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ON SOME PROPERTIES OF GENOMORPHISMS OF C-ALGEBRAS

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ABSTRACT

The genomorphism concept is a generalization of the homomorphism concept. In this article some properties of genomorphisms of connected mono-unary algebras are studied.

1. BASIC CONCEPTS

1.1. Notation.

(1) If A is a set, we denote by $|A|$ the cardinal number of A.

(2) Let A, B be nonempty sets and φ a mapping of A into B. Then we write $\varphi : A \to B$ and, further, we denote by id_A the identity map of A onto A.

(3) Ord denotes the class of all ordinal numbers. If $\alpha \in \mathrm{Ord}$ then we put $W_\alpha = \{\beta \in \mathrm{Ord};\ \beta < \alpha\}$. Finally, we denote by N the set of all finite ordinal numbers.

(4) Let $\infty, \infty_1, \infty_2 \notin \mathrm{Ord}$. If M is an arbitrary set of ordinal numbers, then we denote by $\leqq$ the order relation on $M \cup \{\infty_1, \infty_2\}$ such that its restriction $\leqq \cap (M \times M)$ to M is the natural order relation of ordinal numbers and that for each $\alpha \in M$ is $\alpha < \infty_1 < \infty_2$.

(5) Let $p, q \in N$, $p \neq 0$, then p/q denotes that p is a divisor of q.

(6) Let A, B be nonempty sets, φ a partial map from A into B. Let $\emptyset \neq C \subseteq A$, $\emptyset \neq D \subseteq B$. Then we put:

(a) $\mathrm{dom}\ \varphi = \{x \in A;\ \exists\, y \in B\colon \varphi(x) = y\}$,

(b) $\varphi(C) = \{\varphi(x) \in B;\ x \in C \cap \mathrm{dom}\ \varphi\}$,

(c) $\varphi^{-1}(D) = \{x \in \mathrm{dom}\ \varphi;\ \varphi(x) \in D\}$.

Further, we denote by $\varphi|_C$ the restriction of φ to C (i.e. the mapping of $C \cap \mathrm{dom}\ \varphi$ into B).

(7) Let A be a nonempty set, $\mathscr{A}$ is the family of n-ary partial operations α_n

defined, for $n \in N$, on a nonempty subset of A^n. Then we denote by the ordered pair $(A; \mathscr{A})$ a partial universal algebra.

If $\emptyset \neq M \subseteq A$, then $[M; \mathscr{A}]$ denotes the subalgebra of $(A; \mathscr{A})$ generated by M (in usual sense). If $M = \{a_1, \ldots, a_k\}$ for some $k \in N - \{0\}$, we write $[M; \mathscr{A}]$ as $[a_1, \ldots, a_k; \mathscr{A}]$.

1.2. Definition. Let $\boldsymbol{A} = (A; \mathscr{A})$, $\boldsymbol{B} = (B; \mathscr{B})$ be partial universal algebras. A mapping $\varphi : \boldsymbol{A} \to \boldsymbol{B}$ is said to be *generative* if for each $\alpha_i \in \mathscr{A}$ of arity $r_i > 0$ and each $(a_1, \ldots, a_{r_i}) \in \operatorname{dom} \alpha_i$ it holds that $\varphi(\alpha_i(a_1, \ldots, a_{r_i})) \in [\varphi(a_1), \ldots, \varphi(a_{r_i}); \mathscr{B}]$. φ is said to be *congruential* if for each $\alpha_i \in \mathscr{A}$ of arity $r_i > 0$ and each $(a_1, \ldots, a_{r_i})$, $(a'_1, \ldots, a'_{r_i}) \in \operatorname{dom} \alpha_i$ with the property $\varphi(a_j) = \varphi(a'_j)$ for each $1 \leqq j \leqq r_i$ it follows

$$\varphi(\alpha_i(a_1, \ldots, a_{r_i})) = \varphi(\alpha_i(a'_1, \ldots, a'_{r_i})).$$

φ is said to be a *genomorphism*, if it is both generative and congruential.

1.3. Lemma. *Let* $\boldsymbol{A} = (A; \mathscr{A})$, $\boldsymbol{B} = (B; \mathscr{B})$ *be partial universal algebras,* $\varphi : \boldsymbol{A} \to \boldsymbol{B}$ *be generative,* $\emptyset \neq S \subseteq A$. *Then* $\varphi([S; \mathscr{A}]) = [\varphi(S); \mathscr{B}]$.

Proof see [1], paragraph 2, lemma 2.

2. UNARY ALGEBRAS

2.1. Definition. Let A be a nonempty set, f a partial map from A into A. Then the ordered pair $(A;f) = \boldsymbol{A}$ is called a *mono-unary algebra.*

2.2. Definition. Let $(A;f)$ be a mono-unary algebra. We put $f^0 = \mathrm{id}_A$. Suppose that we have defined a partial map f^{n-1} from A into A for $n \in N - \{0\}$. We denote by f^n the following partial map from A into A: if $x \in \operatorname{dom} f^{n-1}$ and $f^{n-1}(x) \in \operatorname{dom} f$ then we put $f^n(x) = f(f^{n-1}(x))$.

