Ondřej Došlý

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THE BASIC PROPERTIES OF PHASE MATRICES OF LINEAR DIFFERENTIAL SYSTEMS

ONDŘEJ DOŠLÝ, Brno
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Abstract. The basic properties of the phase matrices of selfadjoint linear differential systems of the second order are studied. It is shown that these matrices have many properties similar to those of phase functions of scalar differential equations.

Key words. Selfadjoint differential systems, phase matrix, isotropic solution, trigonometric matrices.

1. Introduction

We consider a linear differential system of the second order

(1.1) $$y'' + P(x)\,y = 0,$$

where $y(x)$ is an n-dimensional vector and $P(x)$ is a continuous symmetric $n \times n$ matrix. Simultaneously with the system (1.1) we consider the associated matrix system

(1.2 $$Y'' + P(x)\,Y = 0.$$

The following notation is used: $C^m(I)$ denotes the space of functions which are m-times continuously differentiable on an interval I, $C^0(I)$ means continuity. If $F(x)$ is an arbitrary matrix, we write $F(x) \in C^m(I)$ if all elements of $F(x)$ belong to $C^m(I)$. F^T denotes the transpose of the matrix F. If F is symmetric (i.e. $F^T = F$), $F > 0$ means that F is positive definite. Furthermore, if $F > 0$, $F^{1/2}$ denotes the (unique) symmetric positive definite matrix for which $F^{1/2}F^{1/2} = F$. E denotes the unit matrix of arbitrary dimension. Troughout the paper the system (1.1) is investigated on some interval I of arbitrary kind (open or closed, bounded or unbounded)

Recall that if $Y_1(x)$, $Y_2(x)$ are two solutions of (1.2) then $Y_1^{T\prime}(x)\,Y_2(x) - Y_1^T(x)\,Y_2'(x) = K$, where K is a constant $n \times n$ matrix. A solution $Y(x)$ of (1.2) is said to be isotropic if $Y^T(x)\,Y'(x) - Y^{T\prime}(x)\,Y(x) = 0$.

Now let us consider the linear Hamiltonian system

$$S' = Q(x)\,C, \qquad C' = -Q(x)\,S, \tag{1.3}$$

where $Q(x) \in C^0(I)$. If for some $x_0 \in I$ $S(x_0) = A$, $C(x_0) = B$ and the matrices A, B satisfy

$$A^T A + B^T B = E, \qquad A^T B - B^T A = 0, \tag{1.4}$$

then

$$S^T(x)\,S(x) + C^T(x)\,C(x) = E, \qquad S^T(x)\,C(x) - C^T(x)\,S(x) = 0. \tag{1.5}$$

The matrices $S(x)$, $C(x)$ which are the solution of (1.3) satisfying (1.5) are called the matrix sines and cosines, see [1].

Let $U(x)$, $V(x)$ be two isotropic solutions of (1.2) for which

$$U^{T\prime}(x)\,V(x) - U^T(x)\,V'(x) = E. \tag{1.6}$$

It was proved in [4] that there exist a symmetric $n \times n$ matrix $A(x) \in C^3(I)$, $A'(x) > 0$ on I and a nonsingular $n \times n$ matrix $R(x)$ satisfying

$$(R^T(x)\,R(x))^{-1} = A'(x), \qquad R^{T\prime}(x)\,R(x) - R^T(x)\,R'(x) = 0 \tag{1.7}$$

such that $U(x)$ and $V(x)$ can be expressed in the form

$$U(x) = R(x)\,S(x), \qquad V(x) = R(x)\,C(x), \tag{1.8}$$

where $(S(x), C(x))$ is a solution of (1.3) with $Q(x) = A'(x)$ for which (1.5) holds. The matrix $A(x)$ was called the phase matrix of (1.1) since it has similar properties as the phase function of the scalar differential equation $y'' + p(x)\,y = 0$, see [2]. Further it was proved that the matrix $R(x)$ and the matrix $P(x)$ from (1.1) are connected by the relation

$$P(x) = -R''(x)\,R^{-1}(x) + (R(x)\,R^T(x))^{-2}. \tag{1.9}$$

The aim of the present paper is to prove some further properties of the phase matrices of (1.1), particularly, to describe the class of the phase matrices of (1.1).

