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Archivum Mathematicum, Vol. 21 (1985), No. 3, 147--157

Persistent URL: `http://dml.cz/dmlcz/107227`

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 21, No. 3 (1985), 147–158

MONOTONICITY PROPERTIES OF THE LINEAR COMBINATION OF DERIVATIVES OF SOME SPECIAL FUNCTIONS

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(Received February 29, 1984)

Abstract. The principal concern here is with monotonicity properties of the zeros and related quantities of the linear combination $\alpha y^{(i)} + \beta y^{(i+1)}$, $i = 0, 1, \ldots$, where α, β are real numbers and $y = y^{(0)}$ is a solution of

$$y'' + a(t)\, y' + b(t)\, y = 0.$$

In particular, the results are formulated for the functions $\alpha Ai(-t) + \beta Ai'(-t)$, $\alpha C_\nu(t) + \beta C'_\nu(t)$ and $\alpha C'_\nu(t) + \beta C''_\nu(t)$, where $Ai(-t)$ and $C_\nu(t)$ denote Airy and Bessel functions, respectively.

Key words. Monotonicity properties – "Bocher-function"-Airy function – Bessel functio n.

1. Introduction

In [4] J. Vosmanský derived certain higher monotonicity properties of i-th derivatives of solutions of

$$y'' + a(t)\, y' + b(t)\, y = 0 \tag{1}$$

in the oscillatoric case. In [2] using the first accompanying equation there are extended results from [4] to the function

$$\alpha y^{(i)} + \beta\left(y^{(i+1)} + \frac{1}{2} a_i(t)\, y^{(i)}\right) \qquad i = 0, 1, \ldots$$

where $y(t)$ is a solution of (1) and functions $a_i(t)$ are defined by the same formulae as $A_i(t)$ below. The used method does not allow to formulate results for the linear combination $\alpha y^{(i)} + \beta y^{(i+1)}$, as there was deduced in [1] for the equation

$$y'' + f(t)\, y = 0 \tag{2}$$

in the case $i = 0$.

The aim of this paper is to investigate monotonicity properties of the zeros of the linear combination $\alpha y^{(i)} + \beta y^{(i+1)}$, where $y = y^{(0)}$ is a solution of (1), and to apply obtained results on Airy and Bessel functions.

Let $a(t)$, $b(t) \in C^\infty(0, \infty)$. The transformation

$$u(t) = y(t) \exp\left[-\frac{1}{2}\int a(t)\,\mathrm{d}t\right]$$

transforms (1) in (2), where

$$f(t) = b(t) - \frac{1}{2}a'(t) - \frac{1}{4}a^2(t). \tag{3}$$

In [3] it is proved that if y is a solution of (1) then the „Bocher-function“ $z = \alpha y + \beta y'$ is a solution of

$$z'' + \left(a + \beta\frac{\alpha a' - \beta b'}{\alpha^2 + \beta^2 b - \alpha\beta a}\right)z' + \left(b + \beta\frac{\alpha b' + \beta(a'b - ab')}{\alpha^2 + \beta^2 b - \alpha\beta a}\right)z = 0, \tag{4}$$

where α, β are real numbers such that $\alpha^2 + \beta^2 > 0$ and $a = a(t)$, $b = b(t)$ are coefficients of (1).

Let us denote

$$K = K(t) = \alpha^2 + \beta^2 b - \alpha\beta a, \tag{5}$$

$$A = A(t) = a + \beta\frac{\alpha a' - \beta b'}{\alpha^2 + \beta^2 b - \alpha\beta a} = a - \frac{K'}{K}, \tag{6}$$

$$B = B(t) = b + \beta\frac{\alpha b' + \beta(a'b - ab')}{\alpha^2 + \beta^2 b - \alpha\beta a} = b + \alpha\beta\frac{b'}{K} + \frac{\beta^2(a'b - ab')}{K}. \tag{7}$$

Analogously as in [4] let $A_0 = A(t)$, $B_0 = B(t)$ and define reccurently for $i = 1, 2, \ldots$ functions $A_i(t)$, $B_i(t) \neq 0$ by formulae

$$\begin{aligned} A_i &= A_{i-1} - B'_{i-1}/B_i, \\ B_i &= B_{i-1} + A'_{i-1} - A_{i-1}B'_{i-1}/B_{i-1} \end{aligned} \tag{8$_i$}$$

and for $i = 0, 1, \ldots$ functions F_i by

$$F_i = B_i - \frac{1}{2}A'_i - \frac{1}{4}A_i^2. \tag{9$_i$}$$

Since the function $F = F_0(t)$ defined by $(9)_0$ plays an important role in our study it is useful to express $F(t)$ using coefficients of (1). By routine computation we get

$$F = f - \frac{3}{4}\left[\frac{K'}{K}\right]^2 + \frac{1}{2}\frac{K''}{K} + \frac{1}{2}a\frac{K'}{K} + \alpha\beta\frac{b'}{K} + \frac{\beta^2(a'b - ab')}{K}, \tag{10}$$

where $f(t)$ and $K(t)$ are defined by (3) and (5), respectively.

