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ANTIPROJECTORS WITH APPLICATIONS IN THE SPECTRAL THEORY

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§ 1. Introduction

Trying to state a sensible analogue of the spectral theorem for normal operators on a real Hilbert space – see e.g. [1] p. 165 – we meet necessarily antiprojectors. Moreover on finite – dimensional spaces the operator need not be normal but merely diagonalizable. Basic concepts for vector spaces are taken from [1].

1.1. Let K be the field R of all real numbers or the field C of all complex numbers. Let $\mathscr{V}$ over K be a vector space, I the identity on $\mathscr{V}$ and $\mathscr{L}(\mathscr{V})$ the space of all linear operators on $\mathscr{V}$.

If N is an index set, then operators $T_\nu \in \mathscr{L}(\mathscr{V})$, $\nu \in N$ will be called pairwise disjoint if $T_{\nu'}T_{\nu''} = 0 = T_{\nu''}T_{\nu'}$ whenever $\nu' \neq \nu'' \in N$.

Lemma 1. *Let $\mathfrak{L}_\tau \subseteqq \mathscr{V}$ for $\tau = 1, \ldots, t$ be subspaces and $R_\tau \in \mathscr{L}(\mathscr{V})$ projectors on $\mathfrak{L}_\tau$ i.e. $R_\tau^2 = R_\tau$ and $\mathfrak{L}_\tau = \operatorname{im} R_\tau$. Then the projectors R_τ are pairwise disjoint and satisfy $\sum_{\tau=1}^{t} R_\tau = I$ if and only if $\mathscr{V} = \sum_{\tau=1}^{t}\cdot \mathfrak{L}_\tau$ and R_τ is the projector on $\mathfrak{L}_\tau$ along $\sum_{\tau\neq\tilde{\tau}=1}^{t}\cdot \mathfrak{L}_{\tilde{\tau}}$ for every $\tau = 1, \ldots, t$ (where by Σ the direct sum of subspaces is meant).*

The following remark will be useful. If $\mathscr{V} = \sum_{\tau=1}^{t}\cdot\mathfrak{L}_\tau$ and $R_\tau \in \mathscr{L}(\mathscr{V})$ is the projector on $\mathfrak{L}_\tau$ along $\sum_{\tau\neq\tilde{\tau}=1}^{t}\cdot \mathfrak{L}_{\tilde{\tau}}$, then for any partition $\{1, \ldots, t\} = \{\tau'\} \mathbin{\underline{\cup}} \{\tau''\}$ the sum $\sum_{\tau'} R_{\tau'}$ is the projector on $\sum_{\tau'}\cdot\mathfrak{L}_{\tau'}$ along $\sum_{\tau''}\cdot \mathfrak{L}_{\tau''}$.

Let $\dim \mathscr{V} = n$ $(\in N)$; let $\gamma_1, \ldots, \gamma_t \in K$ be all the proper values of $C \in \mathscr{L}(\mathscr{V})$ which are pairwise distinct; let $\mathfrak{L}_\tau = \{x \in \mathscr{V} \mid Cx = \gamma_\tau x\}$ be corresponding proper subspaces. Then C will be called diagonalizable if $\sum_{\tau=1}^{t} \dim \mathfrak{L}_\tau = n$. Certainly, C is diagonalizable iff $\mathscr{V} = \sum_{\tau=1}^{t}\cdot \mathfrak{L}_\tau$.

Theorem 1. *Let $\mathscr{V}$ over K be a vector space of dimension $n(\in N)$. An operator $C \in \mathscr{L}(\mathscr{V})$ is diagonalizable if and only if there exist pairwise distinct numbers $\gamma_\tau \in K$ and pairwise disjoint projectors $0 \neq R_\tau \in \mathscr{L}(\mathscr{V})$ (for $\tau = 1, \ldots, t$) such that $I = \sum_{\tau=1}^{t} R_\tau$ and $C = \sum_{\tau=1}^{t} \gamma_\tau R_\tau$. Moreover, the number t, the set $\{\gamma_\tau\}_{\tau=1,\ldots,t}$, the projectors R_τ are determined uniquely and $R_\tau = p_\tau(C)$ where*

$$p_\tau(\lambda) = \frac{\prod_{\tau \neq \tilde{\tau}=1}^{t} (\lambda - \gamma_{\tilde{\tau}})}{\prod_{\tau \neq \tilde{\tau}=1}^{t} (\gamma_\tau - \gamma_{\tilde{\tau}})}.$$