The solutions of all investigated differential equations and systems are considered in the classical sense.

2. Auxiliary results

To simplify our further calculation we prove first several auxiliary statements. Let $U(x)$, $V(x)$ be two isotropic solutions of (1.2). These solutions are said to be independent if every solution $Y(x)$ of (1.2) can be expressed in the form $Y(x) =$

$= U(x)\, C_1 + V(x)\, C_2$, where C_1, C_2 are constant $n\times n$ matrices. It can be proved that $U(x)$ and $V(x)$ are independent if and only if the constant matrix $U^T(x)\, V'(x) - U^{T\prime}(x)\, V(x)$ is nonsingular, see [5].

Now we prove some properties of square matrices.

Lemma 1. *Let K, L, M, N be constant $n\times n$ matrices for which*

$$\begin{aligned} K^TL - L^TK &= 0,\\ M^TN - N^TM &= 0,\\ K^TN - L^TM &= E. \end{aligned} \tag{2.1}$$

Then

$$\begin{aligned} KM^T - MK^T &= 0,\\ LN^T - NL^T &= 0,\\ NK^T - LM^T &= E. \end{aligned} \tag{2.2}$$

Proof. From (2.1) it follows:

$$\begin{pmatrix} K^T & L^T \\ M^T & N^T \end{pmatrix}\begin{pmatrix} N & -L \\ -M & K \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},$$

hence

$$\begin{pmatrix} N & -L \\ -M & K \end{pmatrix}\begin{pmatrix} K^T & L^T \\ M^T & N^T \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

and thus $NK^T - LM^T = E$, $NL^T - LN^T = 0$, $MK^T - KM^T = 0$.

Lemma 2. *Let $(S(x), C(x))$ be a solution of (1.3) for which (1.5) holds. Then in all points where the matrix $C(x)$ is nonsingular the matrix $C^{-1}(x)\, S(x)$ is symmetric and*

$$(C^{-1}(x)\, S(x))' = C^{-1}(x)\, Q(x)\, C^{T-1}(x) \tag{2.3}$$

Proof. From (1.5) it follows

$$\begin{pmatrix} S^T(x) & -C^T(x) \\ C^T(x) & S^T(x) \end{pmatrix}\begin{pmatrix} S(x) & C(x) \\ -C(x) & S(x) \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

and by the same argument as in the proof of Lemma 1 we obtain

$$\begin{aligned} S(x)\, C^T(x) - C(x)\, S^T(x) &= 0,\\ S(x)\, S^T(x) + C(x)\, C^T(x) &= E. \end{aligned} \tag{2.4}$$

From the first relation of (2.4) it follows $C^{-1}(x)\, S(x) = S^T(x)\, C^{T-1}(x)$, hence $C^{-1}(x)\, S(x)$ is symmetric. The second relation of (1.5) gives $S(x)\, C^{-1}(x) = C^{T-1}(x)\, S^T(x)$. Using this equality we have $(C^{-1}S)' = -C^{-1}C'C^{-1}S + C^{-1}S' = C^{-1}QSC^{-1}S + C^{-1}QC = C^{-1}Q(SC^{-1}S + C) = C^{-1}Q(C^{T-1}S^TS + C^{T-1}C^TC) = C^{-1}QC^{T-1}$.

Now let $S(x)$ and $C(x)$ be the matrix sines and cosines. In all points where the matrix $C(x)$ is nonsingular let us define the matrix $T(x) = C^{-1}(x) S(x)$. This matrix will be called the matrix tangents. If it is necessary to emphasize that $(S(x), C(x))$ is a solution of (1.3) with a matrix $Q(x)$, we shall write $(S(x, Q), C(x, Q)$ and also $T(x, Q)$. To unify the matrix notation with the scalar notation it would perhaps be more suitable to denote these matrices $S(x, A)$, $C(x, A)$, $T(x, A)$, where $A(x) = \int^x Q(s)\, ds$, since the functions $s(x) = \sin \alpha(x)$, $c(x) = \cos \alpha(x)$ are the solution of $s' = \alpha'(x)\, c$, $c' = -\alpha'(x)\, s$. But we shall not change the matrix notation introduced by Barrett and many times used by Etgen, Reid and others.