2.3. Lemma. *Let $(A;f)$ be a mono-unary algebra. Then the following assertions hold:*

(a) *If* $n \in N - \{0\}$, $x \in \operatorname{dom} f^n$, *then* $x \in \operatorname{dom} f^m$ *for each* $m \in \{0, \ldots, n\}$ *and* $f^m(x) \in \operatorname{dom} f$ *for each* $m \in \{0, \ldots, n - 1\}$.

(b) *Let* $n \in N$, $x \in A$ *be arbitrary. Then* $x \in \operatorname{dom} f^n$, *if and only if* $f^p(x) \in \operatorname{dom} f^{q-p}$ *for each* $p, q \in N$, $0 \leqq p \leqq q \leqq n$.

(c) *If* $m, n \in N$, $x \in \operatorname{dom} f^m$, $f^m(x) \in \operatorname{dom} f^n$, *then* $x \in \operatorname{dom} f^{m+n}$, $f^{m+n}(x) = f^n(f^m(x))$, $x \in \operatorname{dom} f^n$, $f^n(x) \in \operatorname{dom} f^m$ *and* $f^m(f^n(x)) = f^n(f^m(x))$.

Proof see [2], 1.6.

2.4. Definition. Let $(A;f)$ be a mono-unary algebra and let $x \in A$ be arbitrary. Then we define $[x]_{(A,f)} = \{f^n(x);\ x \in \operatorname{dom} f^n,\ n \in N\}$.

2.5. Remark. From 1.1. (7), 2.1. and 2.4. it follows immediately that $[x;f] = ([x]_{(A,f)}; f|_{[x]_{(A,f)}})$ for each $x \in A$.

2.6. Definition. Let $(A;f)$ be a mono-unary algebra. For arbitrary $x, y \in A$ we put $(x, y) \in \varrho(A,f)$, if and only if there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$ $y \in \operatorname{dom} f^n$ and $f^m(x) = f^n(y)$. If $\varrho(A,f) = A \times A$, then $(A;f)$ is called a *connected* unary algebra and we denote it briefly a *c-algebra.*

2.7. Definition. Let $(A;f)$ be a c-algebra and $x \in A$ be arbitrary. Then we define $Z(x) = \{y \in A;$ there is an infinite set $N(y) \subseteq N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) = y$ for each $n \in N(y)\}$. Now we put $Z(A,f) = Z(x)$, where $x \in A$ is an arbitrary element, $R(A,f) = | Z(A,f) |$.

Remark. The definition 2.7. is correct — see [2], 2.5. and 2.6.

2.8. Lemma. *Let $(A;f)$ be a c-algebra and $Z(A,f) \neq \emptyset$. If $x \in A$ is arbitrary, then $x \in \operatorname{dom} f^n$ for each $n \in N$ and $Z(A,f) \subseteq [x]_{(A,f)}$.*

Proof. By 2.7. $R(A,f) \neq 0$. Suppose first that $x \in Z(A,f)$. Then $x = f^{k \cdot R(A,f)}(x)$ for each $k \in N$ by [2], 2.12.(a). From 2.3.(a) it follows that $x \in \operatorname{dom} f^n$ for each $n \in N$. Now, let $A - Z(A,f) \neq \emptyset$ and $x \in A - Z(A,f)$ be arbitrary. If $x_0 \in Z(A,f)$ is arbitrary, then there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $x_0 \in \operatorname{dom} f^n$ and $f^m(x) = f^n(x_0)$ by 2.6., and we obtain $x \in \operatorname{dom} f^n$ for each $n \in N$ by 2.3.(a). We put $n_0 = \min \{n \in N;\ x_0 \in \operatorname{dom} f^n$ and $f^n(x_0) = f^m(x)\}$. Clearly $n_0 \leqq R(A,f)$ and $f^{n_0}(x_0) \in Z(A,f)$ by [2], 2.10. and 2.11.(a), which implies $x_0 = f^{R(A,f)}(x_0) = f^{R(A,f)+n_0-n_0}(x_0) = f^{R(A,f)-n_0}(f^{n_0}(x_0)) = f^{R(A,f)-n_0}(f^m(x)) = f^{R(A,f)-n_0+m}(x)$ by the above, 2.3.(b), (c) and [2], 2.11.(a). Thus, $x_0 \in [x]_{(A,f)}$ by 2.4.

2.9. Lemma. *Let $(A;f)$ be a c-algebra, $A - Z(A,f) \neq \emptyset$, $x \in A - Z(A,f)$ arbitrary. If $x' \in [x]_{(A,f)}$ and $x \in [x']_{(A,f)}$ for some $x' \in A$, then $x = x'$.*

Proof. By 2.4. there are $k, l \in N$ such that $x \in \operatorname{dom} f^k$, $x' \in \operatorname{dom} f^l$, $x' = f^k(x)$ and $x = f^l(x')$. Thus, $x = f^l(x') = f^l(f^k(x)) = f^{l+k}(x)$ by 2.3.(b), (c). If $l + k > 0$, then $x \in Z(A,f)$ by [2], 2.8.(a) which is a contradiction. Therefore $l + k = 0$ and $x = x'$ by 2.2.