1.2. If $\mathscr{V}_0$ over R is a vector space, then its complexification is the vector space $\mathscr{V} = \mathscr{V}_0 + i\mathscr{V}_0$ over C understood as $\mathscr{V}_0 \times \mathscr{V}_0$ with addition $(x, y) + (\tilde{x}, \tilde{y}) = (x + \tilde{x}, y + \tilde{y})$ written as $(x + iy) + (\tilde{x} + i\tilde{y}) = (x + \tilde{x}) + i(y + \tilde{y})$ and with multiplication $(\alpha + i\beta)(x + iy) = (\alpha x - \beta y, \beta x + \alpha y)$ written as $(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y)$, where $x, \tilde{x}, y, \tilde{y} \in \mathscr{V}_0$ and $\alpha, \beta \in R$. Notice that every $z \in \mathscr{V}$ has a unique representation $z = x + iy$ where $x, y \in \mathscr{V}_0$.

Assume that $\mathscr{V}$ over C is the complexification of $\mathscr{V}_0$ over R. If $\mathfrak{L} \subseteq \mathscr{V}$ is a subspace over C and $\mathfrak{L}_0 \subseteq \mathscr{V}_0$ a subspace over R then $\mathfrak{L}$ is the complexification of $\mathfrak{L}_0$ iff $\mathfrak{L} = \mathfrak{L}_0 + i\mathfrak{L}_0$; then $\mathfrak{L}_0 = \mathfrak{L} \cap \mathscr{V}_0$ and thus any $\mathfrak{L}$ has at most one decomplexification $\mathfrak{L}_0$ such that $\mathfrak{L}_0 \subseteq \mathscr{V}_0$.

To every $z \in \mathscr{V}$, $z = x + iy$ we can assign the vector $\bar{z} = x - iy \in \mathscr{V}$ which may be called the conjugate of z; certainly $\bar{\bar{z}} = z$, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{\gamma z} = \bar{\gamma}\bar{z}$ for $z, z_1, z_2 \in \mathscr{V}$ and $\gamma \in C$.

For any $C_0 \in \mathscr{L}(\mathscr{V}_0)$ we can put $Cz = C_0 x + iC_0 y$ for every $z = x + iy \in \mathscr{V}$; then $C \in \mathscr{L}(\mathscr{V})$, $C/\mathscr{V}_0 = C_0$ so that C is the unique linear extension of C_0 on $\mathscr{V}$ and may be called the complexification of C_0. On the contrary, an operator $C \in \mathscr{L}(\mathscr{V})$ has a (unique) decomplexification $C_0 \in \mathscr{L}(\mathscr{V}_0)$ iff $\mathscr{V}_0$ is invariant under C; then $C_0 = C/\mathscr{V}_0$, ker C_0 and im C_0 are decomplexifications of ker C and im C, respectively.

To every $C \in \mathscr{L}(\mathscr{V})$ we can assign an operator $\bar{C} : \mathscr{V} \to \mathscr{V}$ such that $\bar{C}z = \overline{C\bar{z}}$ for every $z \in \mathscr{V}$. Then $\bar{C} \in \mathscr{L}(\mathscr{V})$ and we may call it the conjugate of C; certainly $\bar{\bar{C}} = C$, $\overline{C_1 + C_2} = \bar{C}_1 + \bar{C}_2$, $\overline{\gamma C} = \bar{\gamma}\bar{C}$, $\overline{C_1 C_2} = \bar{C}_1 \bar{C}_2$ and $\overline{Cz} = \bar{C}\bar{z}$ for C, C_1, $C_2 \in \mathscr{L}(\mathscr{V})$, $z \in \mathscr{V}$ and $\gamma \in C$.

Lemma 2. *Let $\mathscr{V}$ over C be a complexification of $\mathscr{V}_0$ over R. Then $C \in \mathscr{L}(\mathscr{V})$ has a decomplexification $C_0 \in \mathscr{L}(\mathscr{V}_0)$ if and only if $\bar{C} = C$.*

1.3. Let $\mathscr{V}$ over C be a complexification of $\mathscr{V}_0$ over R and $C \in \mathscr{L}(\mathscr{V})$ a complexification of $C_0 \in \mathscr{L}(\mathscr{V}_0)$. It is clear that any Hamel basis $\{a_\nu\}_{\nu \in N}$ of $\mathscr{V}_0$ over R is at once a Hamel basis of $\mathscr{V}$ over C.