We shall study sequences $\{R_k^{(i)}\}_{k=0}^{\infty}$, where $R_k^{(i)}$ is defined for fixed $\lambda > -1$,

$$R_k^{(i)} = R_k^{(i)}(W, \lambda) = \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} W(t) \exp\left\{\frac{\lambda}{2}\int_c^t A_i(\tau)\,\mathrm{d}\tau\right\} |\alpha y^{(i)}(t) + \beta y^{(i+1)}(t)|^\lambda \,\mathrm{d}t, \tag{11$_i$}$$

where $y = y^{(0)}(t)$ is an arbitrary (non-trivial) solution of (1), $\{t_k^{(i)}\}$ denotes any sequence of consecutive zeros of the function $\alpha z^{(i)} + \beta z^{(i+1)}$ ($i = 0, 1, \ldots$), where z is any solution of (1) which may or may not be linearly independent of y; α, β are real numbers such that $\alpha^2 + \beta^2 > 0$ and $W(t)$ sufficiently monotonic function. By special choice of $W(t)$, λ, i and $z(t)$ we can obtain $R_k^{(i)}$ having different meaning.

A function $f(t)$ is said to be monotonic of order n over $(0, \infty)$ if

$$(12) \qquad (-1)^k f^{(k)}(t) \geqq 0 \qquad k = 0, 1, \ldots, n, \; t \in (0, \infty)$$

and we write $f \in M_n$. If (12) holds for $n = \infty$, $f(t)$ is called completely monotonic ($f(t) \in M_\infty$). A sequence $\{t_k\}$ is said to be monotonic of order n if

$$(13) \qquad (-1)^j \Delta^j t_k \geqq 0 \qquad k = 0, 1, \ldots; j = 0, 1, \ldots, n,$$

where $\Delta^0 t_k = t_k$, $\Delta^n t_k = \Delta^{n-1} t_{k+1} - \Delta^{n-1} t_k$ and we denote $\{t_k\} \in M_n$. If (13) holds for $n = \infty$, $\{t_k\}$ is called completely monotonic ($\{t_k\} \in M_\infty$). If strict inequality holds throughout (12) or (13) then we write $f \in M_n^*$ or $\{t_k\} \in M_n^*$, respectively.

2. Preliminaries

Lemma 1. *If $y = y^{(0)}$ is a solution of (1) then $z = \alpha y^{(i)} + \beta y^{(i+1)}$, $i = 0, 1, \ldots$, is a solution of*

$$(14)_i \qquad z'' + A_i(t)\, z' + B_i(t)\, z = 0,$$

where A_i, B_i are defined by $(8)_i$.

Proof. For $i = 0$ Lemma holds according to [3]. Let $i \geqq 1$. Using [4, Lemma 2.1] we get that if $z = z(t)$ is a solution of (4), i.e. of $(14)_0$, then $z = z^{(i)}(t)$ is a solution of $(14)_i$. From this and from the linearity of the derivation a conclusion follows.

Lemma 2. [4, p. 96] *Let $A(t) \in M_{n+2}^*$, $B'(t) \in M_{n+2}^*$, $B(\infty) - A^2(\infty)/4 = \delta > 0$, $B(t) > 0$. Then for $F_1(t)$ defined by $(9)_1$ it holds*

$$F_1' \in M_n^*, \qquad F_1(\infty) = \delta > 0.$$

Lemma 3. *Let $i > 0$, $k < 0$, $x_1 > 0$, $x_2 < 0$ be real numbers such that*

$$(15) \qquad i + j + k > 0,$$

$$(16) \qquad -jx_1 \geqq kx_2 \quad \text{if} \quad j < 0.$$

Let $\varphi(t)$ be for $t > x_1$ defined by

$$(17) \qquad \varphi(t) := \frac{i}{t} + \frac{j}{t - x_1} + \frac{k}{t - x_2}.$$

If $j \geqslant 0$ then $\varphi(t) \in M^*_\infty(x_1, \infty)$.