2.10. Definition. Let $(A;f)$ be a c-algebra. We put $A^\infty = \{x \in A;$ there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N\}$, $A^0 = \{x \in A; f^{-1}(x) = \emptyset\}$. Let $\alpha \in \mathrm{Ord}$, $\alpha > 0$ and suppose that the sets $A^\varkappa$ have been defined for all $\varkappa \in W_\alpha$. Then we put $A^\alpha = \{x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa; f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa\}$.

2.11. Lemma. *Let $(A;f)$ be a c-algebra, $A^\infty \neq \emptyset$ and $x \in A^\infty$ be arbitrary. Then*

(a) $[x]_{(A,f)} \subseteq A^\infty$,

(b) If $(x_i)_{i \in N}$ is such a sequence that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$, then $(x_i)_{i \in N} \subseteq A^\infty$.

Proof.

(a) Since $(A^\infty; f|_{A^\infty})$ is a subalgebra of $(A;f)$ by [2], 2.15.(a), $f^n(x) \in A$ for each $n \in N$ with the property $x \in \operatorname{dom} f^n$ by 2.5., thus, by 2.4., $[x]_{(A,f)} \subseteq A^\infty$.

(b) Let $(x_i)_{i\in N}$ be an arbitrary sequence having required properties (its existence follows from 2.10.).

Now, $x_0 = x \in A^\infty$ by the assumption. Let $n \in N - \{0\}$ be arbitrary. We put $\bar{x}_0 = x_n$, $\bar{x}_j = x_{n+j}$ for each $j \in N$. By the assumption, $\bar{x}_i = x_{i+n} \in \operatorname{dom} f$ and $f(\bar{x}_{i+1}) = f(x_{i+1+n}) = x_{n+i} = \bar{x}_i$ for each $i \in N$ which implies $x_n \in A^\infty$ by 2.10

2.12. Remark. By [2], 2.15.(b), $Z(A,f) \subseteq A^\infty$ for $A^\infty \neq \emptyset$.

2.13. Definition. Let $(A;f)$ be a c-algebra. Then we put $A^{\infty_1} = A^\infty - Z(A,f)$, $A^{\infty_2} = Z(A,f)$.

Notation. Let $(A;f)$ be a c-algebra. Then we put $\vartheta(A,f) = \min\{\vartheta \in \mathrm{Ord}; A^\vartheta = \emptyset\}$. Note that the number $\vartheta(A,f)$ have been defined correctly—see [2], 2.18. and 2.19.

2.14. Theorem. *Let $(A;f)$ be a c-algebra, then* $A = \bigcup_{\varkappa \in W_{\vartheta(A;f)} \cup \{\infty_1,\infty_2\}} A^\varkappa$ *with disjoint terms.*

Proof see [2], 2.22.

2.15. Definition. Let $(A;f)$ be a c-algebra. We define a map $S(A,f) : A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $S(A,f)(x) = \varkappa$ for each $x \in A^\varkappa$, $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty_1, \infty_2\}$. $S(A,f)(x)$ is called the *degree* of x.

Notation. Let $(A;f)$ be a c-algebra, $x \in A - A^0$ arbitrary. If $\alpha \in \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ and $S(A,f)(x') < \alpha$ (or $\leqq$ or $>$ or $\geqq$) for each $x' \in f^{-1}(x)$, then we write $S(A,f)(f^{-1}(x)) < \alpha$ (or $\leqq$ or $>$ or $\geqq$ respectively).

2.16. Lemma. *Let $(A;f)$ be a c-algebra, $\alpha \in \mathrm{Ord}$, $x \in A - A^\infty$. Then the following assertions hold:*

(a) $S(A,f)(x) = \alpha$ if and only if $\alpha \leqq S(A,f)(x)$ and $S(A,f)(f^{-1}(x)) < \alpha$.

(b) If $S(A,f)(f^{-1}(x)) < \alpha$, then $S(A,f)(x) \leqq \alpha$.

Proof see [2], 2.25.(a), (c).

2.17. Lemma. *Let $(A;f)$ be a c-algebra, $x_1 \in A - A^\infty$ and let $x_2 \in [x_1]_{(A,f)} - \{x_1\}$ be arbitrary. Then $S(A,f)(x_1) < S(A,f)(x_2)$.*

Proof. By 2.4. there exists $n \in N$, by 2.2. and by the assumption $n \neq 0$, with the property $x_1 \in \operatorname{dom} f^n$ and $f^n(x_1) = x_2$. Since $S(A,f)(x_1) \in W_{\vartheta(A,f)} \subseteq \mathrm{Ord}$ by 2.13. and 2.14., we have $S(A,f)(x_2) = S(A,f)(f^n(x_1)) = S(A,f)(x_1) + n > S(A,f)(x_1)$ by [2], 2.26.(a).