If $\gamma \in \boldsymbol{R}$ is a proper value of C and $\mathfrak{L} \subseteq \mathscr{V}$ the corresponding proper subspace, then γ is a proper value of C_0 and the corresponding proper subspace $\mathfrak{L}_0 \subseteq \mathscr{V}_0$ is the decomplexification of $\mathfrak{L}$.

If $\gamma \in \boldsymbol{C} \setminus \boldsymbol{R}$ is a proper value of C and $\mathfrak{L} \subseteq \mathscr{V}$ the corresponding proper subspace, then $\bar{\gamma}$ is an other proper value of C and the corresponding proper subspace is $\bar{\mathfrak{L}} = \{\bar{z} \in \mathscr{V} \mid z \in \mathfrak{L}\}$. Certainly, if $\{c_\nu\}_{\nu \in N}$ is a Hamel basis of $\mathfrak{L}$, then $\{\bar{c}_\nu\}_{\nu \in N}$ is a Hamel basis of $\bar{\mathfrak{L}}$, and $\{c_\upsilon\}_{\upsilon \in N} \cup \{\bar{c}_\upsilon\}_{\upsilon \in N}$ a Hamel basis of the direct sum $\mathfrak{L} \dot{+} \bar{\mathfrak{L}}$. If we put $c_\nu = a_\nu + ib_\nu$, where $a_\nu, b_\nu \in \mathscr{V}_0$, then the set $\{a_\nu\}_{\nu \in N} \cup \{b_\nu\}_{\nu \in N}$ is linearly independent over $\boldsymbol{C}$ and thus a Hamel basis of $\mathfrak{L} \dot{+} \bar{\mathfrak{L}}$; we shall call it the induced real basis.

Assume $\dim \mathscr{V}_0 = n$ $(\in N)$ and the complexification C of C_0 is diagonalizable. Let $\gamma_1, \ldots, \gamma_{t_0} \in \boldsymbol{R}$ be all real and pairwise distinct proper values of C, and $\gamma_\tau, \bar{\gamma}_\tau \in \boldsymbol{C} \setminus \boldsymbol{R}$ for $\tau = t_0 + 1, \ldots, t$ be all non-real and pairwise distinct proper values of C.

For $\tau = 1, \ldots, t$ let $\mathfrak{L}_\tau \subseteq \mathscr{V}$ be the proper subspace of C corresponding to the proper value $\gamma_\tau \in \boldsymbol{C}$ so that

$$\mathscr{V} = \sum_{\tau=1}^{t_0}{}^{\cdot}\, \mathfrak{L}_\tau \dot{+} \sum_{\tau=t_0+1}^{t}{}^{\cdot}\, (\mathfrak{L}_\tau \dot{+} \bar{\mathfrak{L}}_\tau). \tag{1}$$

To every $\tau_1 = t_0 + 1, \ldots, t$ there are two distinct proper values $\gamma_\tau, \bar{\gamma}_\tau$ with proper subspaces $\mathfrak{L}_\tau, \bar{\mathfrak{L}}_\tau$; if $\{c_{\nu_\tau}\}$ represents a basis of $\mathfrak{L}_\tau$ where $c_{\nu_\tau} = a_{\nu_\tau} + ib_{\nu_\tau}$ with $a_{\nu_\tau}, b_{\nu_\tau} \in \mathscr{V}_0$, then $\{a_{\nu_\tau}\} \cup \{b_{\nu_\tau}\}$ represents the induced real basis of $\mathfrak{L}_\tau \dot{+} \bar{\mathfrak{L}}_\tau$. Let $\mathfrak{L}_\tau^0 \subseteq \mathscr{V}_0$ be the subspace generated by the set $\{a_{\nu_\tau}\} \cup \{b_{\nu_\tau}\}$ over $\boldsymbol{R}$; then $\mathfrak{L}_\tau^0$ is the decomplexification of $\mathfrak{L}_\nu \dot{+} \bar{\mathfrak{L}}_\tau$ although $\mathfrak{L}_\tau^0$ is no proper subspace of C_0.