If $j < 0$ then $\varphi(t) \in M^*_n(\tau_n, \infty)$, where τ_n denotes the unique zero of the equation

$$(18)_n \quad G_n(t) \equiv i + j\left(\frac{t}{t - x_1}\right)^{n+1} + k\left(\frac{t}{t - x_2}\right)^{n+1} = 0 \qquad t \in (x_1, \infty), n = 0, 1, \ldots$$

Proof. The n-th ($n = 0, 1, \ldots$) derivative of $\varphi(t)$ has the form

$$\varphi^{(n)}(t) = (-1)^n n! [it^{-(n+1)} + j(t - x_1)^{-(n+1)} + k(t - x_2)^{-(n+1)}].$$

It is evidently seen that $\varphi^{(n)}(t) \in C(x_1, \infty)$ and therefore $\varphi^{(n)}$ changes the sign only in the zeros of the equation $\varphi^{(n)}(t) = 0$.

The function $G_n(t)$ defined by $(18)_n$ has the following properties:

$$\lim_{t \to \infty} G_n(t) = i + j + k > 0, \quad G_n'(t) = (n + 1)\, t^n[-jx_1(t - x_1)^{-n-2} - kx_2(t - x_2)^{-n-2}],$$

$$G_n'(t) < 0 \quad \text{and} \quad \lim_{t \to x_1+} G_n(t) = +\infty \quad \text{if} \quad j > 0, t \in (x_1, \infty),$$

$$G_n'(t) > 0 \quad \text{and} \quad \lim_{t \to x_1+} G_n(t) = -\infty \quad \text{if} \quad j < 0, t \in (x_1, \infty).$$

Thus, if $j > 0$ then $G_n(t) > 0$ for $t > x_1$ and therefore $(-1)^n \varphi^{(n)}(t) > 0$ ($n = 0, 1, \ldots$). Let $j < 0$. Then $(18)_n$ has the unique zero τ_n in (x_1, ∞). The rest of the proof is the same as [5, proof of Lemma 2.3].

Corollary 1. Let $\alpha\beta > 0$, $v \geqq 0$ be real numbers and let $P(t)$, $A(t)$ be defined by

$$P(t) := (\alpha^2 + \beta^2)\, t^2 - \alpha\beta t - \beta^2 v^2,$$

$$(19) \qquad A(t) := \frac{1}{t} - \frac{\alpha\beta}{P(t)} - \frac{2\beta^2 v^2}{tP(t)} \qquad \text{for } t > x_1,$$

where

$$(20) \quad x_{1,2} = [\alpha\beta \pm \sqrt{\alpha^2\beta^2 + 4(\alpha^2 + \beta^2)\, \beta^2 v^2}]/2(\alpha^2 + \beta^2), \qquad x_1 > x_2.$$

If $v^2 \geqslant 3/4\beta^2(\alpha^2 + \beta^2)$ then $A(t) \in M^*_\infty(x_1, \infty)$.

If $v^2 < 3/4\beta^2(\alpha^2 + \beta^2)$ then $A(t) \in M^*_n(\tau_n, \infty)$, where τ_n denotes the unique zero of $(18)_n$ with

$$(21) \qquad i = 1 + 2(\alpha^2 + \beta^2), \qquad j = -\frac{\alpha\beta}{x_1 - x_2} + \frac{2\beta^2 v^2}{x_1(x_1 - x_2)},$$

$$k = \frac{\alpha\beta}{x_1 - x_2} + \frac{2\beta^2 v^2}{x_2(x_1 - x_2)}.$$

Proof. The function $A(t)$ can be expressed in the form (17), where i, j, k are defined by (21). It holds $i > 0$, $x_1 > 0$, $x_2 < 0$, $x_1 - x_2 > 0$, $x_1 + x_2 > 0$. By routine computation we get $k < 0$; $j \geqslant 0 \Leftrightarrow v^2 \geqslant 3/4\beta^2(\alpha^2 + \beta^2)$. From the fact $x_1 x_2 = -\beta^2 v^2/(\alpha^2 + \beta^2)$ we have the validity of (15). If $j < 0$ then (16) holds because $2\beta^2 v^2(x_1 + x_2)/x_1 x_2(x_1 - x_2) < 0 < \alpha\beta$. Thus, the conclusion follows directly from Lemma 3.