Remark 1. Some authors (e.g. Reid [6]) define the matrix tangents as the symmetric solution of the Riccati matrix-type equation $T' = TQ(x)\, T + Q(x)$, i.e. $T(x) = S(x)\, C^{-1}(x)$. However, for our purpose it is useful to define $T(x) = C^{-1}(x)\, S(x)$.

Remark 2. Let $U(x)$, $V(x)$ be two isotropic solutions of (1.2) for which $U^{T\prime}(x)\, V(x) - U^T(x)\, V'(x) = K$, where K is a symmetric positive definite or negative definite constant $n \times n$ matrix. In [4] it was shown that these solutions can be also expressed in the form (1.8) but the first relation of (1.5) must be replaced by $S^T(x)\, S(x) + C^T(x)\, C(x) = K$. In this case we must consider the matrix $T(x) = C^{-1}(x)\, S(x)\, K^{-1}$.

Finally, let $P(x) \in C^0(I)$ be a symmetric $n \times n$ matrix and let $M[P(x)]$ denote the class of all differential systems of the second order $y'' + F(x)\, y = 0$ with $F(x) = G^T P(x)\, G$, where G is a constant orthonormal (i.e. $G^{-1} = G^T$) $n \times n$ matrix. In [4] it was proved that if $A(x)$ is a phase matrix of the differential system $y'' + P(x)\, y = 0$, then $A(x)$ is also the phase matrix of $y'' + G^T P(x)\, G = 0$, where G is a constant orthonormal $n \times n$ matrix. Hence instead of saying "$A(x)$ is a phase matrix of $y'' + P(x)\, y = 0$" we shall sometimes say "$A(x)$ is a phase matrix of $M[P(x)]$".

3. The structure of the class of phase matrices

It is known that two scalar functions $\alpha(x)$, $\beta(x) \in C^3(I)$, $\alpha'(x) \neq 0$, $\beta'(x) \neq 0$, are the phase functions of the scalar differential equation of the second order

$$y'' + p(x)\, y = 0 \tag{3.1}$$

if and only if in all points where the following expression is defined we have

$$\operatorname{tg} \beta(x) = (c \operatorname{tg} \alpha(x) + d)^{-1} (a \operatorname{tg} \alpha(x) + b), \tag{3.2}$$

where a, b, c, d are real constants and $ad - bc \neq 0$, see [2]. For the phase matrices we have the following statement.

Theorem 1. *Matrices $A_i(x) \in C^3(I)$, $A_i'(x) > 0$, $i = 1, 2$, are the phase matrices of $M[P(x)]$ if and only if there exist constant $n \times n$ matrices K, L, M, N satisfying (2.1) such that in all points where the following expression is defined we have*

$$T(x, A_2') = (T(x, A_1')\, M + N)^{-1}\, (T(x, A_1')\, K + L). \tag{3.3}$$

Proof. To simplify our further calculation let us denote $C_i(x) = C(x, A_i')$, $S_i(x) = S(x, A_i')$, $T_i(x) = T(x, A_i')$, $Q_i(x) = A_i'(x)$, $i = 1, 2$, and let $R_i(x)$ be the nonsingular matrices for which

$$\begin{aligned} &(R_i^T(x)\, R_i(x))^{-1} = Q_i(x), \\ &R_i^{T'}(x)\, R_i(x) - R_i^T(x)\, R_i'(x) = 0, \qquad i = 1, 2. \end{aligned} \tag{3.4}$$

First, let us suppose that $A_1(x)$ and $A_2(x)$ are the phase matrices of $M[P(x)]$. Then

$$\begin{aligned} Y_1(x) &= R_1(x)\, C_1(x), \\ Y_2(x) &= R_2(x)\, C_2(x) \end{aligned} \tag{3.5}$$

are solutions of (1.2). In all points where the matrices $Y_1(x)$ and $Y_2(x)$ are nonsingular

$$Z_i(x) = Y_i(x) \int_a^x (Y_i^T(s)\, Y_i(s))^{-1}\, ds, \qquad i = 1, 2, \tag{3.6}$$

are also isotropic solutions of (1.2), see [3], and

$$Y_i^T(x)\, Z_i'(x) - Y_i^{T'}(x)\, Z_i(x) = E. \tag{3.7}$$

Thus $Y_1(x)$, $Z_1(x)$ are independent solutions of (1.2) and hence $Y_2(x)$, $Z_2(x)$ can be expressed in the form a „linear combination“ of $Y_1(x)$ and $Z_1(x)$

