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Archivum Mathematicum, Vol. 19 (1983), No. 3, 133--142

Persistent URL: http://dml.cz/dmlcz/107166

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIX: 133—142, 1983

A DOUBLE COMPLEX RELATED WITH A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS II

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(Received June 10, 1982)

17. Preliminaries. The present paper is closely related with the previous part I, and we continue the numeration of sections, formula and references. However, the main results will be derived almost independently here. In the previous part, we look after exactness of certain double complex. We have observed that almost all interesting properties are already reflected in the third column

$$\psi_1 : 0 \to \psi_{1,0} \xrightarrow{\partial} \psi_{1,1} \xrightarrow{\partial} \dots \xrightarrow{\partial} \psi_{1,n} \to 0, \tag{21}$$

so we will consider only this column here.

Now, remind the most essential notions and precise the main task of the present part. We deal with the space J^∞ of ∞-jets of sections of a fibered manifold $\pi : M \to B$ and with a *regular* and *closed* system of partial differential equations $R^\infty \subset J^\infty$. The regularity assumption means that R^∞ is an inverse limit of certain fibered manifolds, and so behaves in the main as a manifold. The closedness is a deeper property which may be formulated as follows: Let $\Phi^l_{1,s}$ be the space of all $(s + 1)$-forms φ on the space J^∞, which may be expressed as

$$\varphi = \Sigma f^j_{I',I} \omega^j_{I'} \wedge \mathrm{d}x^I \qquad (|I'| \leqq l, |I| = s) \tag{22}$$

in a local system of the familiar jet coordinates. (Here, $\omega^j_I = \mathrm{d}y^j_I - \sum y^j_{Ii} \,\mathrm{d}x^i$ are the contact forms, I, I' are multiindices of non-negative integers.) Let $\psi^l_{1,s}$ be the restriction of this space to R^∞. Then, R^∞ is a closed system if the exterior differential considered on R^∞ decomposes into two summands δ, ∂; we will be interested only in the second component $\partial : \Psi^l_{1,s} \to \Psi^{l+1}_{1,s+1}$.

The above mapping ∂ appears in the short exact sequence (23) of complexes,

$$
(23)^{l+n}\qquad
\begin{array}{ccccccccccc}
0 & & & 0 & & 0 & & & & 0 & \\
\downarrow & & & \downarrow & & \downarrow & & & & \downarrow & \\
\Psi(l+n-1): & 0 & \longrightarrow & \Psi^{l-1}_{1,0} & \xrightarrow{\partial} & \Psi^{l}_{1,1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \Psi^{l+n-1}_{1,n} & \longrightarrow 0 \\
\downarrow & & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & \\
\Psi(l+n): & 0 & \longrightarrow & \Psi^{l}_{1,0} & \xrightarrow{\partial} & \Psi^{l+1}_{1,1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \Psi^{l+n}_{1,n} & \longrightarrow 0 \\
\downarrow & & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & & & \downarrow{\scriptstyle j} & \\
\Gamma(l+n): & 0 & \longrightarrow & \Gamma^{l}_{0} & \xrightarrow{\partial} & \Gamma^{l+1}_{1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \Gamma^{l+n}_{n} & \longrightarrow 0 \\
\downarrow & & & \downarrow & & \downarrow & & & & \downarrow & \\
0 & & & 0 & & 0 & & & & 0 &
\end{array}
$$

where i are inclusions, j are natural projections, l is an arbitrary integer; we put $\Psi^l_{1,s} \equiv 0$ for l negative. The factor-spaces $\Gamma^l_s = \Psi^l_{1,s}/\Psi^{l-1}_{1,s}$ of sections appearing in the diagram (23) may be represented more advantageously as the spaces of sections of certain vector bundles G^l_s over the base R^∞, $\Gamma^l_s = C^\infty(G^l_s)$. Moreover, there exists such a complex

$$G(l+n): 0 \to G^l_0 \to G^l_1 \to \dots \to G^l_n \to 0$$

of vector bundles and *vector bundle mappings* that the complex $\Gamma(l+n)$ arises from them by applying the functor C^∞ of taking the sections. The differentials in the complexes $G(l+n)$, $\Gamma(l+n)$ naturally correspond one to the other under the functor C^∞ and will be denoted by the same letter ∂.