3. GENOMORPHISMS OF C-ALGEBRAS

3.1. Notation. Let $\boldsymbol{A} = (A;f)$, $\boldsymbol{B} = (B;g)$ be mono-unary algebras. Then we denote by $G(\boldsymbol{A}, \boldsymbol{B})$ the set of all genomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$.

3.2. Lemma. *Let* $A = (A; f)$, $B = (B; g)$ *be mono-unary algebras. Then* $\varphi \in G(A, B)$ if and only if

1. *for each* $x \in \operatorname{dom} f$ $\varphi(f(x)) \in [\varphi(x); g]$ *holds,*
2. $\varphi(f(x)) = \varphi(f(x'))$ *for each* $x, x' \in \operatorname{dom} f$ *having the property* $\varphi(x) = \varphi(x')$.

Proof. This assertion follows immediately from 1.2. and 2.1.

3.3. Definition. Let $(A; f)$ be a mono-unary algebra, $x \in A$ arbitrary. Then we put $C_k(x) = \{x' \in A;\ x' \in \operatorname{dom} f^k \text{ and } f^k(x') = x\}$ for each $k \in N$ and $C(x) = \bigcup_{k \in N} C_k(x)$.

3.4. Lemma. *Let* $A = (A; f)$, $B = (B; g)$ *be c-algebras,* $\varphi \in G(A, B)$ *and* $x \in A$ *such that there is* $k_0 \in N - \{0\}$ *with the property* $\varphi(f^{k_0}(x)) = \varphi(x)$. *Then the following assertions hold:*

(*a*) $\varphi(f^m(x)) = \varphi(f^{m+k_0}(x))$ *for each* $m \in N$ *such that* $x \in \operatorname{dom} f^{m+k_0}$,

(*b*) $\varphi(f^m(x)) = \varphi(f^{m+nk_0}(x))$ *for each* $m \in N$ *and* $n \in N - \{0\}$ *such that* $x \in \operatorname{dom} f^{m+nk_0}$,

(*c*) *if* $k_0 = 1$, *then* $\varphi([x; f]) = \varphi(x)$.

Proof.

(a) By the assumption, the assertion holds for $m = 0$ by 2.2. Now, let the assertion hold for some $m \in N$ with the property $x \in \operatorname{dom} f^{m+1+k_0}$. Then $\varphi(f^{m+1}(x)) = \varphi(f(f^m(x)))$ by 2.3.(b), (c), (a), $\varphi(f(f^m(x))) = \varphi(f(f^{m+k_0}(x)))$ by 3.2. and the induction hypothesis and $\varphi(f(f^{m+k_0}(x))) = \varphi(f^{m+1+k_0}(x))$ by 2.3.(c).

(b) Let $m \in N$ be arbitrary such that $x \in \operatorname{dom} f^{m+k_0}$. Then $x \in \operatorname{dom} f^m$ by 2.3.(a) and $\varphi(f^m(x)) = \varphi(f^{m+k_0}(x))$ by (a). Let the assertion hold for some $n \in N - \{0\}$ such that $x \in \operatorname{dom} f^{m+(n+1)k_0}$. Then, by 2.3.(a), $x \in \operatorname{dom} f^m$ and $\varphi(f^m(x)) = \varphi(f^{m+nk_0}(x))$ by the induction hypothesis. Further, from 2.3.(a) and (a) it follows $\varphi(f^{m+nk_0}(x)) = \varphi(f^{(m+nk_0)+k_0}(x)) = \varphi(f^{m+(n+1)k_0}(x))$.

(c) By 2.5. it is sufficient to prove that $\varphi(x) = \varphi(f^k(x))$ for each $k \in N$ such that $x \in \operatorname{dom} f^k$. This holds for $k = 1$ by the assumption. Let the assertion hold for some $k \in N - \{0\}$ with the property $x \in \operatorname{dom} f^{k+1}$. Then $x \in \operatorname{dom} f^{k+k_0}$ and we obtain $\varphi(x) = \varphi(f^k(x)) = \varphi(f^{k+k_0}(x)) = \varphi(f^{k+1}(x))$ by (a) and the induction hypothesis.

3.5. Lemma. *Let* $A = (A; f)$, $B = (B; g)$ *be c-algebras and* $\varphi \in G(A, B)$. *Then the following assertions hold:*

(*a*) *If* $x \in A$ *is an arbitrary element, then* $\varphi(C(x)) \subseteq C(\varphi(x))$.

(*b*) *Let* $x_2 \in [x_1]_{(A,f)} - \{x_1\}$ *and* $\varphi(x_1) = \varphi(x_2) \notin Z(B, g)$. *Then* $\varphi([x_1; f]) = \varphi(x_1)$.

(*c*) *If* $Z(A, f) \neq \emptyset$ *and for some* $x \in Z(A, f)$ $\varphi(x) \notin Z(B, g)$, *then* $\varphi(Z(A, f)) = \varphi(x)$.

Proof.