$$\begin{aligned} Y_2(x) &= Y_1(x)\, N + Z_1(x)\, M, \\ Z_2(x) &= Y_1(x)\, L + Z_1(x)\, K, \end{aligned} \tag{3.8}$$

where K, L, M, N are constant $n \times n$ matrices. Since $Y_2(x)$ is isotropic, the following must hold:

$$\begin{aligned} Y_2^T Y_2' - Y_2^{T'} Y_2 &= (N^T Y_1^T + M^T Z_1^T)(Y_1' N + Z_1' M) - (N^T Y^{T'} + M^T Z_1^{T'})(Y_1 N + Z_1 M) = \\ &= N^T (Y_1^T Y_1' - Y_1^{T'} Y_1)\, N + M^T (Z_1^T Z_1' - Z_1^{T'} Z_1)\, M + N^T (Y_1^T Z_1' - Y_1^{T'} Z_1)\, M + \\ &\quad + M^T (Z_1^T Y_1' - Z_1^{T'} Y_1)\, N = N^T M - M^T N = 0. \end{aligned}$$

Similarly we prove that $K^T L - L^T K = 0$ and $K^T N - L^T M = E$, since $Z_2(x)$ is isotropic and (3.7) holds for $i = 2$.

Now the substitution (3.5) and (3.6) in (3.8) yields

$$R_2(x)\, C_2(x) = R_1(x)\, C_1(x)\, (N + \int_a^x C_1^{-1}(s)\, Q_1(s)\, C_1^{T-1}(s)\, ds M), \tag{3.9}$$

$$R_2(x)\,C_2(x)\int_a^x C_2^{-1}(s)\,Q_2(s)\,C_2^{T-1}(s)\,\mathrm{d}s =$$
$$= R_1(x)\,C_1(x)\,(L + \int_a^x C_1^{-1}(s)\,Q_1(s)\,C_1^{T-1}(s)\,\mathrm{d}sK).$$

In all points where both sides of the first relation of (3.9) are nonsingular we have

$$C_2^{-1}(x)\,R_2^{-1}(x) = (N + \int_a^x C_1^{-1}(s)\,Q_1(s)\,C_1^{T-1}(s)\,\mathrm{d}sM)^{-1}\,C_1^{-1}(x)\,R_1^{-1}(x). \tag{3.10}$$

Multiplying both sides of the second relation of (3.9) on the left by the corresponding sides of (3.10) we obtain

$$\int_a^x C_2^{-1}Q_2C_2^{T-1}\,\mathrm{d}s =$$
$$= (Y_1(x)\,(N + \int_a^x (Y_1^TY_1)^{-1}\,\mathrm{d}sM))^{-1}\,(Y_1(x)\,(L + \int_a^x (Y_1^TY_1)^{-1}\,\mathrm{d}sK)) =$$
$$= (N + \int_a^x C_1^{-1}Q_1C_1^{T-1}\,\mathrm{d}sM)^{-1}\,(L + \int_a^x C_1^{-1}Q_1C_1^{T-1}\,\mathrm{d}sK)$$

and from Lemma 2 we have

$$T_2(x) = (T_1(x)\,M + N)^{-1}\,(T_1(x)\,K + L).$$

Now, conversely, let us suppose that $A_1(x)$ is a phase matrix of $M[P(x)]$ and (3.3) holds, where K, L, M, N satisfy (2.1). We wish to prove that $A_2(x)$ is also a phase matrix of $M[P(x)]$. Differentiation of both sides of (3.3) gives

$$T_2'(x) = -(T_1(x)\,M + N)^{-1}\,T_1'(x)\,M(T_1(x)\,M + N)^{-1}\,(T_1(x)\,K + L) +$$
$$+ (T_1(x)\,M + N)^{-1}\,T_1'(x)\,K,$$

