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CONGRUENCE DISTRIBUTIVITY IN VARIETIES WITH CONSTANTS

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Abstract. An algebra A with a nullary operation c is *c-distributive* if $[c]_\Omega = [c]_\Gamma$ for every three congruences $\Phi, \Psi, \Theta \in Con\ \mathfrak{A}$, where $\Omega = \Phi \wedge (\Theta \vee \Psi)$ and $\Gamma = (\Phi \wedge \Theta) \vee (\Phi \wedge \Psi)$. Varieties of c-distributive algebras can be characterized by a Malcev condition in binary polynomials. Such polynomial condition can be easy applied in varieties of semilattices with constants. In weakly regular varieties, the concepts of distributivity and c-distributivity coincide. It implies that the variety of implication algebras is distributive.

Key words. Congruence distributivity, variety with nullary operation, weak regularity, semilattices implication algebras.

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An algebra $\mathfrak{A}$ is *(congruence) distributive* if

$$\Phi \wedge (\Theta \vee \Psi) = (\Phi \wedge \Theta) \vee (\Phi \wedge \Psi)$$

holds for each three congruences $\Phi, \Psi, \Theta \in Con\ \mathfrak{A}$. A variety $\mathscr{V}$ of algebras is *distributive* if each $\mathfrak{A} \in \mathscr{V}$ has this property. For the sake of brevity, denote by $\Omega = \Phi \wedge (\Theta \vee \Psi)$ and $\Gamma = (\Phi \wedge \Theta) \vee (\Phi \wedge \Psi)$ in the whole paper. The foregoing property can be formulated also in the following way:

an algebra $\mathfrak{A}$ is *distributive* if $[z]_\Omega = [z]_\Gamma$ for each element $z \in \mathfrak{A}$.

This formulation enables us to generalize the congruence distributivity by fixing the element z.

Let $\mathfrak{A}$ be an algebra with a nullary operation c. $\mathfrak{A}$ is *c-distributive* if $[c]_\Omega = [c]_\Gamma$ for each three congruences $\Phi, \Psi, \Theta \in Con\ \mathfrak{A}$. A variety $\mathscr{V}$ having a nullary operation c in its type is *c-distributive* provided each $\mathfrak{A} \in \mathscr{V}$ has this property. This property can be characterized by a Mal'cev condition:

Theorem 1. *Let $\mathscr{V}$ be a variety of algebras with a nullary operation c. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is c-distributive;*

(2) *there exist binary polynomials $d_0(x, y), \dots, d_n(x, y)$ such that $d_0(x, y) = c$,*

$d_n(x, y) = x$ and $d_i(c, y) = c$ for $i = 0, \ldots, n$, $d_i(x, c) = d_{i+1}(x, c)$ for i even and $d_i(x, x) = d_{i+1}(x, x)$ for i odd.

Proof goes along the classical scheme by B. Jónsson [3]:

(1) ⇒ (2): Let $\mathscr{V}$ be a variety with a nullary operation c which is c-distributive. Let $F_2(x, y)$ be a free algebra of $\mathscr{V}$ with two free generators x, y. Clearly

$$\langle c, x\rangle \in \Theta(c, x) \wedge [\Theta(c, y) \vee \Theta(x, y)],$$

thus, by c-distributivity, also

$$\langle c, x\rangle \in [\Theta(c, x) \wedge \Theta(c, y)] \vee [\Theta(c, x) \wedge \Theta(x, y)].$$

Hence, there exist elements $b_0, b_1, \ldots, b_n \in F_2(x, y)$ such that $c = b_0$, $x = b_n$ and

$$(*) \qquad \begin{array}{ll} \langle b_i, b_{i+1}\rangle \in \Theta(c, x) & \text{for } i = 0, \ldots, n-1 \\ \langle b_i, b_{i+1}\rangle \in \Theta(c, y) & \text{for } i \text{ even} \\ \langle b_i, b_{i+1}\rangle \in \Theta(x, y) & \text{for } i \text{ odd.} \end{array}$$

Since $b_i \in F_2(x, y)$, there exist binary polynomials $d_0(x, y), \ldots, d_n(x, y)$ such that $b_i = d_i(x, y)$ and $(*)$ implies immediately

$$\begin{array}{ll} d_i(c, y) = d_{i+1}(c, y) & \text{for } i = 0, \ldots, n-1 \\ d_i(x, c) = d_{i+1}(x, c) & \text{for } i \text{ even} \\ d_i(x, x) = d_{i+1}(x, x) & \text{for } i \text{ odd.} \end{array}$$

(2) ⇒ (1): Let $\mathscr{V}$ be a variety with a nullary operation c satisfying the identities (2). Suppose $\mathfrak{A} \in \mathscr{V}$, $\Phi, \Psi, \Theta \in Con\ \mathfrak{A}$. To prove c-distributivity, it clearly satisfies only to prove that for each $a \in \mathfrak{A}$, the inclusion

$$\langle c, a\rangle \in \Phi \wedge (\Theta \vee \Psi)$$

implies the relation

$$\langle c, a\rangle \in (\Phi \wedge \Theta) \vee (\Phi \wedge \Psi) = \Gamma.$$

Suppose the first relationship holds. Then $\langle c, a\rangle \in \Phi$ and there exist elements $c_0, c_1, \ldots, c_k \in \mathfrak{A}$ such that $c_0 = c$, $c_k = a$ and

$$(**) \qquad \begin{array}{ll} \langle c_j, c_{j+1}\rangle \in \Theta & \text{for } j \text{ even} \\ \langle c_j, c_{j+1}\rangle \in \Psi & \text{for } j \text{ odd.} \end{array}$$

