi_1, \ldots, TJ^r\gamma \,.\, T\pi_{r+1} \cdot \xi_p). \end{aligned} \tag{2.2.4}$$

The correspondence $J^{r+1}_x\gamma \to h(\varrho)(J^{r+1}_x\gamma)$ is an element of $\hat{\Omega}^p_X(W')$. The mapping $\varrho \to h(\varrho)$ of $\hat{\Omega}(W)$ into $\hat{\Omega}_X(W')$, defined by (2.2.4), is called the *π-horizontalization* (of odd base forms).

The following elementary properties of the π-horizontalization of forms, and of odd base forms, can be deduced from the definitions.

Theorem 2.1. *Let $W \subset J^rY$ be an open set, $W' = \pi^{-1}_{r+1,r}(W)$. Suppose that either $\varrho, \eta \in \Omega^p(W)$ and $\omega \in \Omega^q(W)$, or $\varrho, \eta \in \hat{\Omega}^p(W)$ and $\omega \in \hat{\Omega}^q(W)$. Then the following conditions hold:*

(a) $h(\varrho + \eta) = h(\varrho) + h(\eta)$, $h(\omega \wedge \varrho) = h(\omega) \wedge h(\varrho)$.

(b) *For any open subset $V \subset W$,*

$$h(\varrho \mid_V) = h(\varrho) \mid_{V'}. \tag{2.2.5}$$

(c) *If $\varrho \in \Omega^p(W)$ (resp. $\varrho \in \hat{\Omega}^p(W)$), then $h(\varrho) \in \Omega^p_X(W')$ (resp. $h(\varrho) \in \hat{\Omega}^p_X(W')$) is a unique form such that for every section γ of Y whose r-jet prolongation $J^r\gamma$ maps its domain of definition into W,*

$$J^r\gamma^*\varrho = J^{r+1}\gamma^*h(\varrho). \tag{2.2.6}$$

(d) *If $p > n$, then $h(\varrho) = 0$.*

(e) *If ϱ is $\pi_{r,r-1}$-horizontal, then $h(\varrho)$ is $\pi_{r+1\,r}$-projectable.*

(f) *Let Y_1, Y_2 be two fibered manifolds, $V \subset Y_1$ an open set, and $\alpha : V \to Y_2$ a homomorphism of fibered manifolds such that pr α is a diffeomorphism. Let ϱ be a form or an odd base form on Y_2. Then*

$$h(J^r\alpha^*\varrho) = J^{r+1}\alpha^*h(\varrho). \tag{2.2.7}$$

Convention 2.1. In the following we sometimes apply a simplifying convention concerning π-projectable forms on Y. If ϱ is a π-projectable form, or a π-projectable odd base form, then its π-projection is denoted, when no misunderstanding may possibly arise, by the same letter, ϱ. Analogously, if η is a form, or an odd form, on X, we write simply ϱ instead of $\pi^*\varrho$. Notice that this convention applies to Theorem 2.1, (e); accordingly, if $h(\varrho)$ is $\pi_{r+1,r}$-projectable, its $\pi_{r+1,r}$-projection is denoted by the same symbol, $h(\varrho)$.

We now establish, in addition to our summation conventions of Section 1, a summation convention for chart expressions of forms on jet prolongations of fibered manifolds. In the next sections, the same convention will also be applied in different situations (e.g. in chart expressions of vector fields). It is enough to explain this summation convention for linear forms.

Convention 2.2. Consider a fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, and a linear form ϱ on V_r. ϱ can uniquely be expressed with respect to this chart (more precisely, with respect to the associated chart (V_r, ψ_r)). The chart expression of ϱ will be denoted by

$$\varrho = f_i\,\mathrm{d}x^i + \Sigma\,\Sigma g_\sigma^{j_1\ldots j_k}\,\mathrm{d}y^\sigma_{j_1\ldots j_k}, \tag{2.2.8}$$

where the first summation sign means the summation over $k = 0, 1, \ldots, r$, and the second one means the summation over all k-tuples $(j_1, \ldots, j_r)$ such that $1 \leqq j_1 \leqq \ldots \leqq j_k \leqq n$.

Sometimes it is necessary to restrict the range of summation over k in (2.2.8); if, for example, ϱ is $\pi_{r,s}$-horizontal, we write

$$\varrho = f_i\,\mathrm{d}x^i + \sum_{k=0}^{s}\sum g_\sigma^{j_1\ldots j_k}\,\mathrm{d}y^\sigma_{j_1\ldots j_k}. \tag{2.2.9}$$

In such cases the range of summation is designated explicitly.

In (2.2.8), f_i and $g_\sigma^{j_1\ldots j_k}$, $j_1 \leqq \ldots \leqq j_k$, are the *components* of ϱ with respect to (V, ψ). If we need summation over *all* k-tuples $(j_1, \ldots, j_k)$, not only over non-decreasing ones, we define the functions $g_\sigma^{j_1\ldots j_k}$ for arbitrary $(j_1, \ldots, j_k)$ on the symmetry requirements, and then proceed as follows. Let $(p_1, \ldots, p_k)$ be any k-tuple such that $1 \leqq p_1, \ldots, p_k \leqq n$. Denote by $N(p_1 \ldots p_k)$ the number of *different* k-tuples $(q_1, \ldots, q_k)$ arising by permuting the set $\{p_1, \ldots, p_k\}$. Obviously,

$$N(p_1 \ldots p_k) = \frac{k!}{s_1! \ldots s_n!}, \tag{2.2.10}$$

where s_i denotes the number of integers i in $\{p_1, \ldots, p_k\}$. Now (2.2.8) takes the form

$$\varrho = f_i \, dx^i + \sum\sum \frac{1}{N(j_1 \ldots j_k)} g_\sigma^{j_1 \ldots j_k} \, dy^\sigma_{j_1 \ldots j_k}, \tag{2.2.11}$$

where the first summation sign means the summation over $k = 1, 2, \ldots, r$, and the second one means the summation over *all* $j_1, \ldots, j_k = 1, 2, \ldots, n$. Putting

$$f_\sigma^{j_1 \ldots j_k} = \frac{1}{N(j_1 \ldots j_k)} g_{\sigma}^{j_1 \ldots j_k} \tag{2.2.12}$$

we can also write

$$\varrho = f_i \, dx^i + \Sigma f_{\sigma}^{j_1 \ldots j_k} \, dy^\sigma_{j_1 \ldots j_k}, \tag{2.2.13}$$

where Σ means summation over $k = 0, 1, \ldots, r$, and summation over δ and $j_1, \ldots, j_k$ is automatically understood.