Lemma 4. *Let $i > 0$, $j < 0$, $k > 0$ be real numbers and let $\psi(t)$ be for $t > 0$ defined by*

$$\psi(t) := \frac{2i}{t^3} + \frac{3j}{t^4} + \frac{4k}{t^5}.$$

Then

$$\psi(t) \in M_n^* \quad on \quad (0, \infty) \quad for \quad 3(n + 3) j^2 < 8(n + 4)\, ik,$$
$$\psi(t) \in M_n^* \quad on \quad (-(n + 3) j/2i, \infty) \quad otherwise.$$

Proof. The n-th derivative of $\psi(t)$ has the form

$$\psi^{(n)}(t) = (-1)^n \frac{1}{t^{5+n}} Q_n(t), \qquad Q_n(t) = i(n + 2)!\, t^2 + j\frac{(n + 3)!}{2} t + k\frac{(n + 4)!}{6}.$$

The function Q_n is positive for $t \in R$ if $3j^2(n + 3) < 8ik(n + 4)$ and in the opposite case is surely positive for $t > -(n + 3)j/2i > x_3$, where x_3 is the root of Q_n, i.e.

$$x_3 = -(n + 3)j/4i + [j^2(n + 3)^2/4 - 2i(n + 3)(n + 4)/3]^{1/2}.$$

Since $Q_k(t) > 0$ on $(-(n + 3)j/2i, \infty)$ for $k = 0, 1, \ldots, n$ the proof is complete.

Corollary 2. Let $\alpha\beta > 0$, $v \geqq 0$ be real numbers and $\omega = 8\beta^2/9\alpha^2 - v^2$. Let $h(t)$ be defined by

$$h(t) := \frac{\beta^2}{t^2} - \frac{2\alpha\beta v^2}{t^3} + \frac{v^2\beta^2}{t^4}. \tag{22}$$

If $\omega > 0$ then $h(t) \in M_\infty^*$ for $t > 0$.

If $\omega \leqq 0$ then $h(t) \in M_n^*$ for $t > (n + 2)\,\alpha v^2/\beta$, $n = 0, 1, \ldots$

Proof. Let us put $i = \beta^2$, $j = -2\alpha\beta v^2$, $k = v^2\beta^2$ in Lemma 4. Let $\omega > 0$. Then it holds $h > 0$ for $t > 0$ and using Lemma 4 $-h' \in M_\infty^*$ for $t > 0$, i.e. $h \in M_\infty^*$ for $t > 0$. Now, let $\omega \leqq 0$. Then we have $h > 0$ for $t > 2\alpha v^2/\beta$ and $-h' \in M_n$ for $t > (n + 3)\,\alpha v^2/\beta$ $n = 0, 1, \ldots$, q.e.d.

3. Statement of principal results

3.1. General theorems. Using Lemma 1 and [4, Theorem 3.5] we have

Theorem 1. *Let $i \geqq 0$ be arbitrary fixed integer and $W(t) \in M_n$, $W(t) > 0$. For the function $F_i(t)$ defined by $(9)_i$ suppose*

$$F_i' \in M_n, \qquad F_i' > 0 \qquad for\ t \in (0, \infty), F_i(\infty) > 0.$$

Then it holds

$$\{R_k^{(i)}\}_{k=0}^{\infty} \in M_n^*,$$

and, in particular

$$\{\Delta t_k^{(i)}\}_{k=0}^{\infty} \in M_n^*,$$

consequently, the sequence of the differences of succesive zeros of a function $\alpha y^{(i)} + \beta y^{(i+1)}$, *where* $y(t)$ *is any solution of* (1), *is monotonic of order n.*

If, in addition, $W(t)$ *is non-constant function, then the hypothesis* $F_i' > 0$ *may be omitted. If* $W(t) \in M_n$ *and the hypothesis* $F_i' > 0$ *is omitted then it holds* $\{R_k^{(i)}\} \in M_n$.

Theorem 2. *Suppose in* (1)

$$a(t) \equiv 0,\ b'(t) \in M_\infty,\ b > 0,\ b' > 0 \quad \text{on} \quad (0, \infty).$$

Let $W(t) \in M_\infty(0, \infty)$ *and let* R_k *be defined by* $(11)_0$.

If $\alpha\beta \leqq 0$ *then* $\{R_k\}_{k=0}^{\infty} \in M_\infty^*$.

If $\alpha\beta > 0$ *suppose, in addition, for some* $p \geqq 0$

$$b^{(n+1)} = O(t^{-(n+p)}) \quad b^{(n+1)} \neq O(t^{-(n+p+1)}) \ \text{as}\ t \to \infty. \tag{23}$$

Then there exists $e = e(n) \in N$ *such that* $\{R_k\}_{k=e(n)}^{\infty} \in M_n^*$.