To every $\tau = 1, \ldots, t_0$ we have the proper value γ_τ $(\in \boldsymbol{R})$ with the proper subspace $\mathfrak{L}_\tau$ which has a decomplexification $\mathfrak{L}_\tau^0 \subseteq \mathscr{V}_0$ being the proper subspace of C_0 (corresponding to γ_τ). Hence

$$\mathscr{V}_0 = \sum_{\tau=1}^{t}{}^{\cdot}\, \mathfrak{L}_\tau^0. \tag{2}$$

Lemma 3. *For $\tau = 1, \ldots, t$ let $R_\tau \in \mathscr{L}(\mathscr{V})$ be the projector on $\mathfrak{L}_\tau$ along the direct sum of the other subspaces in* (1). *Then the linear projector of $\mathscr{V}$ on $\bar{\mathfrak{L}}_\tau$ along the direct sum of the other subspaces in* (1) *is $\bar{R}_\tau$.*

According to Theorem 1 we have then

$$I = \sum_{\tau=1}^{t_0} R_\tau + \sum_{\tau=t_0+1}^{t} (R_\tau + \bar{R}_\tau), \tag{3}$$

where I is the identity on $\mathscr{V}$ and

$$C = \sum_{\tau=1}^{t_0} \gamma_\tau R_\tau + \sum_{\tau=t_0+1}^{t} (\gamma_\tau R_\tau + \bar{\gamma}_\tau \bar{R}_\tau). \tag{4}$$

Clearly $\bar{R}_\iota = R_\iota$ for $\iota = 1, \ldots, t_0$ so that R_ι has a decomplexification $R_\iota^0 = R_\iota/\mathscr{V}_0 \in \mathscr{L}(\mathscr{V}_0)$ which is the projector on $\mathfrak{L}_\iota^0$ along the direct sum of the other subspaces in (2).

For any $\varkappa = t_0 + 1, \ldots, t$ there are two disjoint projectors $R_\varkappa, \bar{R}_\varkappa$ so that $R_\varkappa + \bar{R}_\varkappa \in \mathscr{L}(\mathscr{V})$ is the projector on $\mathfrak{L}_\varkappa \dot{+} \bar{\mathfrak{L}}_\varkappa$ along the direct sum of the other subspaces in (1) and its decomplexification $R_\varkappa^0 \in \mathscr{L}(\mathscr{V}_0)$ is the projector on $\mathfrak{L}_\varkappa^0$ along the direct sum of the other subspaces in (2).

If we put $\tilde{\alpha}_\varkappa = \operatorname{Re} \gamma_\varkappa$, $\tilde{\beta}_\varkappa = \operatorname{Im} \gamma_\varkappa$, then $\gamma_\varkappa R_\varkappa + \bar{\gamma}_\varkappa \bar{R}_\varkappa = \tilde{\alpha}_\varkappa(R_\varkappa + \bar{R}_\varkappa) + i\tilde{\beta}_\varkappa(R_\varkappa - \bar{R}_\varkappa)$ where $S_\varkappa = i(R_\varkappa - \bar{R}_\varkappa) \in \mathscr{L}(\mathscr{V})$ has a decomplexification $S_\varkappa^0 \in \mathscr{L}(\mathscr{V}_0)$. If $\{c_{\nu_\varkappa}\}$ represents a basis of $\mathfrak{L}_\varkappa$ and $\{a_{\nu_\varkappa}\} \cup \{b_{\nu_\varkappa}\}$ the induced real basis of $\mathfrak{L}_\varkappa \dot{+} \bar{\mathfrak{L}}_\varkappa$, then $S_\varkappa^0 a_{\nu_\varkappa} = -b_{\nu_\varkappa}$, $S_\varkappa^0 b_{\nu_\varkappa} = a_{\nu_\varkappa}$ whereas $S_\varkappa^0 c = 0$ for every $c \in \mathfrak{L}_\tau^0$ whenever $\tau \neq \varkappa$, $\tau \in \{1, \ldots, t\}$. Hence $-S_\varkappa^{02} = R_\varkappa^0$, $\operatorname{im} S_\varkappa^0 = \operatorname{im} R_\varkappa^0$, $\ker S_\varkappa^0 = \ker R_\varkappa^0$ and we get the formula

$$C_0 = \sum_{\iota=1}^{t_0} \gamma_\iota R_\iota^0 + \sum_{\varkappa=t_0+1}^{t} (\tilde{\alpha}_\varkappa R_\varkappa^0 + \tilde{\beta}_\varkappa S_\varkappa^0) \tag{5}$$

representing a real spectral decomposition of C_0 which may be considered as a starting point to a real spectral theorem.