It should be pointed out, however, that the coefficients at $dy^\sigma_{j_1 \ldots j_k}$ in (2.2.11) and (2.2.13) are not the components of ϱ, and are related with the components of ϱ by (2.2.12)

Theorem 2.2. (a) *Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y. The π-horizontalization $h : \Omega(V_r) \to \Omega(V_{r+1})$ is a unique mapping, preserving the exterior algebra structure, such that for any function $f : V_r \to R$*

$$h(f) = f \circ \pi_{r+1,r}, \qquad h(df) = d_i f \, . \, dx^i, \tag{2.2.14}$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum\sum \frac{\partial f}{\partial y^\sigma_{j_1 \ldots j_k}} \, y^\sigma_{j_1 \ldots j_k i}. \tag{2.2.15}$$

(b) *The π-horizontalization $h : \hat{\Omega}(V_r) \to \hat{\Omega}_x(V_{r+1})$ is a unique mapping such that for each fiber chart (V, ψ) and each $\varrho \in \hat{\Omega}(V_r)$ expressed with respect to the associated chart (U, φ) on X by $\varrho = \pi_r^* \hat{\varphi} \otimes \varrho_\varphi$, $h(\varrho)$ is expressed by*

$$h(\varrho) = \pi_{r+1}^* \hat{\varphi} \otimes h(\varrho_\varphi) \tag{2.2.16}$$

with respect to (V_{r+1}, ψ_{r+1}).

Proof. (a) To show it, one directly verifies that the mapping h defined by (2.2.2), satisfies (2.2.14). To prove the uniqueness, notice that (2.2.15) implies for each i, k and $j_1, \ldots, j_k$

$$h(dx^i) = dx^i, \qquad h(dy^\sigma_{j_1 \ldots j_k}) = y^\sigma_{j_1 \ldots j_k i} \, dx^i. \tag{2.2.17}$$

Now it is sufficient to check that any two mappings h_1, h_2, preserving the exterior algebra structure, satisfying (2.2.14), agree on functions and linear forms; this follows, however, from (2.2.17).

(b) Transformation formulas show that (2.2.16) defines a mapping $h : \hat{\Omega}(V_r) \to$

$\to \hat{\Omega}(V_{r+1})$. It thus remains to show that the chart expression of the odd base form $h(\varrho)$, defined by (2.2.4), coincides with (2.2.16).

Chart expressions. If $W \subset J^rY$ is an open set and $\varrho \in \Omega^r(W)$, then for any fiber chart (V, ψ), the chart expression of $h(\varrho) \in \Omega^p_X(W')$ with respect to the chart $(V_{r+1} \cap W', \psi_{r+1})$ can be obtained from the chart expression of ϱ with respect to $(V_r \cap W, \psi_r)$ by means of Theorem 2.1. (a), and (2.2.13).

The component $d_if : V_{r+1} \to R$ of $h(\mathrm{d}f)$ (2.2.14) is called the *formal*, or *total*, derivative of f with respect to x^i. Notice that for any two functions $f, g : V_r \to R$,

$$d_i(f \,.\, g) = g \,.\, d_if + f \,.\, d_ig, \tag{2.2.18}$$

where we have used the above convention, and write just f, g instead of $f \circ \pi_{r+1,r}$, $g \circ \pi_{r+1,r}$ on the right-hand side.

2.3. Contact forms. In this section, Y is a fibered manifold with base X and projection π.

Let ϱ be a form or an odd base form defined on an open set $W \subset J^rY$. ϱ is called *π-contact*, or *contact*, or *pseudovertical*, if $h(\varrho) = 0$. By Theorem 2.1 (a) and (1.3.24) (see Remark 1 of Sect. 1.3), π-contact forms (resp. π-contact odd base forms) define an *ideal* (resp. *submodule*) of the exterior algebra $\Omega(W)$ (resp. of the left module $\hat{\Omega}(W)$ over $\Omega(W)$), closed with respect to the exterior derivative. This ideal (resp. submodule) is denoted by $\Omega_p(W)$ (resp. $\hat{\Omega}_p(W)$).

Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart. We shall now study the structure of the ideal $\Omega_p(V_r)$. Put for every σ, $1 \leqq \sigma \leqq m$, k, $0 \leqq k \leqq r - 1$, and $j_1, \ldots, j_k = 1, 2, \ldots, n$

$$\omega^\sigma_{j_1 \ldots j_k} = \mathrm{d}y^\sigma_{j_1 \ldots j_k} - y^\sigma_{j_1 \ldots j_k i}\, \mathrm{d}x^i. \tag{2.3.1}$$

Each of the linear forms (2.3.1) obviously belongs to this ideal (see (2.2.16)).

Theorem 2.3. (a) *The forms* $\mathrm{d}x^i$, $\omega^\sigma_{j_1 \ldots j_k}$, $\mathrm{d}y^\sigma_{j_1 \ldots j_r}$, *where* $1 \leqq i \leqq n$, $1 \leqq \sigma \leqq m$, $0 \leqq k \leqq r - 1$, $1 \leqq j_1 \leqq \ldots \leqq j_k \leqq n$, *are elements of a basis of linear forms on* V_r.

(b) *Let* (U, φ) *be the chart on* X *associated with* (V, ψ), *and let* δ *be a section of the fibered manifold* J^rY *over* U. *Then the following two conditions are equivalent*:

(1) *There exists a section* γ *of* Y *over* U *such that* $\delta = J^r\gamma$.

(2) δ *satisfies the equations*

$$\delta^* \omega^\sigma_{j_1 \ldots j_k} = 0. \tag{2.3.2}$$

Proof. (a) The forms $\mathrm{d}x^i$, $\omega^\sigma_{j_1 \ldots j_k}$, $\mathrm{d}y^\sigma_{j_1 \ldots j_r}$ are obviously linear combinations of the linear forms $\mathrm{d}x^i$, $dy^\sigma_{j_1 \ldots j_k}$, $\mathrm{d}y^\sigma_{j_1 \ldots j_k}$, with regular matrix.