Remark 1. In the case $a(t) \equiv 0$ we can $(11)_0$ rewrite as

$$R_k = \int_{t_k}^{t_{k+1}} W(t) \exp\left\{\frac{\lambda}{2} \int \frac{-\beta^2 b'}{\alpha^2 + \beta^2 b}\right\} |\alpha y + \beta y'|^\lambda \,dt = \int_{t_k}^{t_{k+1}} W(t) \left|\frac{\alpha y + \beta y'}{\sqrt{\alpha^2 + \beta^2 b}}\right|^\lambda dt. \tag{11'}$$

Remark 2. Supposing $\alpha\beta < 0$, $W(t) \equiv 1$ in (11)′ we obtain some results of [1].

3.2. Application for Airy functions

Consider

$$y'' + ct^\mu y = 0 \tag{24}$$

with $t > 0$, where $c > 0$ and $\mu \in (0, 1]$ are parameters. When $c = \mu = 1$, (24) is reduced to the equation

$$y'' + ty = 0,$$

which is satisfied by the linearly independent Airy functions $Ai(-t)$ and $Bi(-t)$ of first and second kind, respectively. Using Theorem 2 we obtain the following result for generalized Airy functions.

Theorem 3. *Let* $\mu \in (0,1]$ *and let* $y(t)$ *be any non-trivial solution of* (24). *Then for* R_k *defined by* (11)′ *it holds*

$$\{R_k\}_{k=0}^{\infty} \in M_{\infty}^* \qquad \text{if} \qquad \alpha\beta \leqq 0,$$

$$\{R_k\}_{k=e(n)}^{\infty} \in M_n^* \qquad \text{if} \qquad \alpha\beta > 0,\ n = 0, 1, \ldots$$

where $e = e(n, \alpha, \beta, c, \mu)$ is sufficiently great integer, i.e. if $c = 1$, $\alpha/\beta = 1$, $n = 0$ it is $e = 2$.

In particular the conclusion holds for the sequence of zeros of the function $\alpha y + \beta y'$.

3.3. Application for Bessel functions

By a Bessel function of order ν we mean any nontrivial solution $C_\nu(t)$ of the Bessel equation

$$(25)_\nu \qquad y'' + \frac{1}{t} y' + \left(1 - \frac{\nu^2}{t^2}\right) y = 0 \qquad t \in (0, \infty).$$

Let us define for $t > \nu$ and $\lambda > -1$

$$(26)_\nu \qquad R_{\nu k} = \int_{d_{\nu k}}^{d_{\nu k+1}} W(t) \frac{t^{3\lambda/2}}{[(\alpha^2 + \beta^2) t^2 - \alpha\beta t - \beta^2 \nu^2]^{\lambda/2}} |\alpha C_\nu + \beta C_\nu'|^\lambda \, dt,$$

$$(27)_\nu \qquad R'_{\nu k} = \int_{d'_{\nu k}}^{d'_{\nu k+1}} W(t) \exp \left| \frac{\lambda}{2} \int_c^t A_1(\tau) \, d\tau \right| |\alpha C_\nu' + \beta C_\nu''|^\lambda \, dt,$$

where $\{d_{\nu k}\}$ and $\{d'_{\nu k}\}$ is a sequence of zeros of the function $\alpha C_\nu + \beta C_\nu'$ and $\alpha C_\nu' + \beta C_\nu''$ respectively and $A_1 = A - B'/B$.

From [5, Theorem 1], [6, Remark 9.1] it follows that the sequence of differences of zeros of C_ν' is completely monotonic for every ν but the sequence of differences of zeros of C_ν is completely monotonic only for $\nu > 1/2$. It is interesting to compare this fact with following theorems.

Theorem 4. *Let $\alpha\beta > 0$, $\nu > 1/2$ be arbitrary numbers. Let $W(t) \in M_n$ and $W(t) > 0$ for $t > \nu$, let $R_{\nu k}$ be defined by $(26)_\nu$.*

Let $m = m(n) := \max(\nu, \alpha\nu^2(n + 2)/\beta)$. Let p and $e = e(n)$ be the smallest integer satisfying $d_{\nu p} \geqq \nu$ and $d_{\nu, e(n)} \geqq m(n)$, respectively. Then

$$\{R_{\nu k}\}_{k=p}^{\infty} \in M_{\infty}^* \qquad \text{if} \qquad \nu^2 < 2\beta^2/3\alpha^2,$$

$$\{R_{\nu k}\}_{k=e(n)}^{\infty} \in M_n^*, \qquad n = 0, 1, \ldots \text{ otherwise,}$$

In particular the conclusion holds for the sequence of differences of zeros of any function $\alpha C_\nu + \beta C_\nu'$.