(a) Let $x' \in C(x)$ be arbitrary. By 3.3., there exists $k \in N$ such that $x' \in \operatorname{dom} f^k$

and $f^k(x') = x$ which implies $x \in [x';f]$ by 2.5. From 3.2. and 1.3. it follows that $\varphi(x) \in [\varphi(x'); g]$, thus, by 2.5., there is $l \in N$ having the property $g^l(\varphi(x')) = \varphi(x)$. Hence, $\varphi(x') \in C(\varphi(x))$ by 3.3.

(b) By the assumption and 2.4. there exists $k \in N - \{0\}$ with the property $x_1 \in \operatorname{dom} f^k$ and $f^k(x_1) = x_2$. It is sufficient to prove that $\varphi(x_1) = \varphi(f(x_1))$ because this implies $\varphi([x_1;f]) = \varphi(x_1)$ by 3.4.(c). Indeed, $x_2 \in [f(x_1);f]$ by 2.5. because $x_2 \neq x_1$ and $\varphi(f(x_1)) \in [\varphi(x_1); g]$ by 3.2. Further, from 3.2. and 1.3. it follows $\varphi(x_1) = \varphi(x_2) \in [\varphi(f(x_1)); g]$, thus, by 2.5. and 2.9., $\varphi(x_1) = \varphi(f(x_1))$.

(c) If $R(A,f) = 1$, then the assertion follows directly from 2.7. Let $R(A,f) > 1$ and $x' \in Z(A,f) - \{x\}$ be arbitrary. From 2.5., 2.8. and [2], 2.10. it follows that $[x';f] = [x;f] = Z(A,f)$. Hence, $x \in [x';f]$ and $x' \in [x;f]$ and $\varphi(x) \in [\varphi(x'); g]$, $\varphi(x') \in [\varphi(x); g]$ by 3.2. and 1.3. which implies $\varphi(x) = \varphi(x')$ by 2.5. and 2.9. Therefore, $\varphi(Z(A,f)) = \varphi(x)$.

3.6. Lemma. *Let $\mathbf{A} = (A;f)$, $\mathbf{B} = (B;g)$ be c-algebras such that $R(B,g) \neq 0$, and let $\varphi \in G(\mathbf{A}, \mathbf{B})$. Let $x \in A$ be such that there exists $x' \in [x]_{(A,f)} - \{x\}$ with the property $\varphi(x) = \varphi(x') \in Z(B,g)$. Then there is $l \in N - \{0\}$ such that $\varphi(x') = \varphi(f^l(x'))$ for each $x' \in [x]_{(A,f)}$ and such that $l \neq 1$ implies that $\varphi(x'), \ldots, \varphi(f^{l-1}(x'))$ are mutually distinct for each $x' \in [x]_{(A,f)}$ and $l / R(A,f)$ for $R(A,f) \neq 0$.*

Proof. By 2.4. there is $k \in N - \{0\}$ such that $x \in \operatorname{dom} f^k$ and $f^k(x) = x'$. From 2.5., [2], 2.10.; 3.2. and 1.3. it follows that $\varphi([x;f]) \subseteq Z(B,g)$. Let us consider $l = \min\{k \in N - \{0\};\ \varphi(f^k(x)) = \varphi(x)\}$ (its existence is evident). We show that l have all required properties:

1. From 3.4.(a) it follows that $\varphi(f^m(x)) = \varphi(f^{m+l}(x))$ for each $m \in N$ with the property $x \in \operatorname{dom} f^{m+l}$ which implies that $\varphi(x') = \varphi(f^l(x'))$ for each $x' \in [x]_{(A,f)}$ by 2.4. and 2.3.(b), (c). Further, if $l = 1$, then for $R(A,f) \neq 0$ $l/R(A,f)$ trivially.

2. Let $l > 1$.

(a) From the minimality of l it follows that $\varphi(x) \neq \varphi(f^k(x))$ for each $k \in \{1, \ldots, l-1\}$.

b) Further, let us admit that there are $i, j \in N$, $1 \leqq i < j \leqq l - 1$, such that $\varphi(f^i(x)) = \varphi(f^j(x))$. Hence $\varphi(f^i(x)) = \varphi(f^j(x)) = \varphi(f^{j-i}(f^i(x)))$ by 2.3.(b), (c) and $\varphi(f^{m+i}(x)) = \varphi(f^m(f^i(x))) = \varphi(f^{m+(j-i)}(f^i(x))) = \varphi(f^{m+j}(x))$ for each $m \in N$ with the property that $x \in \operatorname{dom} f^{m+j}$ by 2.3.(b), (c) and 3.4.(a) which implies, for $m = k$ such that $l = j + k$, that $\varphi(f^{k+i}(x)) = \varphi(f^{k+j}(x)) = \varphi(f^l(x)) = \varphi(x)$ where $x \in \operatorname{dom} f^{i+k}$ by 2.3.(a). Since $k \in N - \{0\}$ (because $j < l$) and $1 \leqq i + k < j + k = l$ by the assumption, $\varphi(f^{k+i}(x)) = \varphi(x)$ is a contradiction to (a). Therefore, $\varphi(x), \ldots, \varphi(f^{l-1}(x))$ are mutually distinct elements of $Z(B, g)$.