The homology $H^{l+s}_s(G)$ of the complex $G(l+n)$ is closely related with the familiar *Spencer* and *Koszul homology of the system* R^∞ (cf. Section 12). For every integers l, s, $H^l_s(G)$ is a family of vector spaces dependent on the parameter point $y \in R^\infty$, not necessarily a vector bundle over R^∞. However, one can easily prove that the inequality $\dim H^l_s(G) \leqq$ given constant is true on an *open subset* of R^∞. Especially, $H^l_s(\Gamma) = 0$ *if and only if* $\dim H^l_s(G) = 0$ *holds on an open dense subset of* R^∞.

Our aim is to relate the homology $H^l_s(G)$ with the homology $H^l_s(\Psi)$ of the complex $\Psi(l+n+s)$ of sections of differential forms, and with the homology of the analogous complex (21).

18. Simple results. (i) $H^l_s(\Gamma) = 0$ *(fixed l, s) if and only if for every* $\varphi \in \Psi^l_{1,s}$ *satisfying* $\partial\varphi \in \Psi^l_{1,s+1}$ *there exists* $\chi \in \Psi^l_{1,s-1}$ *such that* $\varphi - \partial\chi \in \Psi^{l-1}_{1,s}$. This follows by an easy diagram chasing in $(23)^{l+n-s}$.

(ii) *If* $H^l_s(\Gamma) \equiv 0$ *($k \leqq l \leqq L$, fixed s), then for every* $\varphi \in \Psi^L_{1,s}$ *satisfying* $\partial\varphi \in \Psi^k_{1,s+1}$ *there exists* $\chi \in \Psi^{L-1}_{1,s-1}$ *such that* $\varphi - \partial\chi \in \Psi^{k-1}_{1,s}$. This follows by successive application of (i).

(iii) $H^l_s(\Gamma) \equiv 0$ *($k \leqq l \leqq L$, $S \leqq s$) if and only if for every* $\varphi \in \Psi^l_{1,s}$ *($k \leqq l \leqq L$, $S \leqq s$) satisfying* $\partial\varphi \in \Psi^k_{1,s+1}$ *there exists* $\chi \in \Psi^{l-1}_{1,s-1}$ *such that* $\varphi - \partial\chi \in \Psi^{k-1}_{1,s}$. The part "then" follows from (ii). The part "if" may be derived by a diagram chasing in $(23)^{l+n-s}$ running as follows: Let $\gamma \in \Gamma_s^l$, $\partial\gamma = 0$. Choose $\varphi \in \Psi_{1,s}^l$, $\mathrm{j}\varphi = \gamma$. Then $\psi = \partial\varphi \in \Psi_{1,s+1}^l$, $\partial\psi = 0$. Owing to (i), there exists $\chi \in \Psi_{1,s}^{l-1}$, $\psi - \partial\chi \in \Psi_{1,s+1}^{l-1}$. Clearly, we have $\mathrm{j}(\varphi - \chi) = \gamma$, $\partial(\varphi - \chi) \in \Psi_{1,s+1}^{l-1}$. We replace φ by $\varphi - \chi$ and repeat the procedure. At last, we obtain a form $\varphi \in \Psi_{1,s}^l$ with $\mathrm{j}\varphi = \gamma$, $\partial\varphi \in \Psi_{1,s+1}^k$. Owing to the assumption, there exists a form $\chi \in \Psi_{1,s}^{l-1}$ with $\varphi - \partial\chi \in \Psi_{1,s}^{l-1}$. Then, $\beta = \mathrm{j}(\varphi - \partial\chi) \in \Gamma_{1,s}^{l-1}$ satisfies $\partial\beta = \gamma$.

(iv) $H_n^l(G) \equiv 0$, hence $H_n^l(\Gamma) \equiv 0$ $(l \neq 0)$, for every system R^∞ (cf. Section 12). Consequently, *to every* $\varphi \in \Psi_{1,n}^l$ $(l \neq 0)$ *there exists* $\chi \in \Psi_{1,n-1}^{l-1}$ *satisfying* $\varphi - \partial\chi \in \Psi_{1,n}^0$. This follows from (ii) with $k = 1$, because $\partial\varphi \in \Psi_{1,n+1}^{l+1} = 0$.

(v) $H_s^l(\Gamma) \equiv 0$ $(k \leqq l$, *all* $s)$ *if and only if for every* $\varphi \in \Psi_{1,s}^l$ $(k \leqq l)$ *satisfying* $\partial\varphi \in \Psi_{1,s+1}^k$ *there exists* $\chi \in \Psi_{1,s-1}^{l-1}$ *such that* $\varphi - \partial\chi \in \Psi_{1,s}^{k-1}$. A particular case of (iii).