§ 2. Antiprojectors

Let $\mathscr{V}$ over K be a vector space, $Q \in \mathscr{L}(\mathscr{V})$ and I the identity on $\mathscr{V}$. Then $-Q^2 = P$ is a projector iff $Q^2(I + Q^2) = 0$ and then $PQ = -Q = QP$, $\operatorname{im} P \subseteq \operatorname{im} Q$, $\ker Q \subseteq \ker P$.

Definition 1. *Let $\mathscr{V}$ over K be a vector space. An operator $Q \in \mathscr{L}(\mathscr{V})$ will be called antiprojector if $-Q^2 = P$ is a projector and* $\operatorname{im} P = \operatorname{im} Q$, $\ker Q = \ker P$.

If $Q \in \mathscr{L}(\mathscr{V})$ is an antiprojector, then $\mathscr{V} = \ker Q \dot{+} \operatorname{im} Q$ but $Q/\operatorname{im} Q$ is not the identity on $\operatorname{im} Q$ whenever $\operatorname{im} Q \neq 0$; the Q may be called an antiprojector on $\operatorname{im} Q$ along $\ker Q$.

Let $Q \in \mathscr{L}(\mathscr{V})$ be such that $-Q^2 = P$ is a projector; then the assertions (i) Q is an antiprojector (ii) $\operatorname{im} P = \operatorname{im} Q$ (iii) $\ker Q = \ker P$ are equivalent.

Following lemmas are easily prouvable.

Lemma 4. *Let $\mathscr{V}$ over K be a vector space. Then $Q \in \mathscr{L}(\mathscr{V})$ is an antiprojector iff $Q(I + Q^2) = 0$.*

Lemma 5. *If N is a finite set and $Q_\nu \in \mathscr{L}(\mathscr{V})$, $\nu \in N$ are pairwise disjoint antiprojectors, then $\sum_{\nu \in N} Q_\nu$ is an antiprojector on $\sum_{\nu \in N}^{\cdot} \operatorname{im} Q_\nu$ along $\bigcap_{\nu \in N} \ker Q_\nu$.*

Lemma 6. *Let $P \in \mathscr{L}(\mathscr{V})$ be a projector, $Q \in \mathscr{L}(\mathscr{V})$ an antiprojector and $R = -Q^2$. If P and Q commute, then P and R commute as well, PQ is an antiprojector, and* $\operatorname{im} PQ = \operatorname{im} PR$, $\ker PQ = \ker PR$.

Every antiprojector $Q \in \mathscr{L}(\mathscr{V})$ determines uniquely the projector $P = -Q^2$. On the contrary, let $P \in \mathscr{L}(\mathscr{V})$ be a projector, let $\{d_\mu\}$ be a Hamel basis of $\ker P$ and $\{c_v\}$ a Hamel basis of $\operatorname{im} P$. Then for any antiprojector $Q \in \mathscr{L}(\mathscr{V})$ such that $-Q^2 = P$, the $\{d_\mu\}$ is a Hamel basis of $\ker Q$, $\{c_v\}$ a Hamel basis of im Q, and $Q/\operatorname{im} Q$ is a linear bijection such that $I + Q^2 = 0$ on $\operatorname{im} Q$.

In particular, if $\dim \operatorname{im} P = n$ and $K = C$, then all the antiprojectors are obtained by means of all bases $\{c_v\}$ by putting $Qc_v = \pm ic_v$ with all variations of signs and completing $Q = 0$ on $\ker P$.

If $\dim \operatorname{im} P = n$ is odd and $K = R$, then there is no antiprojector Q (such that $-Q^2 = P$). If $\dim \operatorname{im} P = n = 2k$ is even and $K = R$, then all the antiprojectors Q are obtained by means of all real bases $\{a_\varkappa, b_\varkappa\}$ of $\operatorname{im} P$ by puting $Qa_\varkappa = -\sigma_\varkappa b_\varkappa$, $Qb_\varkappa = \sigma_\varkappa a_\varkappa$ with all variations of signs $\sigma_\varkappa = \pm 1$ and completing $Q = 0$ on $\ker P$.

§ 3. The real spectral theorem

Let $\mathscr{V}$ over C be a complexification of $\mathscr{V}_0$ over R, let I denote the identity on $\mathscr{V}$ and I_0 the identity on $\mathscr{V}_0$.