hence

$$T_2'(x) = (T_1(x)\,M + N)^{-1}\,T_1'(x)\,(-M(T_1(x)\,M + N)^{-1}\,(T_1(x)\,K + L) + K) =$$
$$= (T_1(x)\,M + N)^{-1}\,T_1'(x)\,(-M(K^TT_1(x) + L^T)\,(M^TT_1(x) + N^T)^{-1} + K) =$$
$$= (T_1(x)\,M + N)^{-1}\,T_2'(x)\,(-MK^TT_1(x) - ML^T + KM^TT_1(x) + KN^T)\times$$
$$\times (M^TT_1(x) + N^T)^{-1} = (T_1(x)\,M + N)^{-1}\,T_1'(x)\,(T_1(x)\,M + N)^{T-1},$$

where the symmetry of $T_i(x)$, $i = 1, 2$, and (2.2) have been used. By Lemma 2 we have

$$C_2^{-1}(x)\,Q_2(x)\,C_2^{T-1}(x) =$$
$$= (C_1^{-1}(x)\,S_1(x)\,M + N)^{-1}C_1^{-1}(x)\,Q_1(x)\,C_1^{T-1}(x)\,(C_1^{-1}(x)\,S_1(x)\,M + N)^{T-1} =$$
$$= (S_1(x)\,M + C_1(x)\,N)^{-1}\,Q_1(x)\,(S_1(x)\,M + C_1(x)\,N)^{T-1}$$

and hence

$$C_2^T(x)\,Q_2^{-1}(x)\,C_2(x) = (S_1(x)\,M + C_1(x)\,N)^T\,Q_1^{-1}(x)\,(S_1(x)\,M + C_1(x)\,N).$$

The last equality and the fact that $Q_i^{-1}(x) = R_i^T(x)\,R_i(x)$, $i = 1, 2$, give

$$G_2(x)\,R_2(x)\,C_2(x) = G_1(x)\,R_1(x)\,(S_1(x)\,M + C_1(x)\,N), \tag{3.11}$$

where $G_1(x)$ and $G_2(x)$ are certain orthonormal matrices. Let us denote $G(x) = G_2^{-1}(x)\,G_1(x)$, $U(x) = R_2(x)\,C_2(x)$, $V(x) = R_1(x)\,(S_1(x)\,M + C_1(x)\,N)$. Then (3.11) can be rewritten as

$$U(x) = G(x)\,V(x) \tag{3.12}$$

and a simple calculation gives

$$\begin{aligned} U^{T\prime}(x)\,U(x) - U^T(x)\,U'(x) &= 0,\\ V^{T\prime}(x)\,V(x) - V^T(x)\,V'(x) &= 0. \end{aligned} \tag{3.13}$$

Substituting (3.12) in the first expression of (3.13) we obtain $U^{T\prime}U - U^TU' = V^TG^T(G'V + GV') - (V^{T\prime}G + V^TG^{T\prime})\,GV = V^TV' - V^{T\prime}V + V^T(G^TG' - G^{T\prime}G)\,V = V^T(G^TG' - G^{T\prime}G)\,V = 0$, hence $G^TG' - G^{T\prime}G = 0$. As $G(x)$ is orthonormal, we have $G^TG' + G^{T\prime}G = 0$. From the last equalities it follows $G^T(x)\,G'(x) = 0$, hence $G'(x) = 0$ and thus $G(x) = G_0$, where G_0 is a constant orthonormal $n \times n$ matrix.

As $A_1(x)$ is a phase matrix of $M[P(x)]$, $R_1(x)\,S_1(x)$ and $R_1(x)\,C_1(x)$ are independent solutions of $Y'' + G^TP(x)\,GY = 0$, where G is some constant orthonormal matrix. Hence $R_1(x)\,S_1(x)\,M + R_1(x)\,C_1(x)\,N$ is also a solution of this system and in all points where the matrix $(S_1(x)\,M + C_1(x)\,N)$ is nonsingular we have

$$\begin{aligned} G^TP(x)\,G &= -(R_1(x)\,(S_1(x)\,M + C_1(x)\,N))''(R_1(x)\,(S_1(x)\,M + C_1(x)\,N))^{-1} =\\ &= -(G_0^TR_2(x)\,C_2(x))''\,(G_0^TR_2(x)\,C_2(x))^{-1}. \end{aligned}$$