Theorem 5. *Let $\alpha\beta > 0$, $\nu \geqq 0$ be arbitrary numbers. Let $W(t) \in M_n$ and $W(t) > 0$ for $t > \nu$, let $R'_{\nu k}$ be defined by $(27)_\nu$.*

Let τ_n denote the unique zero of $(18)_n$; where i, j, k and x_1 are defined by (21) and (20), respectively. Let $\gamma = \gamma(n) := \max\{x_1, \tau_n, \nu, \alpha\nu^2(n + 3)/\beta\}$. Let p and $q = q(n)$ be the smallest integer satisfying $d'_{\nu p} \geqq \nu$ and $d'_{\nu, q(n)} > \gamma(n)$, respectively.

Then

$$\{R'_{\nu k}\}_{k=p}^{\infty} \in M_{\infty}^{*} \quad \text{if} \quad \frac{3}{4\beta^2(\alpha^2+\beta^2)} \le \nu^2 < \frac{2\beta^2}{3\alpha^2},$$

$$\{R'_{\nu k}\}_{k=q(n)}^{\infty} \in M_{n-2}^{*} \quad (n = 2, 3, \ldots) \text{ otherwise.}$$

In particular the conclusion holds for the sequence of differences of successive zeros of any function $\alpha C'_\nu + \beta C''_\nu$.

4. Proof of Theorems 2, 3, 4, 5

Lemma 5. *Let* $f \in M_\infty$ *in* $(0 < t < \infty)$. *Then* $f^{(k)} = 0(t^{-k})$ *as* $t \to \infty$, $k = 0, 1, \ldots$

Proof. It is similar to [7, proof of Theorem 14a], where we suppose $f \in M_\infty$ in $(0 \leqq t < \infty)$.

Since $f \in M_\infty$ in $(0 < t < \infty)$ it is $f \in M_\infty$ in $(\delta \leqq t < \infty)$, $\delta > 0$. Then from [7, Theorem 3a, pp. 146] $f(t)$ is analytic for $t > \delta$. For any number $a > \delta$

$$f(t) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(t-a)^k}{k!} \qquad (\delta < t < 2a - \delta).$$

Since each term of the series is positive when $t < a$ we have

$$f^{(k)}(a)\frac{(t-a)^k}{k!} \leqq f(t) \leqq f(\delta) \qquad (\delta < t < a)$$

Allowing t to approach δ this becomes

$$f^{(k)}(a)\frac{(\delta-a)^k}{k!} \leqq f(\delta) \qquad (\delta < a < \infty).$$

Hence

$$f^{(k)}(t) = 0\,((t-\delta)^{-k}) = 0\,(t^{-k}) \quad (t \to \infty, k = 0, 1, \ldots).$$

Proof of Theorem 2. According to (10) we get

$$F' = b' - \frac{3}{4}\left[\left(\frac{K'}{K}\right)^2\right]' + \frac{1}{2}\left(\frac{K''}{K}\right)' + \alpha\beta\left(\frac{b'}{K}\right)',$$

where $K = \alpha^2 + \beta^2 b$. It holds $K > 0$, $K' \in M_\infty$. Using [e.g. 4, Lemma 2.3] we have $1/K \in M_\infty$, $b'/K \in M_\infty$, $(K''/K)' \in M_\infty$ on $(0, \infty)$.

1. Let $\alpha\beta \leqq 0$. Then $\alpha\beta(b'/K)' \in M_\infty$ and thus $F' \in M_\infty$ on $(0, \infty)$. Since $b' > 0$ we have $F' > 0$.

2. Let $\alpha\beta > 0$. From l'Hopital rule we get for $i = 0, 1, \ldots, n$ $b^{(i+1)} = 0(t^{-(i+p)})$, $b^{(i+1)} \neq 0(t^{-(i+p+1)})$ as $t \to \infty$.

By Lemma 5 we have $(1/K)^{(i)} = 0(t^{-i})$, $i = 0, 1, \ldots$ and thus

$$(b'/K)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} b^{(j+1)}(1/K)^{(i-j)} = 0(t^{-(j+p)} \cdot t^{-(i-j)}) = 0(t^{-(i+p)}).$$

Hence there exists $\sigma_i = \sigma(i) > 0$ such that

$$(-1)^i t^{i+p+1}(b^{(i+1)} + \alpha\beta(b'/K)^{(i+1)}) > 0 \qquad \text{for } t > \sigma_i,\ i = 0, \ldots, n.$$

It holds $\sigma_{i+1} > \sigma_i$ and thus $b' + \alpha\beta(b'/K)' \in M_n^*$ for $t > \sigma_n$. Together we have $F' \in M_n^*$ for $t > \sigma_n$.