(vi) $H_{n-1}^l(\Gamma) \equiv 0$ $(0 < k \leqq l)$ *if and only if for every* $\varphi \in \Psi_{1,n-1}^l$ *satisfying* $\partial\varphi \in \Psi_{1,n}^k$ *there exists* $\chi \in \Psi_{1,n-2}^{l-1}$ *such that* $\varphi - \partial\chi \in \Psi_{1,n-1}^{k-1}$. Owing to (iv), a particular case of (iii).

(vii) $H_{n-1}^l(\Gamma) \equiv 0$ *(all* l*) if and only if for every* $\varphi \in \Psi_{1,n-1}^l$ *satisfying* $\partial\varphi \in \Psi_{1,n}^0$ *there exists* $\chi \in \Psi_{1,n-2}^{l-1}$ *such that* $\varphi = \partial\chi$. *Especially,* $\varphi = 0$ *in the case* $l = 0$. The first assertion follows from (vi) with $k = 1$, the second assertion from (ii) applied on the assumption $H_{n-1}^0(\Gamma) = 0$.

19. Definition and theorem.. A system R^∞ is called *k-involutive* $(k \geqq 1)$ *on a subset of* R^∞, if $H_s^l(G) \equiv 0$ $(l \geqq k)$ on the mentioned subset. A system R^∞ is called *of essential order* $\leqq k$ on a subset of R^∞, if $H_{n-1}^l(G) \equiv 0$ $(l \geqq k)$ on the mentioned subset. If R^∞ is a system of essential order $\leqq 0$, then $H_s^l(G) \equiv 0$ $(l \neq 0$ or $s \neq 0)$ is true; cf. Section 12.

A system R^∞ *is k-involutive on an appropriate open and everywhere dense subset of* R^∞ *if and only if every homology class in the complexes*

$$\Psi_{1,s-1}^{k-1}/\Theta_{s-1}^{k-1} \to \Psi_{1,s}^k/\Theta_s^k \to \dots \to \Psi_{1,n}^{k+n-s}/\Theta_n^{k+n-s} \to 0 \tag{24}$$

may be represented by a form lying in a space $\Psi_{1,s}^{k-1}$. Here, $\Theta_s^l = \Psi_{1,s}^k \cap \partial\Psi_{1,s-1}^{l-1}$, differentials in (24) are induced by ∂, s is an arbitrary integer, and we put $\Psi_{1,s}^l \equiv 0$ for negative s.

A system R^∞ *is of essential order* $\leqq 0$ *on an appropriate open and everywhere dense subset of* R^∞ *if and only if all complexes*

$$0 \to \Psi_{1,0}^l \to \Psi_{1,1}^{l+1} \to \dots \to \Psi_{1,n-1}^{l+n-1} \to \Psi_{1,n}^{l+n}/\Psi_{1,n}^0 \cap \partial\Psi_{1,n-1}^{l+n-1} \tag{25}$$

are exact.

Proof: The first assertion is a reformulation of the point (v) of Section 18; we take into account a remark at the end of Section 17. As for the second assertion, $H_n^l(G) = H_n^l(\Gamma) \equiv 0$ $(l \neq 0)$ in any case. In the case of the groups $H_{n-1}^l(G)$, $H_{n-1}^l(\Gamma)$, one can apply (vii). The general case follows by (ii), if one choose a negative integer k.

20. Definition. Following the general lines, we come out with the filtration

$$\Psi_{1,s} = \cup \Psi^l_{1,s} \supset \ldots \supset \Psi^{l+1}_{1,s} \supset \Psi^l_{1,s} \supset \ldots \supset \Psi^0_{1,s} \supset \Psi^{-1}_{1,s} = 0 \supset \ldots$$

and introduce the spaces $C^l_{s,r}$, of all forms $\varphi \in \Psi^l_{1,s}$ satisfying $\partial\varphi \in \Psi^{l-r}_{1,s+1}$,

$$E^l_{s,r} = \frac{C^l_{s,r} + \Psi^{l-1}_{1,s}}{\partial C^{l+r-1}_{s-1,r-1} + \Psi^{l-1}_{1,sj}} = \frac{C^l_{s,r}}{\partial C^{l+r-1}_{s-1,r-1} + C^{l-1}_{s,r-1}},$$

where r is an arbitrary integer, see [5], [6], [7]. The differential $\partial : \Psi^l_{1,s} \to \Psi^{l+1}_{1,s+1}$ leads to certain new differentials $\partial_r : E^l_{s,r} \to E^{l-r}_{s+1,r}$, and the corresponding homology is $H^l_s(E_r) \equiv E^l_{s,r+1}$.