(c) Now, we prove that for each $m \in N$ such that $x \in \operatorname{dom} f^{m+l}$ $\varphi(f^m(x)), \ldots, \varphi(f^{m+l-1}(x))$ are mutually distinct: by (b), this assertion holds for $m = 0$. Let $m \in N$ be such that $\varphi(f^m(x)), \ldots, \varphi(f^{m+l-1}(x))$ are mutually distinct and

$x \in \operatorname{dom} f^{m+l}$. Then $\{\varphi(f^{m+1}(x)), \ldots, \varphi(f^{(m+1)+l-2}(x)), \varphi(f^{(m+1)+l-1}(x))\} = \{\varphi(f^{m+1}(x), \ldots, \varphi(f^{m+l-1}(x)), \varphi(f^{m+l}(x))\} = \{\varphi(f^{m+1}(x)), \ldots, \varphi(f^{m+l-1}(x)), \varphi(f^{m}(x))\}$ with the mutually distinct elements by 1., 2.3.(c) and the induction hypothesis. However, from 2.3.(b), (c) and 2.4. it follows that $\varphi(x'), \ldots, \varphi(f^{l-1}(x'))$ are mutually distinct for each $x' \in [x]_{(A,f)}$.

(d) Let $R(A,f) \neq 0$ and $x' \in Z(A,f)$ be arbitrary. Clearly $l \leqq R(A,f)$. (See (c) and [2], 2.11.(a)). If $l = R(A,f)$, then $l/R(A,f)$ trivially. Let $l < R(A,f)$ and suppose on the contrary that $l \not/ R(A,f)$. Then there are $i, j \in N - \{0\}$, $j < l$ such that $R(A,f) = il + j$. Now, from 1., 2.3.(b), (c), 3.4.(b) and [2], 2.11.(a) it follows that $\varphi(x') = \varphi(f^{R(A,f)}(x')) = \varphi(f^{il+j}(x')) = \varphi(f^{il}(f^{j}(x'))) = \varphi(f^{j}(x'))$ which is a contradiction to (c) by 2.8. Thus, $l/R(A,f)$.

3.7. Lemma. *Let $\boldsymbol{A} = (A,f)$, $\boldsymbol{B} = (B; g)$ be c-algebras, $A^{\infty} \neq \emptyset$ and $\varphi \in G(\boldsymbol{A}, \boldsymbol{B})$. Then the following assertions hold*:

(*a*) *If there is a sequence $(x_i)_{i \in N} \subseteq A^{\infty}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $f(x_{i+1}) = x_i$ for each $i \in N$ and if $|\varphi((x_i)_{i \in N})| > 1$, then $B^{\infty} \neq \emptyset$ and $(\varphi(x_i))_{i \in N} \subseteq B^{\infty}$.*

(*b*) *If there exists $x \in A^{\infty}$ with the property $\varphi(x) \notin B^{\infty}$, then $\varphi(A^{\infty}) = \varphi(x)$.*

(*c*) *If $|\varphi(A^{\infty})| > 1$, then $\varphi(A^{\infty}) \subseteq B^{\infty}$.*

Proof.

(a) By 2.5., $x_i \in \lfloor x_{i+1}; f\rfloor$ for each $i \in N$ which implies $\varphi(x_i) \in [\varphi(x_{i+1}); g]$ for each $i \in N$ by 3.2. and 1.3., thus for each $i \in N$ there is $l_i \in N$ such that $\varphi(x_{i+1}) \in \operatorname{dom} g^{l_i}$ and $\varphi(x_i) = g^{l_i}(\varphi(x_{i+1}))$. By the assumption, there are $i_1, i_2 \in N$, $i_1 \neq i_2$ such that $\varphi(x_{i_1}) \neq \varphi(x_{i_2})$. Let, for example, $i_1 < i_2$. Then $x_{i_1} \in [x_{i_2}]_{(A,f)} - \{x_{i_2}\}$ by 2.3.(b) and 2.4. and there exists $i \in N$, $i_1 \leqq i \leqq i_2 - 1$ such that $l_i \neq 0$. We put $n_0 = \min\{i \in N; l_i \neq 0\}$. Then, for $n_0 \neq 0$, $l_k = 0$ for each $k \in \{0, \ldots, n_0 - 1\}$ by the above, i.e. $\varphi(x_k) = \varphi(x_{n_0})$ for each $k \in \{0, \ldots, n_0\}$, and $l_k \neq 0$ for each $k \geqq n_0$: suppose on the contrary that there is $j \in N$, $j \geqq n_0$, with the property $l_j = 0$. Since $l_{n_0} \neq 0$, then $j > n_0$. By the above and 2.2. $\varphi(f(x_{j+1})) = \varphi(x_j) = g^{l_j}(\varphi(x_{j+1})) = g^{0}(\varphi(x_{j+1})) = \varphi(x_{j+1})$ which implies $\varphi([x_{j+1}; f]) = \varphi(x_{j+1})$ by 3.4.(c), thus $l_k = 0$ for each $k \in N$, $k \leqq j$, by 2.2. and 2.5. Hence $l_{n_0} = 0$ which is a contradiction. Now, we may put $y = y_0$, $m_0 = 0$, $m_i = m_{i-1} + l_{n_0+i-1}$ for each $i \in N - \{0\}$ and $\varphi(x_{n_0+i}) = y_{m_i}$ for each $i \in N$, $y_{m_i-k} = g^{k}(y_{m_i})$ for each $k \in \{1, \ldots, l_{n_0+i-1}\}$ and each $i \in N - \{0\}$ in virtue of 2.3.(a), (b). From 2.3.(a), (b) and the above it follows that $y_i \in \operatorname{dom} g$ for each $i \in N - \{0\}$. Further, if $j \in N$ is arbitrary, then there is $i \in N$ such that $j = m_i - k$ for some $k \in \{1, \ldots, l_{n_0+i-1}\}$ and we obtain $y_j = y_{m_i-k} = g^{k}(y_{m_i}) = g(g^{k-1}(y_{m_i})) = g(y_{m_i-k+1}) = g(y_{j+1})$ by 2.3.(a), (c). Thus, $y_i = g(y_{i+1})$ for each $i \in N$. From the above it follows that $\varphi(x_0) = \varphi(x_{n_0}) = y_0 = y \in B^{\infty}$, hence $B^{\infty} \neq \emptyset$. Finally, $(\varphi(x_i))_{i \in N} \subseteq (y_i)_{i \in N} \subseteq B^{\infty}$ by 2.11.(b).