3. It remains to prove $F(\infty) > 0$. It holds $K'(\infty) = \beta^2 b'(\infty)$. If $b'(\infty) = 0$ then $K''(\infty) = 0$ and $F(\infty) = b(\infty) > 0$. If $b'(\infty) = c > 0$ then it holds $c < \infty$, $K(\infty) = \infty$, $K''(\infty) = 0$. Therefore $F(\infty) = b(\infty) > 0$.

Now, the conclusion follows from Theorem 1 for $i = 0$.

Proof of Theorem 3. In the case of the equation (24) there are $a(t) \equiv 0$, $b(t) = ct^{\mu}$ and thus $b' \in M_\infty$, $b > 0$, $b' > 0$ for $t > 0$, $\mu \in (0, 1]$.

1. Let $\alpha\beta \leqq 0$ and $\mu \in (0, 1]$. Then conclusion follows directly from Theorem 2.

2. Let $\alpha\beta > 0$ and $\mu \in (0, 1)$. Then (23) is fulfilled for $p = 1 - \mu$ and by Theorem 2 we have $\{R_{\mu k}\}_{k=e(n)}^{\infty} \in M_n^*$. Let us compute $e = e(0)$. By routine computation we get

$$b' + \alpha\beta\left(\frac{b'}{K}\right)' = b' - \alpha\beta\mu c \frac{\beta^2 c + \alpha^2(1-\mu)t^{-\mu}}{(\beta^2 ct + \alpha^2 t^{1-\mu})^2} > c\mu t^{\mu-1} - \frac{2\alpha\mu}{\beta t^2} > 0$$

for $t > T$, where $T = \max\left\{\left(\frac{2\alpha}{\beta c}\right)^{1/(\mu+1)}, \left(\frac{\alpha^2(1-\mu)}{\beta^2 c}\right)^{1/\mu}\right\}$. If $c = 1$, $\alpha/\beta = 1$ then $e(0) = 2$.

3. It remains to prove the limit case $\mu = 1$ for $\alpha\beta > 0$. The functions $A(t)$, $B(t)$ in $(14)_0$ are

$$A_\mu(t) = A(t) = -\mu\beta^2 t^{\mu-1}/(\alpha^2 + \beta^2 t^{\mu})$$

$$B_\mu(t) = B(t) = t^{\mu} + \alpha\beta\mu t^{\mu-1}/(\alpha^2 + \beta^2 t^{\mu}), \qquad \mu \in (0, 1].$$

Since $A_\mu(t) \to A_1(t)$, $B_\mu(t) \to B_1(t)$ uniformly on $[\delta, \infty)$, $\delta > 0$ as $\mu \to 1_-$ we have $\alpha y_\mu(t) + \beta y'_\mu(t) \to \alpha y_1(t) + \beta y'_1(t)$ uniformly on compact subintervals of $[\delta, \infty)$. It follows $t_{\mu k} \to t_{1k}$ as $\mu \to 1_-$, $k = 0, 1, \ldots$, where $t_{\mu k}$ denotes k-th zero points of $\alpha y_\mu + \beta y'_\mu$ and y_μ is a solution of (24). From this we obtain for $k = 0, 1, \ldots$

$$\lim_{\mu \to 1-} R_{\mu k} = \lim_{\mu \to 1-} \int_{t_{\mu k}}^{t_{\mu k+1}} W(t) \left| \frac{\alpha y_\mu + \beta y'_\mu}{\sqrt{\alpha^2 + \beta^2 ct^{\mu}}} \right|^{\lambda} dt = \int_{t_{1k}}^{t_{1,k+1}} W(t) \left| \frac{\alpha y_1 + \beta y'_1}{\sqrt{\alpha^2 + \beta^2 ct}} \right|^{\lambda} dt = R_{1k}.$$

Finally, because $\{R_{\mu k}\}_{k=e(n)}^{\infty} \in M_n^*$, we have for $k = e(n), e(n) + 1, \ldots$

$$0 \leqq \lim_{\mu \to 1-} (-1)^n \Delta^n R_{\mu k} = (-1)^n \Delta^n \lim_{\mu \to 1-} R_{\mu k} = (-1)^n \Delta^n R_{1k}, \qquad \text{q.e.d.}$$

Remark 3. In the limit case $\mu = 1$ the function F defined by (10) is

$$F = t + \alpha\beta/(\alpha^2 + \beta^2 t) - 3\beta^2/4(\alpha^2 + \beta^2 t)^2.$$

Thus, if $\alpha\beta > 0$ there exists $t_n = t(n)$ such that $F > 0$, $F' > 0$, $F'' \in M_n$ for $t > t_n$, $n = 2, 3, \ldots$ It shows „strength" of the sufficient condition in Theorem 1.