In our case, one can easily verify that

$$\ldots = E^l_{s,-2} = E^l_{s,-1} = \Gamma^l_s, E^l_{s,0} = H^l_s(\Gamma),$$

and the general term E_r ($r \geqq 0$) may be described as follows: A vector from the space $E^l_{s,r}$ is represented by a form $\varphi \in \Psi^l_{1,s}$ satisfying $\partial\varphi \in \Psi^{l-r}_{1,s+1}$ which is taken modulo the differentials $\partial\psi$ of all forms $\psi \in \Psi^{l+r-1}_{1,s-1}$ satisfying $\partial\psi \in \Psi^l_{1,s}$, and also modulo the lower order forms from the spaces $\Psi^{l-1}_{1,s}, \ldots, \Psi^0_{1,s}$.

(A local and *non-invariant* presentation seems to be very instructive: Refine the formula (22) by writting

$$\varphi = \sum_{|I'|=l} + \sum_{|I'|=l-1} + \ldots + \sum_{|I'|=0} = \varphi^l + \varphi^{l-1} + \ldots + \varphi^0.$$

The differential ∂ may be decomposed into the sum $\partial = \partial_s + \partial_c$ (s = *simple*, c = *complicated*), where

$$\partial_s\varphi = -\Sigma f^j_{I',I}\omega^j_{iI'} \wedge \mathrm{d}x^{iI}, \qquad \partial_c = \Sigma \partial_i f^j_{I',I} \cdot \omega^j_{I'} \wedge \mathrm{d}x^{iI};$$

$\partial_i = \partial/\partial x^i + \sum y^j_{iI}\, \partial/\partial y^j_I$ are the formal derivative operators. Then

$$\partial\varphi = (\partial_s + \partial_c)(\varphi^l + \ldots + \varphi_0),$$

and we have the conditions

$$\partial_s\varphi^l = 0,\ \partial_c\varphi^l + \partial_s\varphi^{l-1} = 0,\ \ldots,\ \partial_c\varphi^{l-r+1} + \partial_s\varphi^{l-r} = 0$$

for a form φ representing a vector from the space $E^l_{s,r}$. However, only the highest order term φ^l is essential for this representation, and it is taken modulo the forms $\partial_c\psi^l + \partial_s\psi^{l-1}$, where the form

$$\psi = \psi^{l+r-1} + \psi^{l+r-2} + \ldots + \psi_0$$

(a development analogous to (25)) satisfies

$$\partial_s\psi^{l+r-1} = 0,\ \partial_c\psi^{l+r-1} + \partial_s\psi^{l+r-2} = 0,\ \ldots,\ \partial_c\psi^{l+1} + \partial_s\psi^l = 0.$$

The differential ∂_r maps the class of φ^l into the class corresponding to $\partial_c\varphi^{l-r} + \partial_s\varphi^{l-r-1}$. All is restricted to R^∞.)

21. Convergence. We introduce the spaces $C^l_{s,\infty}$, of all forms $\varphi \in \Psi^l_{1,s}$ satisfying $\partial\varphi = 0$,

$$E^l_{s,\infty} = \frac{C^l_{s,\infty} + \Psi^{l-1}_{1,s}}{\partial\Psi_{1,s-1} \cap \Psi^l_{1,s} + \Psi^{l-1}_{1,s}} = \frac{C^l_{s,\infty}}{\partial\Psi_{1,s-1} \cap \Psi^l_{1,s} + C^{l-1}_{s,\infty}},$$

Evidently, $C^l_{s,\infty} = C^l_{s,l+1} = C^l_{s,l+2} = \ldots$. It follows that $E^l_{s,R+1} = E^l_{s,R+2} = \ldots = E^l_{s,\infty}$ for certain R ($R \geqq l$), if and only if

$$\partial\Psi_{1,s-1} \cap \Psi^l_{1-s} + \Psi^{l-1}_{1,s} = \partial C^{l+R}_{s-1,R} + \Psi^{l-1}_{1,s}$$

or, equivalently,

$$\partial\Psi_{1,s-1} \cap \Psi^l_{1,s} + C^{l-1}_{s,\infty} = \partial C^{l+R}_{s-1,R} + C^{l-1}_{s,R}.$$

It is a highly important fact that these conditions are certainly satisfied in all current cases, moreover, $E^l_{s,r} \equiv 0$ ($r \geqq 0$) for all sufficiently large l, independently on the value of r. All these conditions will be referred as the *strong convergence conditions*, see also Section 22.