(b) Let δ be of the form $J^r\gamma$. Then (2.3.2) follows from the definition of coordinates $y^\sigma_{j_1\dots j_k}$ (2.1.1). Conversely, suppose that δ satisfies the system (2.3.2). Then we get

$$\begin{aligned} &y^\sigma_{j_1} \circ \delta = D_{j_1}(y \circ \delta),\\ &y^\sigma_{j_1 j_2} \circ \delta = D_{j_2}(y^\sigma_{j_1} \circ \delta) = D_{j_1} D_{j_2}(y^\sigma \circ \delta),\\ (2.2.3)\qquad &\dots\\ &y^\sigma_{j_1\dots j_r} \circ \delta = D_{j_r}(y^\sigma_{j_1\dots j_{r-1}} \circ \delta) = \dots = D_{j_1} \dots D_{j_r}(y^\sigma \circ \delta). \end{aligned}$$

Putting $\gamma = \pi_{r,0} \circ \delta$ we get a section of Y over U for which $y^\sigma(\delta(x)) = y^\sigma(\gamma(x))$ for each $x \in U$; then (2.3.3) means that $J^r\gamma = \delta$.

Theorem 2.3, (a) shows, in particular, that the forms dx^i, $\omega^\sigma_{j_1\dots j_k}$ are elements of a basis of linear $\pi_{r,r-1}$-horizontal forms; (b) characterizes those of sections of J^rY over U which are prolongations of sections of Y over U.

The following simple observations show that the ideal $\Omega_p(V_r)$ has a rather complicated structure.

Remark 2.3. (a) A form $\varrho \in \Omega^1(V_r)$ is contact if and only if it is a linear combination of the forms (2.3.1), i.e.,

$$(2.3.4)\qquad \varrho = \sum_{k=0}^{r-1} \sum g_\sigma^{j_1\dots j_k} \omega^\sigma_{j_1\dots j_k}.$$

(b) The 2-form $d\omega^\sigma_{j_1\dots j_{r-1}}$ is contact by (2.2.6), but it is obviously not generated by linear contact forms (see (a)). More generally, it can be shown by a direct calculation that a form $\varrho \in \Omega^2(V_r)$ is contact if and only if

$$(2.3.5)\qquad \varrho = \sum_{k=0}^{r-1} \sum P_\sigma^{j_1\dots j_k} \wedge \omega^\sigma_{j_1\dots j_k} + \sum Q_\sigma^{j_1\dots j_{r-1}}\, d\omega^\sigma_{j_1\dots j_{r-1}},$$

where $P_\sigma^{j_1\dots j_k}$ are some linear forms and $Q_\sigma^{j_1\dots j_{r-1}}$ are some functions on V_r.

(c) Let $r = 1$, $n \geqq 2$, and denote for each i, $1 \leqq i \leqq n$,

$$(2.3.6)\qquad \omega_i = (-1)^{i-1}\, dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n.$$

The n-form $\varrho \in \Omega^n(V_1)$ defined by

$$(2.3.7)\qquad \varrho = dy^\sigma_i \wedge \omega_j - dy^\sigma_j \wedge \omega_i$$

is contact for any i, j. This form is not generated by ω^σ and $d\omega^\sigma$. Clearly, analogous examples can be constructed for arbitrary $r > 1$.

We shall determine the transformation properties of the forms (2.3.1). Consider another fiber chart $(\bar V, \bar\psi)$, $\bar\psi = (\bar x^i, \bar y^\sigma)$, on Y and denote by $\bar\omega^\sigma_{j_1\dots j_k}$ the forms (2.3.1) related to this fiber chart.

Theorem 2.4. *Let (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\bar V, \bar\psi)$, $\bar\psi = (\bar x^i, \bar y^\sigma)$ be two fiber charts on Y such that $V \cap \bar V \neq \emptyset$. Then*

$$\bar{\omega}^{\nu}_{p_1 \dots p_l} = \sum_{k=0}^{l} \sum \frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_1 \dots j_k}} \omega^{\sigma}_{j_1 \dots j_k}, \tag{2.3.8}$$

and the coefficients on the right satisfy the recurrent formulas

$$\begin{aligned} &\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l q}}{\partial y^{\sigma}} \frac{\partial \bar{x}^q}{\partial x^s} = d_s\left(\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}}\right), \\ &\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l q}}{\partial y^{\sigma}_{j_1 \dots j_k}} \frac{\partial \bar{x}^q}{\partial x^s} = d_s\left(\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_1 \dots j_k}}\right) + \\ &+ \frac{N(j_1 \dots j_k)}{k}\left(\frac{1}{N(j_1 \dots j_{k-1})} \frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_1 \dots j_{k-1}}} \delta^{j_k}_s + \dots + \right. \\ &\left. + \frac{1}{N(j_2 \dots j_k)} \frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_2 \dots j_k}} \delta^{j_1}_s\right), \qquad 1 \leqq k \leqq l + 1. \end{aligned} \tag{2.3.9}$$

Proof. Let d_s denote the formal derivative with respect to $\bar{x}^s$. Since $\bar{y}^{\nu}_{p_1 \dots p_l s} = d_s \bar{y}_{p_1 \dots p_l}$, we get by (2.1.3)

$$\begin{aligned} \bar{\omega}^{\nu}_{p_1 \dots p_l} = &\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial x^j} \mathrm{d}x^j + \\ &+ \sum_{k=0}^{l} \sum \frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_1 \dots j_k}} \mathrm{d}y^{\sigma}_{j_1 \dots j_k} - d_j \bar{y}^{\nu}_{p_1 \dots p_l} \frac{\partial x^j}{\partial \bar{x}^k} \mathrm{d}\bar{x}^k. \end{aligned} \tag{2.3.10}$$

Expressing the right side in terms of (2.3.1) and computing the formal derivatives $d_j \bar{y}_{p_1 \dots p_l}$ we get at once (2.3.8).