To simplify the further calculation let us denote $H = GG_0^T$, $S(x) = S_2(x)$, $R(x) = R_2(x)$ and $C(x) = C_2(x)$. The matrix H is orthonormal and $H^TPH = -(RC)''(RC)^{-1} = -(R''C + 2R'C' + RC'')\,C^{-1}R^{-1} = -R''R^{-1} - RC''C^{-1}R^{-1} + 2R'(R^TR)^{-1}SC^{-1}R^{-1} = -R''R^{-1} + 2R'R^{-1}R^{T-1}SC^{-1}R^{-1} - R(-Q_2'S - Q_2^2C)\,C^{-1}R^{-1} = -R''R^{-1} + 2R'R^{-1}R^{T-1}SC^{-1}R^{-1} - R(R^TR)^{-1}(R^{T\prime}R + R^TR')\,(R^TR)^{-1}SC^{-1}R^{-1} + R^{T-1}(R^TR)^{-1}R^{-1} = -R''R^{-1} + 2R'R^{-1}R^{T-1} \times SC^{-1}R^{-1} + R^{T-1}(R^TR)^{-1}R^{-1} - 2R^{T-1}R^TR'R^{-1}R^{T-1}SC^{-1}R^{-1} = -R''R^{-1} + (RR^T)^{-2}$. Thus $H^TP(x)\,H = -R_2''(x)\,R_2^{-1}(x) + (R_2(x)\,R_2^T(x))^{-2}$ and according to (1.9) $A_2(x)$ is also a phase matrix of $M[P(x)]$. The proof is complete.

Now let us define a certain relation on the class of all phase matrices. Recall that this class is identical with the class of all $n \times n$ matrix functions $A(x) \in C^3(I)$ for which $A'(x) > 0$.

Definition. Let $A_i(x) \in C^3(I)$, $A_i(x) > 0$, $i = 1, 2$. We write $A_1(x) \cong A_2(x)$ if there exist constant $n \times n$ matrices K, L, M, N satisfying (2.1) such that (3.3) holds.

Theorem 2. *The relation $\cong$ is an equivalence on the class of all phase matrices.*

Proof. i) The reflexivity of $\cong$ is obvious by choosing $L = M = 0$ and $K = N = E$.

ii) The symmetry: From (3.3) we have

$$T(x, A_1') = (-NT(x, A_2') + L)(MT(x, A_2') - K)^{-1}$$

The symmetry of $T(x, A_i')$, $i = 1, 2$, gives

$$T(x, A_1') = (T(x, A_2') M^T - K^T)^{-1}(-T(x, A_2') N^T + L)$$

and (2.1) is satisfied by Lemma 1.

iii) The transitivity: For simplification let us denote $T_i = T(x, A_i')$, $i = 1, 2, 3$, and let

$$T_2 = (T_1 M_1 + N_1)^{-1} (T_1 K_1 + L_1),$$
$$T_3 = (T_2 M_2 + N_2)^{-1} (T_2 K_2 + L_2),$$

where K_i, L_i, M_i, N_i, $i = 1, 2$, satisfy (2.1). Then

$$\begin{aligned} T_3 = {} & ((T_1 M_1 + N_1)^{-1} (T_1 K_1 + L_1) M_2 + N_2)^{-1} ((T_1 M_1 + N_1)^{-1} \times \\ & \times (T_1 K_1 + L_1) K_2 + L_2) = \\ = {} & ((T_1 M_1 + N_1)^{-1}(T_1 K_1 M_2 + L_1 M_2 + T_1 M_1 N_2 + N_1 N_2))^{-1} \times \\ & \times ((T_1 M_1 + N_1)^{-1}(T_1 K_1 K_2 + L_1 K_2 + T_1 M_1 L_2 + N_1 L_2)) = \\ = {} & (T_1 M + N)^{-1}(T_1 K + L), \end{aligned}$$