At last, following the general theory of spectral sequences, we consider certain filtration of the homology space H_s ($= H_s(\Psi_1)$, an abbreviation for the homology of the complex (21)) given by

$$H_s = \cup H^l_s \supset \ldots \supset H^{l+1}_s \supset H^l_s \supset \ldots \supset H^0_s \supset H^{-1}_s = 0 \supset \ldots,$$

where H^l_s is the space of all homology classes which may be represented by a form from the space $\Psi^l_{1,s}$. After the general theory, we obtain the isomorphism $H^l_s/H^{l-1}_s \equiv E^l_{s,\infty}$ and, if the above strong convergence conditions are satisfied,

$$H^l_s/H^{l-1}_s = E^l_{s,r}, \qquad H_s = \oplus\, H^l_s/H^{l-1}_s = \oplus\, E^l_{s,r}$$

(finite sum), for all sufficiently large r. This final result need not any comment. Unfortunately, the above spectral sequence can be explicitly determined only rarely, but these difficulties may be expected in analogy with algebraic geometry or homology theory of fibered spaces.

22. Theorem. *Assume that $H^l_s(\Gamma) \equiv 0$ ($l \geqq L$). Then, $E^l_{s,r} \equiv 0$ ($l \geqq L$, $r \geqq 0$), $E^l_{s,L} = E^l_{s,L+1} = \cdots = E^l_{s,\infty}$.*

Proof: $0 \equiv H^l_s(\Gamma) = E^l_{s,0}$, and if $E^l_{s,r} = 0$, then $E^l_{s,r+1} = H^l_s(E_r) = 0$, too. Moreover, $\partial_r \equiv 0$ ($r \geqq L$) which implies the second assertion.

At the end of the paper note that the assumption of Theorem 22 is satisfied if $H^l_s(G) \equiv 0$ ($l \geqq L$) on R^∞. It is a well-known fact that $H^l_s(G) \equiv 0$ at every fixed point of R^∞ (and hence in a neighbourhood), for all large l, cf. Section 7. Consequently, the above homology groups vanish on every compact subset of R^∞. But more is true: If a system R^∞ appears as a prolongation of a fixed finite system of partial differential equations, then all homology groups $H_s^l(G)$ vanish for sufficiently large integer l dependent only on the order of the system under consideration and the dimension of B and M. In any case, the assumption of Theorem 22 is satisfied in all current cases.

APPENDIX

23. Essential order. We shall state another definition than in the main text; Theorem 24 asserts that they are wholly equivalent. Let $\overline{\Psi}^{(l)}_{0,0}$ be the space of all functions locally dependent only on the variables x^i, y^j_I ($i = 1, \ldots, n; j = 1, \ldots, m$; $|I| \leqq l$) and vanishing on the set R^∞. The system R^∞ is called *of essential order* $\leqq k$, if every function $f \in \overline{\Psi}^{(l)}_{0,0}$ ($l \geqq k$) belongs to the ideal generated by the functions of the type $\partial_I g$ ($|I| \leqq l - k$, $g \in \overline{\Psi}^{(k)}_{0,0}$). Especially, R^∞ is of order $\leqq 0$ if and only if every above mentioned function f belongs to the ideal generated by $\partial_I g$ ($|I| \leqq l$, $g \in \overline{\Psi}^{(0)}_{0,0}$). The condition $|I| \leqq l$ may be omitted in this particular case; one can prove this by using an appropriate local coordinate system and partition of unity.

We say that the *property* $\{l + 1\}$ takes place, if every function $f \in \overline{\Psi}^{(l+1)}_{0,0}$ belongs to the ideal with generators g, $\partial_i g$ ($i = 1, \ldots, n$), where $g \in \overline{\Psi}^{(l)}_{0,0}$. Clearly, R^∞ is of essential order $\leqq k$ if and only if all properties $\{k + 1\}$, $\{k + 2\}$, ... are true.