(b) Let $x \in A^{\infty}$ with the property $\varphi(x) \notin B^{\infty}$ be arbitrary but fixed and let us consider an arbitrary element $x' \in A^\infty - \{x\}$. By 2.6. there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $x' \in \operatorname{dom} f^n$ and $f^m(x) = f^n(x')$. We put $f^m(x) = f^n(x') = \bar{x}$. By 2.4., $\bar{x} \in [x]_{(A,f)}$, $\bar{x} \in [x']_{(A,f)}$ and by 2.11.(a) $\bar{x} \in A^\infty$. Further, there is a sequence $(\bar{x}_i)_{i \in N} \subseteq A^\infty$ such that $\bar{x} = \bar{x}_0$, $\bar{x}_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $f(\bar{x}_{i+1}) = \bar{x}_i$ for each $i \in N$ and $x \in \{\bar{x}_i; i \in N\}$. To prove it we take an arbitrary sequence $(x_i)_{i \in N} \subseteq A^\infty$ such that $x = x_0$, $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$ and $f(x_{i+1}) = x_i$ for each $i \in N$ (its existence follows from 2.10. and 2.11.(b)) and in virtue of 2.11(a) and 2.4. we put $\bar{x}_i = f^{m-i}(x)$ for each $i \in \{0, \ldots, m\}$ and $\bar{x}_{m+i} = x_i$ for each $i \in N - \{0\}$. Now, if there is $i_0 \in N - \{0\}$ such that $\varphi(\bar{x}_0) \neq \varphi(\bar{x}_{i_0})$, then $(\varphi(\bar{x}_i))_{i \in N} \subseteq B^\infty$ by (a), thus $\varphi(x) = \varphi(\bar{x}_m) \in B^\infty$ which is a contradiction. Therefore, $\varphi((\bar{x}_i)_{i \in N}) = \varphi(\bar{x}_0)$ which implies $\varphi(\bar{x}) = \varphi(\bar{x}_0) = \varphi(\bar{x}_m) = \varphi(x) \notin B^\infty$. Similarly we can prove that $\varphi(x') = \varphi(\bar{x})$ because, by the above, $\varphi(\bar{x}) \notin B^\infty$. Thus, $\varphi(x) = \varphi(\bar{x}) = \varphi(x')$. Since x' has been selected arbitrary, we have $\varphi(A^\infty) = \varphi(x)$.

(c) Suppose on the contrary that there is $x \in A^\infty$ such that $\varphi(x) \notin B^\infty$. Then $\varphi(A^\infty) = \varphi(x)$ by (b), thus $|\varphi(A^\infty)| = 1$ which is a contradiction. Therefore $\varphi(A^\infty) \subseteq B^\infty$.

3.8. Lemma. *Let* $\mathbf{A} = (A, f)$, $\mathbf{B} = (B, g)$ *be c-algebras,* $\varphi \in G(\mathbf{A}, \mathbf{B})$. *Then the following assertions hold*:

(a) If $x_2 \in [x_1]_{(A,f)}$, $S(B, g)(\varphi(x_1)) = S(B, g)(\varphi(x_2)) \neq \infty$, *then* $\varphi(x_1) = \varphi(x_2)$.

(b) If $x_2 \in [x_1; f]$, *then* $S(B, g)(\varphi(x_1)) \leqq S(B, g)(\varphi(x_2))$.