We shall now show that (2.3.9) holds; to prove this formulas together for $l < r$ and $l = r$ it is convenient to work on $J^{r+1}Y$ instead of J^rY. We get, using (2.3.8)

$$\begin{aligned} \pi^{*}_{r+1,r}\, d\bar{\omega}^{\nu}_{p_1 \dots p_l} &= -\bar{\omega}^{\nu}_{p_1 \dots p_l q} \wedge d\bar{x}^q = \\ &= -\sum_{k=0}^{l+1} \sum \frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l q}}{\partial y^{\sigma}_{j_1 \dots j_k}} \frac{\partial \bar{x}^q}{\partial x^s} \omega^{\sigma}_{j_1 \dots j_k} \wedge \mathrm{d}x^s. \end{aligned} \tag{2.3.11}$$

On the other hand, (2.3.8) can be directly differentiated. Since

$$\begin{aligned} \pi^{*}_{r+1,r} d\left(\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_1 \dots j_k}}\right) = &d_s\left(\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\sigma}_{j_1 \dots j_k}}\right) \mathrm{d}x^s + \\ &+ \sum_{q=0}^{l} \sum \frac{\partial^2 \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial y^{\varrho}_{i_1 \dots i_q} \partial y^{\sigma}_{j_1 \dots j_k}} \omega^{\varrho}_{i_1 \dots i_q}, \end{aligned} \tag{2.3.12}$$

we get

$$\begin{aligned} &\pi^{*}_{r+1,r}\, \mathrm{d}\bar{\omega}^{\nu}_{p_1 \dots p_l} = \\ &= \sum_{k=0}^{l} \sum \left(d_s\left(\frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{d y^{\sigma}_{j_1 \dots j_k}}\right) \mathrm{d}x^s \wedge \omega^{\sigma}_{j_1 \dots j_k} - \frac{\partial \bar{y}^{\nu}_{p_1 \dots p_l}}{\partial\, y^{\sigma}_{j_1 \dots j_k}} \omega^{\sigma}_{j_1 \dots j_k s} \wedge \mathrm{d}x^s\right). \end{aligned} \tag{2.3.13}$$

We obtain for the second term, up to the minus sign,

$$\sum_{k=1}^{l+1}\sum \frac{\partial \bar{y}^{\nu}_{p_1\ldots p_l}}{\partial y^{\sigma}_{j_1\ldots j_{k-1}}} \omega^{\sigma}_{j_1\ldots j_{k-1}s} \wedge dx^s = \tag{2.3.14}$$

$$= \sum_{k=1}^{l+1}\sum \frac{1}{N(j_1 \cdots j_{k-1})} \frac{\partial \bar{y}^{\nu}_{p_1\ldots p_l}}{\partial y^{\sigma}_{j_1\ldots j_{k-1}}} \delta^{j_k}_s \omega^{\sigma}_{j_1\ldots j_k} \wedge dx^s =$$

$$= \sum_{k=1}^{l+1}\sum \frac{1}{k}\left(\frac{1}{N(j_1 \cdots j_{k-1})} \frac{\partial \bar{y}^{\nu}_{p_1\ldots p_l}}{\partial y^{\sigma}_{j_1\ldots j_{k-1}}} \delta^{j_k}_s + \ldots + \right.$$

$$\left. + \frac{1}{N(j_2 \cdots j_k)} \frac{\partial \bar{y}^{\nu}_{p_1\ldots p_l}}{\partial y^{\sigma}_{j_2\ldots j_k}} \delta^{j_1}_s \right) \omega^{\sigma}_{j_1\ldots j_k} \wedge dx^s =$$

$$= \sum_{k=1}^{l+1}\sum \frac{N(j_1 \cdots j_k)}{k}\left(\frac{1}{N(j_1 \cdots j_{k-1})} \frac{\partial \bar{y}^{\nu}_{p_1\ldots p_l}}{\partial y^{\sigma}_{j_1\ldots j_{k-1}}} \delta^{j_k}_s + \ldots + \right.$$

$$\left. + \frac{1}{N(j_2 \cdots j_k)} \frac{\partial \bar{y}^{\nu}_{p_1\ldots p_l}}{\partial y^{\sigma}_{j_2\ldots j_k}} \delta^{j_1}_s \right) \omega^{\sigma}_{j_1\ldots j_k} \wedge dx^s,$$

where we have passed from the summation over non-decreasing $(k-1)$-tuples $(j_1, \ldots, j_{k-1})$ (the first expression) to the summation over all k-tuples $(j_1, \ldots, j_k)$ (the second and the third expressions) and then to the summation over non-decreasing k-tuples $(j_1, \ldots, j_k)$ (the last expression) according to our summation conventions. Substituting (2.3.14) in (2.3.13) and comparing the result with (2.3.11) we obtain (2.3.9).

Corollary 1. *Let q be an integer such that $0 \leqq q \leqq r-1$.*

(a) *The linear forms $\omega^{\sigma}_{j_1\ldots j_k}$, where $1 \leqq \sigma \leqq m$, $1 \leqq j_1 \leqq \ldots \leqq j_k \leqq n$, $0 \leqq k \leqq q$, locally span a submodule of the module $\Omega^p(J^rY)$, for each $p \leqq 1$. This submodule consists of contact, $\pi_{r,q}$-horizontal forms.*

(b) *The linear forms $\omega^{\sigma}_{j_1\ldots j_k}$ generate an ideal of forms on J^rY. This ideal consists of contact forms.*

Proof. This follows from (2.3.8).

Let $W \subset J^rY$ be an open set, and let $\varrho \in \Omega^p(W)$, or $\varrho \in \hat{\Omega}^p(W)$. Put $W' = \pi^{-1}_{r+1,r}(W)$. Then $\pi^*_{r+1,r}\varrho$ has a unique decomposition

$$\pi^*_{r+1,r}\varrho = h(\varrho) + p(\varrho). \tag{2.3.15}$$

By (2.2.6), $p(\varrho)$ is a contact form, or a contact odd base form. One can directly deduce the following elementary properties of the mapping $\varrho \to p(\varrho)$.