We say that the *property* $[l + 1]$ takes place, if every form $\alpha \in \overline{\Psi}^{l+1}_{1,n}$ is (locally) expressible by a sum $\alpha = \partial\beta + \gamma$, where $\beta \in \overline{\Psi}^l_{1,n-1}$, $\gamma \in \overline{\Psi}^l_{1,n}$. Using partition of unity, one can prove that both variants (local and global) are equivalent.

24. Theorem. *Supposing* $l \geqq 0$, *the following three properties are equivalent*: $H^l_{n-1}(G) = 0$, $\{l + 1\}$, $[l + 1]$. *Especially*, R^∞ *is of essential order* $\leqq k$ *if and only if* $H^l_{n-1}(G) \equiv 0$ ($l \geqq k$).

Proof: Owing to the relations between Dedecker and Koszul homologies (Section 5) and a result of Section 6, one can see that $H^l_{n-1}(G) = 0$ if and only if $H^l_1(F) = 0$, that means if and only if $\partial' : F^l_1 \to F^{l+1}_0$ and hence $\partial : \bar{G}^l_{n-1} \to \bar{G}^{l+1}_n$ is a surjective mapping. This is the case if and only if $\partial : \bar{\Gamma}^l_{n-1} = C^\infty(\bar{G}^l_{n-1}) \to$ $\to \bar{\Gamma}^{l+1}_n = C^\infty(\bar{G}^{l+1}_n)$ is a surjective mapping. Then the following commutative diagram with exact rows implies that the above case takes place if and only if $[l + 1]$ is valid.

$$\begin{array}{ccccccc}
0 \to & \overline{\Psi}^{l-1}_{1,n-1} & \to & \overline{\Psi}^l_{1,n-1} & \to & \bar{\Gamma}^l_{n-1} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \overline{\Psi}^l_{1,n} & \to & \overline{\Psi}^{l+1}_{1,n} & \to & \bar{\Gamma}^{l+1}_n & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

Consequently, it is sufficient to verify that $[l + 1]$, $\{l + 1\}$ are equivalent properties.

We shall use the local coordinates and denote $dx = dx^1 \wedge \ldots \wedge dx^n$, $dx^{(i)} = -(-1)^i dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n$, γ will be an arbitrary form from the space $\overline{\Psi}^l_{1,n}$. It is sufficient to consider only the particular forms

$$\alpha = A\delta a \wedge dx = A\, da \wedge dx \in \overline{\Psi}^{l+1}_{1,n}, \qquad \beta = \Sigma B_i \delta b \wedge dx^{(i)} \in \overline{\Psi}^l_{1,n-1},$$

where $a \in \overline{\Psi}^{(l+1)}_{0,0}$, $b \in \overline{\Psi}^{(l)}_{0,0}$, see the formula (10). Then,

$$\partial\beta = -\Sigma B_i \delta\, \partial b \wedge dx^{(i)} + \Sigma\, \partial B_i \wedge \delta b \wedge dx^{(i)} = -\Sigma B_i\, d\, \partial_i b \wedge dx + \gamma.$$

Assume $\{l + 1\}$. Then a belongs to the ideal generated by certain functions functions $g_\varkappa$, $\partial_i h_\varkappa$ ($g_\varkappa, h_\varkappa \in \overline{\Psi}^{(l)}_{0,0}$; $\varkappa$ varies in a finite index set), hence

$$da = \Sigma F_{i\varkappa}\, d\, \partial_i h_\varkappa + \Sigma G_\varkappa\, dg_\varkappa \qquad (F_{i\varkappa} \in \overline{\Psi}^{(l)}_{0,0}),$$
$$A\, da \wedge dx = \Sigma F_{i\varkappa}\, d\, \partial_i h_\varkappa \wedge dx + \gamma = \partial\beta + \gamma, \qquad \beta = \Sigma A F_{i\varkappa} \delta h_\varkappa \wedge dx^{(i)}.$$

So, $[l + 1]$ is true.

Assume $[l + 1]$. Then a form $\alpha = da \wedge dx$ (we choose $A = 1$ in the above formula) is of the type $\alpha = \partial\beta + \gamma$, where $\beta = \sum B_{i\varkappa} \delta b_\varkappa \wedge dx^{(i)}$. It follows that

$$da = \Sigma B_{i\varkappa}\, d\, \partial_i b_\varkappa + (\gamma),$$

where (γ) is certain form expressible by the differentials dx^i, dy^j_I ($|I| \leqq l$) and vanishing on R^∞. Hence, $(\gamma) = \gamma$, and using the regularity of R^∞ and the implicite function theorem, the function $a \in \overline{\Psi}^{(l)}_{0,0}$ appears as an element of the ideal generated by $\partial_i b_\varkappa$ and certain functions from $\overline{\Psi}^{(l)}_{0,0}$. The property $\{l + 1\}$ is verified.