Proof of Theorem 4. In the case of the equation $(25)_\nu$ there is $a(t) = 1/t$, $b(t) = 1 - \nu^2/t^2$. It holds $a \in M_\infty$, $b' \in M_\infty$, $b > 0$ for $t > \nu$. Let us denote $\omega = 8\beta^2/9\alpha^2 - \nu^2$. In the proof we use Theorem 1 for $i = 0$. We have by (3), (5)

$$K = \alpha^2 + \beta^2(1 - \nu^2/t^2) - \alpha\beta/t, \qquad K' = \alpha\beta/t^2 + 2\nu^2\beta^2/t^3,$$
$$K'' = -2\alpha\beta/t^3 - 6\nu^2\beta^2/t^4, \qquad f = 1 - (\nu^2 - 1/4)/t^2, \qquad f' = 2(\nu^2 - 1/4)/t^3.$$

It holds $f' \in M_\infty^*$ for $\nu > \frac{1}{2}$, $K' \in M_\infty$ on (ν, ∞) and $K > 0$ on (ν, ∞) because K increases and $K(\nu) = \alpha^2 + \alpha\beta/\nu \geqq 0$. Let us define the functions $G(t)$, $H(t)$ as

$$(28) \qquad \begin{aligned} G(t) &= (K'' + aK')/2K \\ H(t) &= \alpha\beta b'/K + \beta^2(a'b - ab')/K. \end{aligned}$$

Then,

$$G(t) = \frac{1}{2K}\left(-\frac{\alpha\beta}{t^3} - \frac{4\nu^2\beta^2}{t^4}\right), \qquad H(t) = \frac{1}{K}\left(-\frac{\beta^2}{t^2} + \frac{2\alpha\beta\nu^2}{t^3} - \frac{\nu^2\beta^2}{t^4}\right),$$

$$H' = -\left(\frac{1}{K}\right)' h + \frac{1}{K} h', \qquad \text{where } h(t) \text{ is defined by (22).}$$

According to [4, Lemma 2.3] we have

$$1/K \in M_\infty, \qquad -(1/K)' \in M_\infty, \qquad -\frac{3}{4}[(K'/K)^2]' \in M_\infty, \qquad G' \in M_\infty \text{ for } t > \nu.$$

Using Corollary 2 we get $H' \in M_\infty^*$ for $t > \nu$ and $H' \in M_n^*$ for $t > \alpha\nu^2(n + 2)/\beta$, if $\omega > 0$ and $\omega \leqq 0$, respectively.

Since we can write the derivation of F defined by (10) as

$$F' = f' - \frac{3}{4}\left[\left(\frac{K'}{K}\right)^2\right]' + G' + H',$$

we obtain $F' \in M_\infty^*$ and $F' \in M_n^*$ for $t > \nu$ and $t > \alpha\nu^2(n + 2)/\beta$, if $\omega > 0$ and $\omega \leqq 0$, respectively.

It is easy to verify $F(\infty) = f(\infty) = 1$.

The proof is complete.

Proof of Theorem 5. According to Theorem 1 and Lemma 2 it suffices to prove $A \in M_n^*$, $B' \in M_n^*$, $B > 0$ for $t > \gamma(n)$ and $B(\infty) - A^2(\infty)/4 = \delta > 0$.

It holds $B' = b' + H'$, where $H(t)$ is defined by (28). Evidently $b' \in M_\infty^*$ for $t > \nu$ and by the same way as in the proof of Theorem 4 we prove $H' \in M_\infty^*$ for $t > \nu$ and $H' \in M_n^*$ for $t > \alpha\nu^2(n + 2)/\beta$ if $\omega > 0$ and $\omega \leqq 0$, respectively. Together

$B' \in M_\infty^*$ for $t > v$ and $B' \in M_n^*$ for $t > \alpha v^2(n + 2)/\beta$ if $\omega > 0$ and $\omega \leqq 0$, respectively.

In the case of the equation $(25)_v$ the function $A(t)$ has the form (19). From Corollary 1 it follows $A \in M_\infty^*$ for $t > x_1$ and $A \in M_n^*$ for $t > \tau_n$ if $v^2 \geqslant 3/4\beta^2 \times \times (\alpha^2 + \beta^2)$ and $v^2 < 3/4\beta^2(\alpha^2 + \beta^2)$, respectively.

It is easy to see that $B(\infty) - A^2(\infty)/4 = 1$. The proof is complete.

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