Theorem 2.5. *Let $W \subset J^rY$ be an open set, $W' = \pi^{-1}_{r+1,r}(W)$. Suppose that either $\varrho, \eta \in \Omega^p(W)$ and $\omega \in \Omega^q(W)$, or $\varrho, \eta \in \hat{\Omega}^p(W)$ and $\omega \in \hat{\Omega}^q(W)$. Then the following conditions hold:*

(a) *$p(\varrho + \eta) = p(\varrho) + p(\eta)$ and $p(\omega \wedge \varrho) = p(\omega) \wedge p(\varrho) + p(\omega) \wedge h(\varrho) + h(\omega) \wedge \wedge\, p(\varrho)$. In particular, if $\omega = f \in \Omega^0(W)$,*

$$p(f \cdot \varrho) = f \cdot p(\varrho). \tag{2.3.16}$$

(b) *For any subset* $V \subset W$,

$$p(\varrho\,|\,_V) = p(\varrho)\,|\,_{V'}. \tag{2.3.17}$$

(c) *For all sections* γ *of* π,

$$J^{r+1}\gamma^* p(\varrho) = 0. \tag{2.3.18}$$

(d) If $p > n$, *then* $p(\varrho) = \pi^*_{r+1,r}\varrho$.

(e) If ϱ is $\pi_{r,r-1}$*-horizontal, then* $p(\varrho)$ *is* $\pi_{r+1,r}$*-projectable.*

(f) $p(\varrho)$ *is* $\pi_{r+1,r}$*-projectable if and only if* $h(\varrho)$ *is* $\pi_{r+1,r}$*-projectable.*

(g) ϱ *is* π_r*-horizontal (resp. contact) if and only if* $p(\varrho) = 0$ *(resp.* $h(\varrho) = 0$*).*

(h) *Let* Y_1, Y_2 *be two fibered manifolds,* $V \subset Y_1$ *an open set,* $\alpha : V \to Y_2$ *a homomorphism of fibered manifolds such that* $\operatorname{pr}\alpha$ *is a diffeomorphism. Let* ϱ *be a form or an odd base form on* Y_2*. Then*

$$p(J^r\alpha^*\varrho) = J^{r+1}\alpha^* p(\varrho). \tag{2.3.19}$$

Remark 3.4. If $f : V_r \to R$ is a function, (2.3.15) gives

$$\pi^*_{r+1,r}df = h(df) + p(df) = d_i f\,.dx^i + \sum_{k=0}^{r} \frac{\partial f}{\partial y^\sigma_{j_1\ldots j_k}}\,\omega^\sigma_{j_1\ldots j_k}. \tag{2.3.20}$$

Decompositions of this kind will be frequently used to simplify various coordinate computations.

Chart expressions. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y. By (2.3.15),

$$\begin{aligned} p(dx^i) &= dx^i - h(dx^i) = 0,\\ p(dy^\sigma_{j_1\ldots j_k}) &= dy^\sigma_{j_1\ldots j_k} - h(dy^\sigma_{j_1\ldots j_k}) = \omega^\sigma_{j_1\ldots j_k}. \end{aligned} \tag{2.3.21}$$

Decomposition (2.3.15) of $\pi^*_{r+1,r}\varrho$, where $\varrho \in \Omega^p(W)$, thus consists in substituting the expressions

$$dy^\sigma_{j_1\ldots j_k} = h(dy^\sigma_{j_1\ldots j_k}) + p(dy^\sigma_{j_1\ldots j_k}) = y^\sigma_{j_1\ldots j_k s}\,dx^s + \omega^\sigma_{j_1\ldots j_k} \tag{2.3.22}$$

in the chart expression of ϱ. In this way we get the expression of $\pi^*_{r+1,r}\varrho$ with respect to the basis of linear forms dx^i, $\omega^\sigma_{j_1\ldots j_k}$, $dy^\sigma_{j_1\ldots j_{r+1}}$ (Theorem 2.3 (a)); the coefficients at $dy^\sigma_{j_1\ldots j_{r+1}}$ will obviously be equal to zero. The chart expression of $p(\varrho)$ is then obtained by subtracting the chart expression of $h(\varrho)$.

If $\varrho \in \hat{\Omega}^p(W)$ is expressed by

$$\varrho = \pi^*_r\hat{\varphi} \otimes \varrho_\varphi, \tag{2.3.23}$$

then

$$\begin{aligned} p(\varrho) &= \pi^*_{r+1,r}\varrho - h(\varrho) =\\ &= \pi^*_{r+1}\hat{\varphi} \otimes \pi^*_{r+1,r}\varrho_\varphi - \pi^*_{r+1}\hat{\varphi} \otimes h(\varrho_\varphi) = \pi^*_{r+1}\hat{\varphi} \otimes p(\varrho_\varphi) \end{aligned} \tag{2.3.24}$$

where $p(\varrho_\varphi)$ can be expressed as above.

We shall now study the modules $\Omega^p_{J^{r-1}Y}(W)$ of $\pi_{r,r-1}$-horizontal forms. The reason for our interest in these modules consists in the fact that the restriction of the π-horizontalization $h : \Omega^p(W) \to \Omega^p(W')$ to $\pi_{r,r-1}$-horizontal forms can be regarded as a mapping from $\Omega^p_{J^{r-1}Y}(W)$ to $\Omega^p_X(W)$ (see Theorem 2.1 (e), and Theorem 2.5 (e)), and in their relatively simple algebraic structure (see Corollary 1 (a) of Theorem 2.4).

Let $p \geqq 1$, and let $\varrho \in \Omega^p_{J^{r-1}Y}(W)$, or $\varrho \in \hat{\Omega}^p_{J^{r-1}Y}(W)$, be a *contact* form. We say that ϱ is 1-*contact* if for each π_r-vertical vector field ξ on W the interior product $i_\xi\varrho$ is a π_r-horizontal form; we say that ϱ is q-*contact*, where $2 \leqq q \leqq p$, if $i_\xi\varrho$ is $(q-1)$-contact. π_r-horizontal forms, and odd base forms, are also called 0-*contact*.

For each pair (p, q), where $0 \leqq q \leqq p$, q-contact p-forms (resp. q-contact odd base p-forms) define a submodule of $\Omega^p_{J^{r-1}Y}(W)$ (resp. $\hat{\Omega}^p_{J^{r-1}Y}(W)$), denoted by $\Omega^{p-q,q}(W)$ (resp. $\hat{\Omega}^{p-q,q}(W)$).

Theorem 2.6. (a) *Let p, q be integers such that $1 \leqq q \leqq p$, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y. A form $\varrho \in \Omega^p_{J^{r-1}Y}(V_r)$ (resp. an odd base form $\varrho \in \hat{\Omega}^p_{J^{r-1}Y}(V_r)$) is q-contact if and only if*

$$\varrho = \sum_{k=0}^{r-1} \sum \omega^\sigma_{j_1 \dots j_k} \wedge \varrho^{j_1 \dots j_k}_\sigma, \tag{2.3.25}$$

where $\varrho^{j_1 \dots j_k}_\sigma$ are $(q-1)$-contact forms (resp. odd base forms).