25. Automorphisms of the space J^∞. These are such ono-to-one mappings F which preserves the fibration $x^i = \text{const.}$ and the set of linear contact forms. Consequently, the formulae

$$F^* dx^i = \Sigma A^{ii'} dx^{i'}, \qquad F^* \omega^j = \Sigma B^{jj'} \omega^{j'},$$

are true, where $A^{ii'}$ are functions only of the variables $x^1, \ldots, x^n$ and $B^{jj'}$ are functions of $x^1, \ldots, x^n, y^1, \ldots, y^m$; $\det(A^{ii'}) \neq 0$, $\det(B^{jj'}) \neq 0$. We claim a formula

$$(27)^{|I|} \qquad F^* \omega^j_I = \Sigma B^{jj'}_{II'} \omega^{j'}_{I'} (|I'| \leqq |I|),$$

where $B^{jj'}_{II'}$ are functions of the variables $x^i, y^j_{I''}$ ($|I'| + |I''| \leqq |I|$). Suppose that (27) $|I|$ is true. Then

$$F^* d\omega^j_I = F^* \Sigma\, \omega^j_{iI} \wedge dx^i = \Sigma F^* \omega^j_{iI} \wedge A^{ii'} dx^{i'} =$$
$$= dF^* \omega^j_I = \Sigma dB^{jj'}_{II'} \wedge \omega^{j'}_{I'} + \Sigma B^{jj'}_{II'} \omega^{j'}_{iI'} \wedge dx^i.$$

It follows,

$$\Sigma A^{ii'} F^* \omega^j_{iI} = -\Sigma \partial_{i'} B^{jj'}_{II'} \omega^{j'}_{I'} + \Sigma B^{jj'}_{II'} \omega^{j'}_{i'I'}$$

(and $\Sigma\, \delta B^{jj'}_{II'} \wedge \omega^{j'}_{I'} \equiv 0$), which is the formula (27) $|I| + 1$.

As a consequence, all filtrations of Section 9 posses an invariant meaning.

26. An application. We shall only briefly look on certain questions related with the familiar Lagrange problem. Denote by P an s-dimensional compact and oriented submanifold of B with boundary Q, $\mathrm{i} : Q \to P$ be the natural inclusion of the boundary. (The case $\dim P = \dim B = n$ is the *classical* one. Denote by $p : P \to R^\infty$ a section over P (i.e. $\pi^\infty \circ p : P \to P$ is the identity), and let $q = p \circ \mathrm{i}$. We introduce the functional

$$\mathscr{F}(p) = \int_P p^*\varphi + \int_Q q^*\psi \qquad (\varphi \in \Psi_{0,s}, \psi \in \Psi_{0,s-1})$$

considered on the set $\boldsymbol{P}$ of all sections p satisfying

$$(28) \qquad p(t) \in R^\infty (t \in P), \qquad p^*\omega_I^j \equiv 0$$

and, may be, certain boundary conditions on Q which we do not precise here. (For the classical case, (28) means that p is a solution of R^∞. In general, and especially for the involutive case, p may be considered as a submanifold of such a solution.) The fundamental concept is as follows: $p \in \boldsymbol{P}$ is called a *critical point of the functional* $\mathscr{F}$ *on the set* $\boldsymbol{P}$, if

$$(29) \qquad \mathrm{d}\mathscr{F}(p(\lambda))/\mathrm{d}\lambda\,|_{\lambda=0} \equiv 0$$

is true for every one-parameter family $p(\lambda) \in \boldsymbol{P}$ with $p(0) = p$. The submanifold P may, or may not depend on the parameter λ; we assume the first possibility.