(c) Let $x \in A$ *be such that* $S(A, f)(x) \in \mathrm{Ord} - \{0\}$ *and* $S(A, f)(x) > S(B, g)(\varphi(x))$. *Then there exists* $x' \in f^{-1}(x)$ *having the property* $\varphi(x') = \varphi(x)$.

Proof.

(a) From 2.4., 3.2. and 1.3. it follows that $\varphi(x_2) \in [\varphi(x_1)]_{(B,g)}$. If $\varphi(x_1) \neq \varphi(x_2)$, then $S(B, g)(\varphi(x_1)) \neq S(B, g)(\varphi(x_2))$ by the assumption, 2.14, 2.15. and 2.17. which is a contradiction. Thus $\varphi(x_1) = \varphi(x_2)$.

(b) By 2.5. there is $k \in N$ such that $x_1 \in \operatorname{dom} f^k$ and $f^k(x_1) = x_2$. If $k = 0$. then the assertion holds trivially. Let $k \in N - \{0\}$. Then $\varphi(x_2) \in [\varphi(x_1); g]$ by 3.2, and 1.3. If $S(B, g)(\varphi(x_1)) \in \{\infty_1, \infty_2\}$ then the assertion follows from 2.5., 2.13., 2.11.(a), 2.12. and [2], 2.10.

If $S(B, g)(\varphi(x_1)) \in W_{\vartheta(A,f)}$, then the assertion follows from 2.5., 2.14., 2.15. and [2], 2.26.(a).

(c) Let $S(A, f)(x) = 1$. Then $S(B, g)(\varphi(x)) = 0$ and, by 2.10, 2.15. and 3.5.(a), $\varphi(x') = \varphi(x)$ for each $x' \in f^{-1}(x)$. Let $S(A, f)(x) \in \mathrm{Ord} - \{0, 1\}$ and $S(A, f)(x) > S(B, g)(\varphi(x))$. We denote by α the ordinal number $S(A, f)(x)$. Suppose that the assertion holds for each $x' \in A$ with the property $S(A, f)(x') < \alpha$. By 2.16.(a), $S(A, f)(f^{-1}(x)) < \alpha$. Assume first that there is $x' \in f^{-1}(x)$ with the property $S(A, f)(x') > S(B, g)(\varphi(x'))$. Now, the induction hypothesis implies that there exists $x_0 \in f^{-1}(x')$ with the property $\varphi(x_0) = \varphi(x')$, thus $\varphi([x_0; f]) = \varphi(x_0) = \varphi(x')$ by 3.4.(c). Hence $\varphi(x) = \varphi(f^2(x_0)) = \varphi(x')$ by 2.5. and 2.3.(c). Let

$S(A,f)(x') \leqq S(B,g)(\varphi(x'))$ for each $x' \in f^{-1}(x)$. By (b) and 2.5. we obtain $S(B,g)(\varphi(x')) \leqq S(B,g)(\varphi(x))$ for each $x' \in f^{-1}(x)$. If there exists $x' \in f^{-1}(x)$ such that $S(B,g)(\varphi(x)) = S(B,g)(\varphi(x'))$, then $\varphi(x) = \varphi(x')$ by the assumption, (a) and 2.14. Finally we prove that $S(B,g)(\varphi(x')) < S(B,g)(\varphi(x))$ for each $x' \in f^{-1}(x)$ cannot occur: in this case, $S(A,f)(x) > S(B,g)(\varphi(x)) > S(B,g)(\varphi(x')) \geqq \geqq S(A,f)(x')$ for each $x' \in f^{-1}(x)$, thus $S(A,f)(f^{-1}(x)) < S(B,g)(\varphi(x))$ and from 2.16.(b) it follows that $S(A,f)(x) \leqq S(B,g)(\varphi(x))$ which is a contradiction to the assumption that $S(A,f)(x) > S(B,g)(\varphi(x))$.

3.9. Lemma. *Let* $\mathbf{A} = (A; f)$, $\mathbf{B} = (B; g)$ *be c-algebras,* $\varphi \in G(\mathbf{A}, \mathbf{B})$ *and* $x \in A$ *be such that* $S(A,f)(x) > S(B,g)(\varphi(x))$. *Then* $\varphi([x; f]) = \varphi(x)$.

Proof. By 1.1.(4), 2.10., 2.13. and 2.15. the following cases can occur:

(1) $S(B,g)(\varphi(x)) = \infty_1$. Then $S(A,f)(x) = \infty_2$ and the assertion follows from 2.5., 3.5.(c) and [2], 2.10.

(2) $S(B,g)(\varphi(x)) \in \mathrm{Ord}$.

(a) If $S(A,f)(x) \in \{\infty_1, \infty_2\}$, then the assertion follows from 2.5., 2.8., 2.10., 2.11.(a), 2.12., 2.13., 2.14., 3.7.(b) and from [2], 2.10., 2.15.(a).

(b) If $S(A,f)(x) \in \mathrm{Ord}$, then $S(A,f)(x) \neq 0$ by the assumption, from 3.8.(c) it follows that there is $x' \in f^{-1}(x)$ with the property $\varphi(x') = \varphi(x)$ and the assertion follows from 3.4.(c).

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