(b) *Let $W \subset J^rY$ be an open set, $\varrho \in \Omega^p_{J^{r-1}Y}(W)$ (resp. $\varrho \in \hat{\Omega}^p_{J^{r-1}Y}(W)$). ϱ is uniquely expressible in the form*

$$\varrho = \sum_{q=0}^{p} \varrho_q, \tag{2.3.26}$$

where $\varrho_q \in \Omega^{p-q,q}(W)$ (resp. $\varrho_q \in \hat{\Omega}^{p-q,q}(W)$). In other words,

$$\begin{aligned} \Omega^p_{J^{r-1}Y}(W) &= \Omega^{p,0}(W) \oplus \Omega^{p-1,1}(W) \oplus \dots \oplus \Omega^{0,p}(W), \\ \hat{\Omega}^p_{J^{r-1}Y}(W) &= \hat{\Omega}^{p,0}(W) \oplus \hat{\Omega}^{p-1,1}(W) \oplus \dots \oplus \hat{\Omega}^{0,p}(W) \end{aligned} \tag{2.3.27}$$

(the direct sum of submodules).

Proof. (a) Suppose that $\varrho \in \Omega^p_{J^{r-1}Y}(W)$ is expressible in the form (2.3.25), where $\varrho^{j_1 \dots j_k}_\sigma$ are $(q-1)$-contact. Then $h(\varrho) = 0$ and it remains to show that $i_\xi\varrho$ is $(q-1)$-contact for each π_r-vertical vector field ξ on W. Let ξ be such a vector field,

$$\xi = \sum \sum \xi^\sigma_{j_1 \dots j_l} \frac{\partial}{\partial y^\sigma_{j_1 \dots j_l}} \tag{2.3.28}$$

its chart expression with respect to (V, ψ). Then

$$i_\xi\varrho = \sum_{k=0}^{r-1} \sum (\xi^\sigma_{j_1 \dots j_k} \varrho^{j_1 \dots j_k}_\sigma - \omega^\sigma_{j_1 \dots j_k} \wedge i_\xi \varrho^{j_1 \dots j_k}_\sigma). \tag{2.3.29}$$

By hypothesis, the forms $i_\xi \varrho^\sigma_{j_1 \dots j_k}$ are $(q - 2)$-contact, and it is sufficient to show that the form

$$\varrho^{(1)} = \sum_{k=0}^{r-1} \sum \omega^\sigma_{j_1 \dots j_k} \wedge i_\xi \varrho^{j_1 \dots j_k}_\sigma \tag{2.3.30}$$

is $(q - 1)$-contact. Continuing this procedure we see that it is sufficient to show that for any π_r-vertical vector fields $\xi_1, \dots, \xi_{q-1}$ the form

$$\varrho^{(q-1)} = \sum_{k=0}^{r-1} \sum \omega^\sigma_{j_1 \dots j_k} \wedge i_{\xi_{q-1}} \dots i_{\xi_1} \varrho^{j_1 \dots j_k}_\sigma \tag{2.3.31}$$

is 1-contact; this is, however, obviously true.

Conversely, suppose that we have a q-contact form $\varrho \in \Omega^p_{J^{r-1}Y}(W)$. ϱ is expressible as a linear combination of exterior products of the forms dx^i, $\omega^\sigma_{j_1 \dots j_k}$, $0 \leqq k \leqq \leqq r - 1$. We write

$$\varrho = \varrho_0 + \varrho_1 + \dots + \varrho_p, \tag{2.3.32}$$

where ϱ_k contains precisely k factors (2.3.1). For any π_r-vertical vector field ξ on W

$$i_\xi \varrho = i_\xi \varrho_1 + \dots + i_\xi \varrho_p. \tag{2.2.33}$$

By definition, $i_\xi \varrho$ is $(q - 1)$-contact. We first consider the case $q = 1$. In this case $i_\xi \varrho_s = 0$ for all $s = 2, \dots, p$ and all ξ. We want to show that $\varrho_s = 0$ for $s > 1$. It is convenient to introduce multi-indices $K = (^\sigma_{j_1 \dots j_k})$, and to consider the set of these multi-indices with some, for example lexicographical, ordering. Then with the obvious notation

$$\varrho_s = \Sigma f_{i_1 \dots i_{p-s} K_1 \dots K_s} dx^{i_1} \wedge \dots \wedge dx^{i_{p-s}} \wedge \omega^{K_1} \wedge \dots \wedge \omega^{K_s} \tag{2.3.34}$$

summation over increasing sequences $(i_1, \dots, i_{p-s})$ and $(K_1, \dots, K_s)$). Applying the condition $i_\xi \varrho_s = 0$, with $\xi^K = i_\xi \omega^K$, to (2.3.34) one directly gets $f_{i_1 \dots i_{p-s} K_1 \dots K_s} = 0$ as desired. Returning to (2.3.32) we obtain $\varrho = \varrho_1$, and ϱ has the form (2.3.25). We now suppose q to be arbitrary, and proceed by induction.

The same proof applies to the case of an odd base form ϱ.

(b) By (a), a form $\varrho \in \Omega^p_{J^{r-1}Y}(W)$ belongs to $\Omega^{p-q,q}(W)$ if and only if each term in its chart expression with respect to a fiber chart (V, ψ), $\psi = (x^i, y^\sigma)$, contains precisely q-factors (2.3.1). Thus the desired decomposition of ϱ is given by (2.3.32). Invariance of this decomposition has already been proved (Corollary 1 (a) of Theorem 2.4), and its uniqueness follows from Theorem 2.3 (a).

The same applies to the case of an odd base form ϱ.

Decomposition (2.3.26) defines linear mappings of modules $p_i : \Omega^p_{J^{r-1}Y}(J^r Y) \to \to \Omega^{p-q,q}(J^r Y)$ by $p_i(\varrho) = \varrho_i$; evidently $p_0 = h$. For $i \geqq 1$, p_i is called the i-th *contact projection*.