By (2), we have immediately

$$d_i(a, c_j)\, \Phi d_i(c, c_j) = c$$

for each $i = 0, \ldots, n$ and $j = 0, \ldots, k$. By the transitivity of Φ, we obtain

$$d_i(a, c_j)\, \Phi d_i(a, c_{j+1}).$$

Hence and by $(**)$ we have

$$d_i(a, c_0)\, (\Phi \wedge \Theta)\, d_i(a, c_1)\, (\Phi \wedge \Psi)\, d_i(a, c_2) \ldots d_i(a, c_k),$$

thus

$$d_i(a, c)\, \Gamma\, d_i(a, a) \qquad \text{for } i = 0, \ldots, n.$$

By (2) it implies

$$c = d_0(a, c) = d_1(a, c)\, \Gamma\, d_1(a, a) = d_2(a, a)\, \Gamma\, d_2(a, c) = d_3(a, c)\, \Gamma \ldots = a,$$

i.e.

$$\langle c, a \rangle \in \Gamma.$$

Example 1. Evidently, each distributive variety $\mathscr{V}$ with a nullary operation c is c-distributive. Formal polynomials $d_i(x, y)$ can be constructed from Jónssons' ternary polynomials $t_0, \ldots, t_n$ by the formula:

$$d_i(x, y) = t_{n-i}(x, y, c).$$

Then $d_0(x, y) = t_n(x, y, c) = c$, $d_n(x, y) = t_0(x, y, c) = x$ and

$$\begin{aligned} d_i(c, y) &= t_{n-i}(c, y, c) = t_{n-i-1}(c, y, c) = d_{i+1}(c, y) && \text{for } i = 0, \ldots, n-1 \\ d_i(x, c) &= t_{n-i}(x, c, c) = t_{n-i-1}(x, c, c) = d_{i+1}(x, c) && \text{for } i \text{ even} \\ d_i(x, x) &= t_{n-i}(x, x, c) = t_{n-i-1}(x, x, c) = d_{i+1}(x, x) && \text{for } i \text{ odd.} \end{aligned}$$

The following examples show that there exist c-distributive varieties which are not distributive:

Example 2. *A variety $\mathscr{V}$ of idempotent groupoids with zero* 0 *is* 0-*distributive.* We can put $n = 2$, $d_0(x, y) = 0$, $d_1(x, y) = x \cdot y$, $d_2(x, y) = x$. Then

$$\begin{aligned} &d_0(0, y) = 0, \qquad d_1(0, y) = 0 \cdot y = 0, \qquad d_2(0, y) = 0 \\ &d_0(x, 0) = 0 = x \cdot 0 = d_1(x, 0) \qquad (i \text{ even}) \\ &d_1(x, x) = x \cdot x = x = d_2(x, x) \qquad (i \text{ odd}). \end{aligned}$$

Example 3. *The variety of all join* (*meet*) *semilattices with* 1 (*with* 0) *is* 1-*distributive* (0-*distributive*).

It follows directly from Example 2.

In the next part we show how the concept of c-distributivity can be applied in weakly regular varieties.

An algebra $\mathfrak{A}$ with a nullary operation c is *weakly regular* (see [1], [2], [4]) if $[c]\Theta = [c]\Phi$ implies $\Theta = \Phi$ for each two congruences $\Theta, \Phi \in Con\ \mathfrak{A}$. A variety $\mathscr{V}$ with a nullary operation c is *weakly regular* if each $\mathfrak{A} \in \mathscr{V}$ has this property. Clearly, every regular variety with a nullary operation is also weakly regular.

Lemma. *A variety $\mathscr{V}$ with nullary operation c is weakly regular if and only if there exist binary polynomials $q_1, \ldots, q_n$ such that*

$$[q_1(x, y) = c \text{ and } \ldots \text{ and } q_n(x, y) = c] \Leftrightarrow x = y.$$

For the proof, see e.g. Theorem B in [5].

Theorem 2. *Let $\mathcal{V}$ be a variety with nullary operation c which is weakly regular. $\mathcal{V}$ is distributive if and only if $\mathcal{V}$ satisfies the condition* (2) *of Theorem* 1.

The proof follows immediately from Theorem 1 and the definitions of c-distributivity and weak regularity.

Remark. Theorem 2 enables us to characterize the congruence distributivity in weakly regular varieties by Malcev condition using binary polynomials only.

An implication algebra (introduced by J. C. Abbott) is an algebra with one binary and one nullary operation, denoted by . and 1, satisfying the identities

$$(a \,.\, b) \,.\, a = a$$
$$(a \,.\, b) \,.\, b = (b \,.\, a) \,.\, a$$
$$a \,.\, (b \,.\, c) = b \,.\, (a \,.\, c)$$
$$a \,.\, a = 1.$$

Corollary. *A variety $\mathcal{V}$ of implicative algebras is distributive.*

Proof. Put $n = 2$, $q_1(x, y) = x \,.\, y$, $q_2(x, y) = y \,.\, x$. It is well known that

$$(x \,.\, y = 1 \text{ and } y \,.\, x = 1) \text{ if and only if } x = y,$$

thus, by the Lemma, $\mathcal{V}$ is weakly regular. Moreover, every implicative algebra is a join semilattice with the greatest element 1 with respect to the induced order

$$x \leqq y \text{ if ond only if } x \,.\, y = 1.$$

By Example 3, $\mathcal{V}$ is 1-distributive and, by Theorem 2, $\mathcal{V}$ is distributive.

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