If $T, T_v \in \mathscr{L}(\mathscr{V})$ (where $v \in N$, N finite) are complexifications of T_0, $T_v^0 \in \mathscr{L}(\mathscr{V}_0)$, respectively, then: T_v are pairwise disjoint iff T_v^0 are pairwise disjoint; $\operatorname{im} T$ and $\ker T$ are complexifications of $\operatorname{im} T_0$ and $\ker T_0$, respectively; T is a projector iff T_0 is a projector; T is an antiprojector iff T_0 is an antiprojector.

Assume $\dim \mathscr{V}_0 = n$ $(\in N)$ and let $C \in \mathscr{L}(\mathscr{V})$ be the complexification of $C_0 \in$ $\in \mathscr{L}(\mathscr{V}_0)$.

3.1. If C is diagonalizable then according to **1.3** we have the formula (5) where $0 \neq R_\tau^0$ (for $\tau = 1, \ldots, t$) are pairwise disjoint projectors, $0 \neq S_\varkappa^0$ (for $\varkappa =$ $= t_0 + 1, \ldots, t$) are pairwise disjoint antiprojectors commuting with all R_τ^0 and such that $-S_\varkappa^{02} = R_\varkappa^0$.

Put $\tilde{\alpha}_\iota = \gamma_\iota$ also for $\iota = 1, \ldots, t_0$. If $\{\alpha_\varrho\}_{\varrho=1,\ldots,r} = \{\tilde{\alpha}_\tau\}_{\tau=1,\ldots,t}$, where α_ϱ are pairwise distinct, and if $\{\beta_\sigma\}_{\sigma=1,\ldots,s} = \{\tilde{\beta}_\varkappa\}_{\varkappa=t_0+1,\ldots,t}$, where β_σ are pairwise distinct, then to every ϱ there is a unique set $\{\tau_\varrho\}$ of all $\tau \in \{1, \ldots, t\}$ satisfying $\tilde{\alpha}_\tau = \alpha_\varrho$, and to every σ there is a unique set $\{\varkappa_\sigma\}$ of all $\varkappa \in \{t_0 + 1, \ldots, t\}$ satisfying $\tilde{\beta}_\varkappa = \beta_\sigma$.

If we put $P_\varrho^0 = \sum_{\tau_\varrho} R_{\tau_\varrho}^0$, $Q_\sigma^0 = \sum_{\varkappa_\sigma} S_{\varkappa_\sigma}^0$ then $0 \neq P_\varrho^0$ are pairwise disjoint projectors satisfying $I_0 = \sum_{\varrho=1}^{r} P_\varrho^0$ and $0 \neq Q_\sigma^0$ are pairwise disjoint antiprojectors commuting with all P_ϱ^0 and such that

$$C_0 = \sum_{\varrho=1}^{r} \alpha_\varrho P_\varrho^0 + \sum_{\sigma=1}^{s} \beta_\sigma Q_\sigma^0. \tag{6}$$

The set of all indices σ is empty iff the set of all indices $\varkappa$ is empty iff the numbers $\alpha_\varrho \in R$ are all the proper values of C iff C_0 is diagonalizable.

3.2. Let there be given a non empty set $\{\alpha_\varrho\}_{\varrho=1,\dots,r}$ of pairwise distinct real numbers, a set $\{\beta_\sigma\}_{\sigma=1,\dots,s}$ of pairwise distinct positive numbers, pairwise disjoint projectors $0 \neq P_\varrho^0 \in \mathscr{L}(\mathscr{V}_0)$ satisfying $I_0 = \sum_{\varrho=0}^{r} P_\varrho^0$, and pairwise disjoint antiprojectors $0 \neq Q_\sigma^0 \in \mathscr{L}(\mathscr{V}_0)$ commuting with all P_ϱ^0. Let C_0 be defined by formula (6).

We wish to show that C_0 has a diagonalizable complexification $C \in \mathscr{L}(\mathscr{V})$. We may assume that the set $\{\beta_\sigma\}_{\sigma=1,\dots,s}$ is not empty.

Then to every σ there is at least one ϱ such that $P_\varrho^0 Q_\sigma^0 \neq 0$. Thus the domain of all the ordered pairs (ϱ, σ) such that $P_\varrho^0 Q_\sigma^0 \neq 0$ can be enumerated by $\varkappa = t_0 + 1, \dots, t$ where t_0 is still unknown.