If $\varrho \in \Omega^{p-q,q}(W)$, we say that ϱ has the *order of contact* q; if $\varrho \in \Omega^{p-q,q}(W) \oplus$

$\oplus \ldots \oplus \Omega^{0,p}(W)$ (resp. $\varrho \in \Omega^{p,0}(W) \oplus \ldots \oplus \Omega^{p-q,q}(W)$) we say that ϱ has the *order of contact* $\geqq q$ (resp. $\leqq q$). The *order of contact* of an base form ϱ is defined in the same way.

The following assertion is the Poincaré lemma for contact forms. Its proof is similar to a standard one for (ordinary) forms on a smooth manifold [5].

Theorem 2.7. *Let $U \subset R^n$ (resp. $V \subset R^m$) be an open ball with center $0 \in R^n$ (resp. $0 \in R^m$), $W = U \times V$, and $\tau : W \to U$ the first canonical projection. Let p, k be positive integers, $k \leqq p$, and let $\varrho \in \Omega^{p-k,k}(J^rW)$ (resp. $\varrho \in \hat{\Omega}^{p-k,k}(J^rW)$) be a closed form (resp. a closed odd base form). Then there exists a form $\eta \in \Omega^{p-k,k-1}(J^rW)$ (resp. an odd base form $\eta \in \hat{\Omega}^{p-k,k-1}(J^rW)$) such that $\varrho = d\eta$.*

Proof. Let (x^i, y^σ) be the canonical coordinates on W. We define a mapping $\chi : [0, 1] \times J^rW \to J^rW$ by

$$\chi(t, (x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_r})) = (x^i, ty^\sigma, \ldots, ty^\sigma_{j_1 \ldots j_r}). \tag{2.3.35}$$

We have

$$\begin{aligned} \chi(0, (x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_r})) &= (x^i, 0, \ldots, 0), \\ \chi(1, (x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_r})) &= (x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_r}) \end{aligned} \tag{2.3.36}$$

and

$$\begin{aligned} \chi^* \, dx^i &= dx^i, \\ \chi^* \omega^\sigma_{j_1 \cdots j_q} &= y^\sigma_{j_1 \ldots j_q} \, dt + t\omega^\sigma_{j_1 \ldots j_q}, \qquad 0 \leqq q \leqq r - 1. \end{aligned} \tag{2.3.37}$$

Let $k \geqq 1$, and let $\varrho \in \Omega^{p-k,k}(J^rW)$ be a k-contact form. Then ϱ is uniquely expressible as a linear combination of exterior products of $p - k$ factors dx^i and k factors $\omega^\sigma_{j_1 \ldots j_q}$. Hence

$$\chi^* \varrho = dt \wedge \varrho_0 + t^k \cdot \varrho'_t, \tag{2.3.38}$$

where ϱ_0, ϱ'_t do not contain dt, and $dy^\sigma_{j_1 \ldots j_r}$, ϱ_0 contains $k - 1$ factors of type $\omega^\sigma_{j_1 \ldots j_q}$, and ϱ'_t contains k of these factors. Moreover, by (2.3.36), if $t = 1$, we have

$$\varrho'_1 = \varrho. \tag{2.3.39}$$

In order to study decomposition (2.3.38) in more detail we introduce multi-indices I, J, K as follows. We let I label the coordinates $x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_r}$ on J^rW; these coordinates will be denoted by z_I. J (resp. K) will label the $(p - 1)$-forms (resp. p-forms) defined by all different exterior products of 1-forms $dx^i, dy^\sigma, \ldots, dy^\sigma_{j_1 \ldots j_{r-1}}$; these $(p - 1)$-forms (resp. p-forms) will be denoted by $dz^{(p-1)}_J$ (resp. $dz^{(p)}_K$. Using these forms we can write

$$\varrho_0 = \sum_J a_J \, dz^{(p-1)}_J, \qquad \varrho'_t = \sum_K b_K \, dz^{(p)}_K, \tag{2.3.40}$$

where a_J, b_K are functions on $[0, 1] \times J^rW$.

We set

$$(2.3.41) \qquad A\varrho = \sum_J \int_0^1 a_J \, dt \,.\, dz_J^{(p-1)}.$$

$A\varrho$ is a $(p - 1)$-form on $J^r W$, and the mapping $\varrho \to A\varrho$ is R-linear. Let us consider the form $dA\varrho + Ad\varrho$. We have

$$(2.3.42) \qquad dA\varrho = \sum_{I,J} \left(\int_0^1 \frac{\partial a_J}{\partial z_I} dt \right) . \, dz_I \wedge dz_J^{(p-1)}.$$

On the other hand, by (2.3.38) and (2.3.40)

$$(2.3.43) \qquad \chi^* \, d\varrho = d\chi^* \varrho =$$

$$= -dt \wedge \left(\sum_J da_J \wedge dz_J^{(p-1)} - kt^{k-1} . \sum_K b_K \,.\, dz_K^{(p)} - t^k \sum_K \frac{\partial b_K}{\partial t} \,.\, dz_K^{(p)} \right) + $$
$$+ t^k \sum_K \frac{\partial b_K}{\partial z_I} dz_I \wedge dz_K^{(p)}.$$

Using this expression, (2.3.39), and (2.3.40) we obtain, since $k \geqq 1$,

$$(2.3.44) \qquad A \, d\varrho = \varrho - \sum_{I,J} \left(\int_0^1 \frac{\partial a_J}{\partial z_I} dt \right) . \, dz_I \wedge dz_J^{(p-1)}.$$

Collecting (2.3.42) and (2.3.44) together we get

$$(2.3.45) \qquad dA\varrho + A \, d\varrho = \varrho.$$

But each summand in the form ϱ_0 contains exactly $k - 1$ factors (2.3.1), and $p - k$ factors dx^i. Thus, since the form $A\varrho$ (2.3.41) is defined by means of integration of coefficients in ϱ_0 with respect to the variable t, which does not change the coordinates $y^\sigma_{j_1 \ldots j_q}$ in (2.3.1), $A\varrho$ also contains, in each summand, precisely $k - 1$ factors $\omega^\sigma_{j_1 \ldots j_q}$. Since the factors dx^i remain unchanged by the integration, we conclude that $A\varrho \in \Omega^{p-k, k-1}(J^r W)$.

If now $d\varrho = 0$, we have $\varrho = d\eta$, where $\eta = A\varrho$, and the proof is complete.