where $K = K_1 K_2 + M_1 L_2$, $L = L_1 K_2 + N_1 L_2$, $M = K_1 M_2 + M_1 M_2$, $N = L_1 M_2 + N_1 N_2$. Let us verify the validity of (2.1). $K^T L - L^T K = (K_2^T K_1^T + L_2^T M_1^T)(L_1 K_2 + N_1 L_2) - (K_2^T L_1^T + L_2^T N_1^T)(K_1 K_2 + M_1 L_2) = K_2^T K_1^T L_1 K_2 + K_2^T K_1^T N_1 L_2 + L_2^T M_1^T L_1 K_2 + L_2^T M_1^T N_1 L_2 - K_2^T L_1^T K_1 K_2 - K_2^T L_1^T M_1 L_2 - L_2^T N_1^T K_1 K_2 - L_2^T N_1^T M_1 L_2 = K_2^T(K_1^T L_1 - L_1^T K_1) K_2 + L_2^T(M_1^T N_1 - N_1^T M_1) L_2 + K_2^T(K_1^T N_1 - L_1^T M_1) L_2 - L_2^T(N_1^T K_1 - M_1^T L_1) K_2 = 0$. Similarly we prove that $M^T N - N^T M = 0$ and $K^T N - L^T N = E$. Hence the relation $\cong$ is transitive and the proof is complete.

Theorem 3. *Let A_0, A_1, A_2 be arbitrary symmetric constant $n \times n$ matrices and $A_1 > 0$. Then for every $x_0 \in I$ there exists the only one phase matrix $A(x)$ of* (1.1) *for which*

$$A(x_0) = A_0$$
$$A'(x_0) = A_1$$
$$A''(x_0) = A_2.$$

Proof. If $A(x)$ is a phase matrix of (1.1) then $A(x) + K$, where K is a constant symmetric $n \times n$ matric, is also a phase matrix of (1.1). Thus, without loss of generality, we can suppose that $A_0 = 0$. Recall that if a phase matrix $A(x)$ is determined by a pair of isotropic solutions $U(x)$, $V(x)$, then these solutions can be expressed in the form (1.8) and

$$A(x) = \int_{x_1}^{x} (R^T(s)\, R(s))^{-1}\, ds \qquad x_1 \in I \tag{3.14}$$

where $R(x)$ is a nonsingular $n \times n$ matrix satisfying

$$\begin{aligned} R(x)\, R^T(x) &= U(x)\, U^T(x) + V(x)\, V^T(x), \\ R^{T\prime}(x)\, R(x) - R^T(x)\, R'(x) &= 0. \end{aligned} \tag{3.15}$$

(The matrix $R(x)$ is determined by (3.15) uniquely up to a right multiple by some constant orthonormal $n \times n$ matrix, see [4]). Let $U_0(x)$, $V_0(x)$ be two solutions of (1.2) for which $U_0(x_0) = 0$, $U_0'(x_0) = E$, $V_0(x_0) = E$, $V_0'(x_0) = 0$ and let

$$U(x) = U_0(x)\,(A_1^{1/2})$$

$$V(x) = -\frac{1}{2}\, U_0(x)\,(A_1^{1/2})^{-1}\, A_2 A_1^{-1} + V_0(x)\,(A_1^{1/2})^{-1}$$

Then $U(x)$, $V(x)$ are both isotropic solutions of (1.2) for which (1.6) holds. Hence $U(x)$ and $V(x)$ can be expressed in the form (1.8) and the matrices $S(x)$, $C(x)$ can be chosen such that $S(x_0) = 0$ and $C(x_0) = E$. Now let $R(x)$ be the matrix from this expression, then $R(x_0) = R(x_0)\, C(x_0) = V(x_0) = (A_1^{1/2})^{-1}$ and $R'(x_0) = R'(x_0)\, C(x_0) - R(x_0)\, Q(x_0)\, S(x_0) = V'(x_0) = -\frac{1}{2}(A_1^{1/2})^{-1} A_2 A_1^{-1}$. From (3.14) it follows $A'(x_0) = (R^T(x_0)\, R(x_0))^{-1} = A_1$ and $A''(x_0) = -(R^T(x_0)\, R(x_0))^{-1} \times (R^T(x_0)\, R'(x_0) + R^{T\prime}(x_0)\, R(x_0))\, (R^T(x_0)\, R(x_0))^{-1} = -A_1 \left(-\frac{1}{2} A_1^{-1} A_2 A_1^{-1} - \frac{1}{2} A_1^{-1} A_2 A_1^{-1}\right) A_1 = A_2$. Choosing $x_1 = x_0$ in (3.14) we obtain $A(x_0) = 0$. The proof is complete.

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O. Došlý,
Department of Mathematics,
Faculty of Science, J. E. Purkyně University,
Janáčkovo nám. 2a,
662 95 Brno
Czechoslovakia