In this domain the mapping $(\varrho, \sigma) \mapsto P_\varrho^0 Q_\sigma^0$ is an injection and thus we can put $S_\varkappa^0 = P_\varrho^0 Q_\sigma^0$, $\gamma_\varkappa = \alpha_\varrho + i\beta_\sigma$ and denote $\tilde{\alpha}_\varkappa = \operatorname{Re}\gamma_\varkappa$; $\tilde{\beta}_\varkappa = \operatorname{Im}\gamma_\varkappa$; then the operators $S_\varkappa^0$ $(\neq 0)$ are pairwise disjoint antiprojectors commuting with all $P_{\tilde{\varrho}}^0$, $Q_{\tilde{\sigma}}^0$, and also the corresponding projector $(0 \neq)$ $R_\varkappa^0 = -S_\varkappa^{02}$ are pairwise disjoint and commuting with all $P_{\tilde{\varrho}}^0$, $Q_{\tilde{\sigma}}^0$.

If for every ϱ the $P_\varrho^0(I_0 - \sum_{\varkappa=t_0+1}^{t} R_\varkappa^0) = 0$, then we put $t_0 = 0$. Otherwise the domain of all ϱ such that $P_\varrho^0(I_0 - \sum_{\varkappa=t_0+1}^{t} R_\varkappa^0) \neq 0$ can be enumerated by $\iota = 1, \dots, t_0$. In this domain the mapping $\varrho \mapsto P_\varrho^0(I_0 - \sum_{\varkappa=t_0+1}^{t} R_\varkappa^0)$ is an injection and thus we can put $R_\iota^0 = P_\varrho^0(I_0 - \sum_{\varkappa=t_0+1}^{t} R_\varkappa^0)$ and put $\tilde{\alpha}_\iota = \alpha_\varrho$; then the operators R_ι^0 are pairwise disjoint projectors commuting with all $P_{\tilde{\varrho}}^0$, $Q_{\tilde{\sigma}}^0$.

Moreover, all the projectors R_τ^0 $(\tau = 1, \dots, t)$ are pairwise disjoint and for $\iota = 1, \dots, t_0$; $\varkappa = t_0 + 1, \dots, t$ we have a) $P_\varrho^0 R_\iota^0 \neq 0$ iff $P_\varrho^0(I_0 - \sum_{\varkappa=t_0+1}^{t} R_\varkappa^0) = R_\iota^0$ iff $P_\varrho^0 R_\iota^0 = R_\iota^0$ b) $P_\varrho^0 R_\varkappa^0 \neq 0$ iff there exists exactly one σ such that $S_\varkappa^0 = P_\varrho^0 Q_\sigma^0$ iff $P_\varrho^0 R_\varkappa^0 = R_\varkappa^0$ c) $I_0 = \sum_{\tau=1}^{t} R_\tau^0$ d) every R_τ^0 commutes with every $S_\varkappa^0$ and $R_\tau^0 S_\varkappa^0 = S_\varkappa^0$ iff $\tau = \varkappa$ while $R_\tau^0 S_\varkappa^0 = 0$ iff $\tau \neq \varkappa$ e) $Q_\sigma^0 R_\iota^0 = 0$ f) $Q_\sigma^0 R_\varkappa^0 \neq 0$ iff there exists exactly one ϱ such that $S_\varkappa^0 = P_\varrho^0 Q_\sigma^0$ iff $Q_\sigma^0 R_\varkappa^0 = S_\varkappa^0$.

It holds $\{\alpha_\varrho\}_{\varrho=1,\dots,r} = \{\tilde{\alpha}_\tau\}_{\tau=1,\dots,t}$ and $\{\beta_\sigma\}_{\sigma=1,\dots,s} = \{\tilde{\beta}_\varkappa\}_{\varkappa=t_0+1,\dots,t}$. For every ϱ let $\{\tau_\varrho\}$ be the set of all $\tau \in \{1, \dots, t\}$ satisfying $\tilde{\alpha}_\tau = \alpha_\varrho$, and for every σ let $\{\varkappa_\sigma\}$ be the set of all $\varkappa \in \{t_0 + 1, \dots, t\}$ satisfying $\tilde{\beta}_\varkappa = \beta_\sigma$. Then $P_\varrho^0 R_{\tau_\rho}^0 = R_{\tau_\rho}^0$ while $P_\varrho^0 R_\tau^0 = 0$ iff $\tau \notin \{\tau_\varrho\}$ and thus $P_\varrho^0 = \sum_{\tau_\rho} R_{\tau_\rho}^0$. Similarly $Q_\sigma^0 R_{\varkappa_\sigma}^0 = S_{\varkappa_\sigma}^0$ while $Q_\sigma^0 R_\varkappa^0 = 0$ iff $\varkappa \notin \{\varkappa_\sigma\}$ and thus $Q_\sigma^0 = \sum_{\varkappa_\sigma} S_{\varkappa_\sigma}^0$.

