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ON ASSOCIATIVE DEVELOPABLE SURFACES

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Abstract. Under some regularity assumptions all developable surfaces of associative binary operation on the positive real line are found.

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MS Classification. 39 B 40.

Our chief concern in this paper is to find, under some regularity assumptions, which associative operations $z = F(x, y)$ on the positive real line have developable surfaces.

Let F be a two-place function from $[0, \infty) \times [0, \infty)$ into $[0, \infty)$ satisfying the following conditions for all x, y and z in $[0, \infty)$:

(i) $F(x, 0) = x$;
(ii) F is reducible;
(iii) $F(x, y) = F(y, x)$;
(iv) $F(x, F(y, z)) = F(F(x, y), z)$;
(v) F is continuous.

This class of topological semigroups have been characterized in [1] where the following representation is showed: there exist a continuous strictly increasing function f from $[0, \infty)$ into $[0, \infty)$ with $f(0) = 0$ such that

$$F(x, y) = f^{-1}(f(x) + f(y)), \tag{1}$$

for all x, y in $[0, \infty)$. The function f is called an additive generator of F and it is unique up to a multiplicative constant. Our aim is to find functions of type (1) which have developable surfaces, i.e., where the Gauss curvature vanishes.

Theorem. *Let F be a binary operation on $[0, \infty)$ representable in the form (1) where the additive generator f is such that f' and f'' exist and are continuous functions on $(0, \infty)$ with $f''(x) \neq 0$ for all $x > 0$. Then the surface $z = F(x, y)$ is developable if and only if there exists a positive constant K such that*

$$F(x, y) = (x^K + y^K)^{1/K}. \tag{2}$$

Proof. It is immediate to show that the function given by (2) satisfies all the conditions required in the theorem and has a developable surface. Conversely, let us assume that F as given by (1) has a developable surface. In view of the conditions of differentiability assumed on the additive generator f it follows that F has continuous partial derivatives of order 2 which are given by the following expressions

$$\frac{\partial^2 F(x, y)}{\partial x^2} - \frac{f''(x)}{f\,(F(x, y))} - \frac{f'(x)^2 f''(F(x, y))}{f\,(F(x, y))^3}, \tag{2}$$

$$\frac{\partial^2 F(x, y)}{\partial y^2} = \frac{f''(y)}{f\,(F(x, y))} - \frac{f'(y)^2 f''(F(x, y))}{f'(F(x, y))^3}, \tag{3}$$

$$\frac{\partial^2 F(, y)}{\partial x\, \partial y} = \frac{-f'(x)\, f'(y)\, f''(F(x, y))}{f\,'(F(x, y))^3}. \tag{4}$$

If F have a developable surface then its Gauss curvature must be zero, i.e.

$$\frac{\partial^2 F(x, y)}{\partial x^2} \cdot \frac{\partial^2 F(x, y)}{\partial y^2} - \left(\frac{\partial^2 F(x, y)}{\partial x\ \partial y}\right)^2 = 0, \tag{5}$$

for all x, y. Substitution of (2), (3) and (4) in (5) yields, after appropiate simplifications:

$$f''(x) f''(y)\, f'(F(x, y))^2 - f''(x) f'(y)^2 f''(F(x, y)) - f''(y) f'(x)^2 f''(F(x, y)) = 0$$

or, equivalently,

$$\frac{f'(F(x, y))^2}{f''(F(x, y))} = \frac{f'(y)^2}{f''(y)} + \frac{f'(x)^2}{f''(x)}. \tag{6}$$

Define the function g from $(0, \infty)$ into $\boldsymbol{R}$ by

$$g(z) = \frac{f'(f^{-1}(z))^2}{f''(f^{-1}(z))}.$$

Since f^{-1}, f' and f'' are assumed to be continuous and f'' does not vanish in any point of $(0, \infty)$ the function g is continuous and, moreover, using (6) we can show the following:

$$g(x) + g(y) = \frac{f'(f^{-1}(x))^2}{f''(f^{-1}(x))} + \frac{f'(f^{-1}(y))^2}{f''(f^{-1}(y))} =$$

$$= \frac{f'(F(f^{-1}(x), f^{-1}(y)))^2}{f''(F(f^{-1}(x), f^{-1}(y)))} = \frac{f'(f^{-1}(x + y))^2}{f''(f^{-1}(x + y))} = g(x + y),$$

i.e. g is a continuous solution of Cauchy functional equation (see [1]) and consequently g must be of the form

$$g(z) = c \,.\, z,$$

where c is an arbitrary constant. Thus

$$\frac{f'(f^{-1}(z))^2}{f''(f^{-1}(z))} = c \,.\, z,$$

and this yields the ordinary differential equation

$$f'(x)^2 = c \,.\, f''(x) \,.\, f(x). \tag{7}$$

Obviously $c \neq 0$ because f cannot be constant. If $c = 1$ the general solution of (7) is given by $f(x) = e^{Ax+B}$, which cannot satisfy the requirement $f(0) = 0$, or $f(x) \equiv 0$. Thus in the case $c \neq 0, 1$ we obtain that the solution of (7) must verify

$$f(x)^{\frac{c-1}{c}} = \frac{c-1}{c}(Ax + B).$$

Since $f(0) = 0$ we can conclude that necessarily $B = 0$ and $\frac{c-1}{c} > 0$. Whence $K = \frac{c}{c-1} > 0$ and $f(x) = \left(\frac{A}{K}x\right)^K$,

i.e.,

$$F(x, y) = f^{-1}(f(x) + f(y)) = (x^K + y^K)^{1/K}.$$

The theorem is proved.

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