If ϱ is an odd base form, we denote by (U, φ) the canonical chart on U, and obtain an (ordinary) form ϱ_φ, defined by $\varrho = \tau_r^* \hat{\varphi} \otimes \varrho_\varphi$. Our assertion now follows from the definition of exterior derivative (Sec. 1.3).

We shall now consider symmetries of the ideal $\Omega_p(W)$ of contact forms.

Theorem 2.8. *Let $W_r \subset J^r Y$ be an open set, $\alpha_r : W_r \to J^r Y$ a homomorphism of fibered manifolds. Suppose that the projection α_0 of α_r is a diffeomorphism of $\pi_r(W_r)$ onto $\alpha_0(\pi_r(W_r))$. Then the following two conditions are equivalent:*

(1) *There exists a homomorphism of fibered manifolds $\alpha : \pi_{r,0}(W_r) \to Y$ whose projection is α_0, such that $\alpha_r = J^r \alpha$.*

(2) *For each contact form ϱ, defined on an open set in $J^r Y$, $\alpha_r^* \varrho$ is a contact form.*

Proof. 1. Suppose that (1) holds. Let ϱ be a contact form, γ a section of Y.

We have $(\alpha_0^{-1})^* J^r\gamma^* \alpha_r^* \varrho = (\alpha_r \circ J^r\gamma \circ \alpha_0^{-1})^* \varrho = (J^r\alpha\gamma\alpha_0^{-1})^* \varrho = 0$ since ϱ is contact (see (2.1.5), and (2.2.6)). As α_0 is a diffeomorphism, this implies $J^r\gamma^*\alpha_r^*\varrho = 0$, and $\alpha_r^*\varrho$ is contact.

2. Suppose that (2) holds. Let γ be a section of Y, ϱ any contact form on an open subset of J^rY. Then $J^r\gamma^*\alpha_r^*\varrho = 0$. Since $\delta = \alpha_r \circ J^r\gamma \circ \alpha_0^{-1}$ is a section of J^rY and δ satisfies (2.3.2), there exists a unique section $\bar{\gamma}$ of Y such that $\delta = J^r\bar{\gamma}$ (Theorem 2.3). This implies $J^r(\pi_{r,0}\delta) = \delta$. Thus for each $J_x^r\gamma$ from the domain of definition of α_r,

$$\alpha_r(J_x^r\gamma) = J_{\alpha_0(x)}^r(\pi_{r,0}\alpha_r \circ J^r\gamma \circ \alpha_0^{-1}). \tag{2.3.46}$$

We shall show that this condition implies that the mapping $\pi_{r,0}\alpha_r$ is constant on the fibers of the projection $\pi_{r,0}$. Let (V, ψ), $\psi = (x^i, y^\sigma)$ (resp. $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$) be a fiber chart on Y. Suppose that $J_x^r\gamma \in V_r \cap W$, $\alpha_r(J_x^r\gamma) \in \bar{V}_r$. In order to express the coordinates of the r-jet $\alpha_r(J_x^r\gamma)$, we should differentiate the functions

$$\begin{aligned} &(\bar{x}^1, \ldots, \bar{x}^n) \to f^\sigma(\bar{x}^1, \ldots, \bar{x}^n) = \\ &= (\bar{y}^\sigma\pi_{r,0}\alpha_r\psi^{-1} \circ \psi J^r\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\bar{\varphi}^{-1})(\bar{x}^1, \ldots, \bar{x}^n), \end{aligned} \tag{2.3.47}$$

where γ is any representative of the r-jet $J_x^r\gamma$. We have

$$\begin{aligned} \frac{\partial \bar{f}^\sigma}{\partial \bar{x}^i} = \Bigg(&\frac{\partial \bar{y}^\sigma\pi_{r,0}\alpha_r\psi^{-1}}{\partial x^k} + \\ &+ \sum_{q=0}^{r}\sum \frac{\partial \bar{y}^\sigma\pi_{r,0}\alpha_r\psi^{-1}}{\partial y^\nu_{p_1\ldots p_q}}(y^\nu_{p_1\ldots p_q k} \circ J^r\gamma)\Bigg) \cdot \frac{\partial x^k\alpha_0^{-1}\bar{\varphi}^{-1}}{\partial \bar{x}^i}. \end{aligned} \tag{2.3.48}$$

Thus we get a linear function in $y^\nu_k, y^\nu_{p_1k}, \ldots, y^\nu_{p_1\ldots p_rk}$. Since the left-hand side does not depend on $y^\nu_{p_1\ldots p_{r+1}}$ (see (2.3.46)), the coefficient at $y^\nu_{p_1\ldots p_{r+1}}$ is equal to zero. Let us consider the coefficient at $y^\nu_{p_1\ldots p_r}$. If it is non-zero, then $\partial^2\bar{f}^\sigma/(\partial\bar{x}^i\,\partial\bar{x}^{i_2})$ depends linearly on $y^\nu_{p_1\ldots p_{r+1}}$, with the same coefficient; thus we get a contradiction because of (2.3.46). Continuing we obtain

$$\frac{\partial \bar{y}^\sigma\pi_{r,0}\alpha_r\psi^{-1}}{\partial y^\nu_{p_1\ldots p_q}} = 0, \qquad 1 \leqq q \leqq r. \tag{2.3.49}$$

Thus the mapping $\pi_{r,0}\alpha_r$ is constant on the fibers of $\pi_{r,0}$ and there exists a mapping $\alpha : \pi_{r,0}(W_r) \to Y$ such that

$$\alpha \circ \pi_{r,0} = \pi_{r,0} \circ \alpha_r. \tag{2.3.50}$$

Since for any section w of J^rY over an open subset of Y, $\alpha = \pi_{r,0} \circ \alpha_r \circ w$, α is smooth. From this representation it follows that α is a homomorphism of fibered manifolds, and α_0 is its projection. Now substituting (2.3.50) in (2.3.46) we get

$$\alpha_r(J_x^r\gamma) = J_{\alpha_0(x)}^r(\alpha\gamma\alpha_0^{-1}) = J^r\alpha(J_x^r\gamma), \tag{2.3.51}$$

and condition (1) holds.

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D. Krupka
Department of Mathematics,
Faculty of Science,
J. E. Purkyně University
Janáčkovo nám. 2a, 662 95 Brno
Czechoslovakia