Hence we obtain the formula (5) and we are to show that its complexification is exactly the formula (4). Indeed, if R_ι, $A_\varkappa$, $B_\varkappa \in \mathscr{L}(\mathscr{V})$ is the complexification of

R^0_ι, $R^0_\varkappa$, $S^0_\varkappa$, respectively, then $0 \neq R_\iota$, $A_\varkappa \neq 0$ are pairwise disjoint projectors commuting with all pairwise disjoint antiprojectors $B_\varkappa \neq 0$, and we have $-B_\varkappa^2 = A_\varkappa$, $R_\iota B_\varkappa = 0$, $A_{\varkappa'} B_{\varkappa''} = 0$ iff $\varkappa' \neq \varkappa''$, $A_\varkappa B_\varkappa = B_\varkappa$. If we put $R_\varkappa = \frac{1}{2}(A_\varkappa - iB_\varkappa)$, then $R_\varkappa$ and $\bar{R}_\varkappa = \frac{1}{2}(A_\varkappa + iB_\varkappa)$ are projectors, $A_\varkappa = R_\varkappa + \bar{R}_\varkappa$, $B_\varkappa = i(R_\varkappa - \bar{R}_\varkappa)$ and the formula (4) is valid; moreover, the numbers $\gamma_\iota = \tilde{\alpha}_\iota \in \boldsymbol{R}$, $\gamma_\varkappa = \tilde{\alpha}_\varkappa + i\tilde{\beta}_\varkappa \in \boldsymbol{C} \setminus \boldsymbol{R}$, $\bar{\gamma}_\varkappa \in \boldsymbol{C} \setminus \boldsymbol{R}$ are pairwise distinct, the projectors $0 \neq R_\tau$, $\bar{R}_\varkappa \neq 0$ are pairwise disjoint and satisfy $I = \sum_{\iota=1}^{t_0} R_\iota + \sum_{\varkappa=t_0+1}^{t} (R_\varkappa + \bar{R}_\varkappa)$.

According to Theorem 1 the operator C is diagonalizable and the $\gamma_\iota \in \boldsymbol{R}$ are all the real proper values of C, $\gamma_\varkappa, \bar{\gamma}_\varkappa \in \boldsymbol{C} \setminus \boldsymbol{R}$ are all the non-real proper values of C, and R_τ is the projector corresponding to γ_τ. Hence the sets $\{\alpha_\varrho\}_{\varrho=1,\dots,r}$, $\{\beta_\sigma\}_{\sigma=1,\dots,s}$, the corresponding projectors P^0_ϱ and antiprojectors Q^0_σ are determined uniquely. This yields the asked real spectral

Theorem 2. *Let the vector space $\mathscr{V}$ over $\boldsymbol{C}$ be the complexification of a vector space $\mathscr{V}_0$ over $\boldsymbol{R}$ where $\dim \mathscr{V}_0 = n$ $(\in N)$. Let $C \in \mathscr{L}(\mathscr{V})$ be the complexification of $C_0 \in \mathscr{L}(\mathscr{V}_0)$. Then C is diagonalizable if and only if there exists a non empty set $\{\alpha_\varrho\}_{\varrho=1,\dots,r}$ of pairwise distinct real numbers, a set $\{\beta_\sigma\}_{\sigma=1,\dots,s}$ of pairwise distinct positive numbers, a set of pairwise disjoint projectors $0 \neq P^0_\varrho \in \mathscr{L}(\mathscr{V}_0)$ satisfying $\sum_{\varrho=1}^{r} P^0_\varrho = I_0$ and a set of pairwise disjoint antiprojectors $0 \neq Q^0_\sigma \in \mathscr{L}(\mathscr{V}_0)$ commuting with all P^0_ϱ such that*

$$C_0 = \sum_{\varrho=1}^{r} \alpha_\varrho P^0_\varrho + \sum_{\sigma=1}^{s} \beta_\sigma Q^0_\sigma. \tag{7}$$

The sets $\{\alpha_\varrho\}$, $\{\beta_\sigma\}$, the projectors P^0_ϱ and the antiprojectors Q^0_σ are determined uniquely.

It is worth noting that the Jordan representation yields a visible form of Theorem 2, see [2].

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