Using some fundamental tools of the differential calculus on manifolds, one can easily see that (29) looks more explicitly as follows:

$$(30) \qquad 0 \equiv \int_P p^*\mathscr{L}_Z\varphi + \int_Q q^*\mathscr{L}_Z\psi = \int_P p^*Z \lrcorner\, \mathrm{d}\varphi + \int_Q q^*Z \lrcorner\, (\varphi + \mathrm{d}\psi).$$

Here, $\mathscr{L}_Z$ is the Lie derivative, Z is a vector field on J^∞ satisfying $Z_{p(t)} = \mathrm{d}p(\lambda)/\mathrm{d}\lambda\,|_{\lambda=0}$ at every point $t \in P$. The last condition and (28) imply

$$(31) \qquad Z \text{ is tangent to } R^\infty,\ \mathscr{L}_Z\omega_I^j \equiv 0,$$

at every point $p(t)$ $(t \in P)$. Now, we *postulate* (or *prove*, for particular problems) that also conversely, every vector field Z satisfying (44) corresponds to certain above mentioned family $p(\lambda) \in \boldsymbol{P}$, $p(0) = p$. (This is the common approach, but it can be avoided; see [11].) Under this assumption $p \in \boldsymbol{P}$ *is a critical point of* $\mathscr{F}$ *on* $\boldsymbol{P}$ *if and only if* (30) *holds for every vector* peld Z *satisfying* (31). This result may be considerably improved if we deal with the case of an involutive system.

For simplicity, we shall omit the boundary terms and use the local coordinates. We start with the observation that $\mathscr{F}$ does not change its value, if φ is replaced by $\varphi - \alpha$ $(\alpha \in \Psi_{1,s})$. Moreover, decompose $Z = \sum z_i\, \partial/\partial x^i + \sum z_I^j\, \partial/\partial y_I^j$ into the *horizontal* and the *vertical component,*

$$Z = Z_H + Z_V = \Sigma z_i\, \partial_i + \Sigma \bar{z}_I^j\, \partial/\partial y_I^j, \ (\bar{z}_I^j = z_I^j - \Sigma z_i y_{iI}^j),$$

and use the decomposition $\mathrm{d} = \delta + \partial$. As a result, (30) looks as follows:

(32) $$0 \equiv \int p^*(Z_H \lrcorner\, \partial\varphi + Z_V \lrcorner\, (\delta\varphi - \partial\alpha)) + \int (\ldots).$$

It is natural to assume $\partial\varphi = 0$. (This identity is trivially satisfied in the classical case $\dim P = \dim B$. In general, we have only $0 \equiv \int_P p^* Z_H \lrcorner\, \partial\varphi$, but the horizontal vector field Z_H is almost arbitrary here provided p is a submanifold of an n-dimensional solution of R^∞.) Then $\partial\delta\varphi = -\delta\,\partial\varphi = 0$ and, owing to the involutiveness of R^∞, there exists such form $\alpha \in \Psi_{1,s-1}$ that $\delta\varphi - \partial\alpha \in \Psi^0_{1,s}$. In local coordinates, $\partial\varphi - \partial\alpha = \sum \varepsilon^j_I \omega^j \wedge \mathrm{d}x^I$ ($|I| = s$; especially, $\delta\varphi - \partial\alpha = \sum \varepsilon^j \omega^j \wedge \mathrm{d}x \wedge \ldots \wedge \mathrm{d}x^n$ in the classical case), hence (32) gives the equation

(33) $$0 \equiv \int_P p^* \Sigma \varepsilon^j_I z^j \,\mathrm{d}x^I + \int_Q (\ldots).$$

This equation is valid for every vector field $Z = Z_V$ satisfying (31).

The mentioned conditions (31) may be expressed as a system of linear partial differential equations for the coefficients z^I_j of the vector Z_V. If the set of the lower order coefficients $z^1, \ldots, z^n$ of these solutions is *dense* in the space of all m-tuples of functions, then (33) implies the equations

(34) $$p^* \sum \varepsilon^j_I \,\mathrm{d}x^I \equiv 0 \qquad (p^*\varepsilon \equiv 0 \text{ in the classical case}).$$

These conditions are a generalisation of the *Euler–Lagrange system*. At the same time, the form $\varphi - \alpha$ appears as a generalisation of the famous *Poincaré–Cartan form*. Note that $\mathrm{d}(\varphi - \alpha) - (\delta\varphi - \partial\alpha) \in \Psi_{2,s+1}$ is a 2-contact form, so the condition (33) may be written as

$$0 \equiv \int_P p^* Z_V \lrcorner\, \mathrm{d}(\varphi - \alpha) + \int_Q (\ldots);$$

this is the familiar property of the Poincaré–Cartan form.

The case $J^\infty = R^\infty$, $\dim P = \dim B$, exactly coincides with the *higher order multiple integral variational problem* of the classical calculus of variations. Lately, it was studied in fine details. We refer to [1], [2], and to the forthcomming papers [8], [9].

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