Jan Chrastina

A double complex related with a system of partial differential equations. I

Archivum Mathematicum, Vol. 19 (1983), No. 2, 83--97

Persistent URL: `http://dml.cz/dmlcz/107159`

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIX: 83—98, 1983

A DOUBLE COMPLEX RELATED WITH A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS I

JAN CHRASTINA, Brno

(Received June 10, 1982)

1. Introduction. We shall deal with the infinite-dimensional space J^∞, where coordinates are

$$x^i,\ y^j,\ y^j_{i_1},\ y^j_{i_1 i_2},\ \dots,$$

$(i, i_1, i_2, \dots = 1, \dots, n;\ i_1 \leqq i_2 \leqq \dots;\ j = 1, \dots, m)$. We shall use the following convention: If a multiindex $I = i_1 \dots i_s$ is only a reordering of a non-decreasing multiindex $I' = i'_1 \dots i'_s$ $(i'_1 \leqq \dots \leqq i'_s)$, we put

$$y^j_I = y^j_{i_1 \dots i_s} = y^j_{I'} = y^j_{i'_1 \dots i'_s}.$$

Also, we employ the advantageous multiindex notation illustrated by the examples

$$|I| = s, \qquad |J| = r, \qquad |\mathscr{I}| = |I_1| + \dots + |I_r|,$$
$$iI = ii_1 \dots i_s, \qquad I\mathscr{I} = (I, I_1, \dots, I_r),$$
$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_s}, \qquad \odot\, \partial^s/\partial t^I = \partial/\partial^{\,i_1} \odot \dots \odot \partial/\partial^{\,i_s},$$
$$\partial^s/\partial x^I = \partial^s/\partial x^{i_1} \dots \partial x^{i_s}, \qquad \partial_I = \partial_{i_1} \circ \dots \circ \partial_{i_s}, \qquad \omega^J_{\mathscr{I}} = \omega^{j_1}_{I_1} \wedge \dots \wedge \omega^{j_r}_{J_r},$$

where $I = i_1 \dots i_s$, $J = J_1 \dots j_r$ are multiindices and $\mathscr{I} = (I_1, \dots, I_r)$ is a multiindex of multiindices.

We consider only functions and differential forms dependent C^∞-smoothly on a finite number of the above variables. Especially, we have the *contact forms* $\omega^j_I = \mathrm{d}y^j_I - \Sigma y^j_{iI}\,\mathrm{d}x^i$ and the *base forms* $\mathrm{d}x^i$. Denoting by $\Phi_{r,s}$ the space of *r-contact* $(r + s)$*-forms*

$$\varphi = \Sigma f^{J,I}_{\mathscr{I}} \omega^J_{\mathscr{I}} \wedge \mathrm{d}x^I \qquad (|I| = s,\ |J| = r,\ \mathscr{I} = (I_1, \dots, I_r)), \tag{1}$$

the space Φ of all differential forms on the space J^∞ is the direct sum of these spaces $\Phi_{r,s}$.

Since $\mathrm{d}\omega^j_I = -\Sigma\omega^j_{iI} \wedge \mathrm{d}x^i$, the exterior differential maps every space $\Phi_{r,s}$ into the direct sum $\Phi_{r+1,s} \oplus \Phi_{r,s+1}$. We get two commuting differentials

$$(2) \qquad \delta : \Phi_{r,s} \to \Phi_{r+1,s}, \qquad \partial : \Phi_{r,s} \to \Phi_{r,s+1}, \qquad \mathrm{d} = \delta + \partial.$$

They are explicitly determined by $\delta f = \Sigma\, \partial f/\partial y^j_I \,.\, \omega^j_I$, $\partial f = \Sigma\, \partial_i f \,.\, \mathrm{d}x^i$, $\delta\omega^j_I = \delta\, \mathrm{d}x^i = \partial\, \mathrm{d}x^i \equiv 0$, $\partial\omega^j_I = -\Sigma\, \omega^j_{iI} \wedge \mathrm{d}x^i$, where

$$(3) \qquad \partial_i = \frac{\partial}{\partial x^i} + \Sigma\, y^j_{iI} \frac{\partial}{\partial y^j_I}$$

are the formal derivative operators.

At last, let Ξ_s ($s = 1, \ldots, n$) be the space of all s-forms dependent only on the variables $x^1, \ldots, x^n$, and denote by i various inclusion mappings. The main result of the paper [1] asserts that *the diagram* (4) *is an exact double complex, with an exception of the bottom parts* $\Phi_{r,n-1} \to \Phi_{r,n} \to 0$.

$$(4) \qquad \begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \mathbf{R} & & \mathbf{R} & & 0 & & 0 & & \\
 & & \downarrow{\scriptstyle \mathrm{i}} & & \downarrow{\scriptstyle \mathrm{i}} & & \downarrow & & \downarrow & & \\
0 & \to & \Xi_0 & \xrightarrow{\mathrm{i}} & \Phi_{0,0} & \xrightarrow{\delta} & \Phi_{1,0} & \xrightarrow{\delta} & \Phi_{2,0} & \xrightarrow{\delta} & \ldots \\
 & & \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \to & \Xi_1 & \xrightarrow{\mathrm{i}} & \Phi_{0,1} & \xrightarrow{\delta} & \Phi_{1,1} & \xrightarrow{\delta} & \Phi_{2,1} & \xrightarrow{\delta} & \ldots \\
 & & \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \to & \Xi_n & \xrightarrow{\mathrm{i}} & \Phi_{0,n} & \xrightarrow{\delta} & \Phi_{1,n} & \xrightarrow{\delta} & \Phi_{2,n} & \xrightarrow{\delta} & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

2. Setting of the problem. We will consider the diagram (4) under the presence of certain relations among the coordinates. So, let f_ι (ι varies in an index set) be given functions on the space J^∞; we try to restrict the diagram (4) to the subset $R^\infty \subset J^\infty$ consisting of all points where the functions f_ι vanish. Assuming the differentials $\mathrm{d}f_\iota$ linearly independent at every point of R^∞, the meaning of the restriction of the spaces $\Phi_{r,s}$, Ξ_s to the subset R^∞ is quite clear, however, some ambiguities concerning the differentials δ, ∂ still remain.

At this place, we remind the following property of the exterior differential d: If a form φ identically vanishes as a consequence of certain relations $f_\iota \equiv 0$ (denote this fact briefly by $\varphi^\sim = 0$), then $(\mathrm{d}\varphi)^\sim = 0$, too. Now, we will impose the *requirement* that $\varphi^\sim = 0$ implies in addition also $(\delta\varphi)^\sim = 0$. Owing to $\mathrm{d} = \delta + \partial$, the last condition is equivalent to $(\partial\varphi)^\sim = 0$. It is clear that, under this requirement, the differentials δ, ∂ turn into intrinsic operators on the set R^∞: If $\varphi = \psi$ on R^∞, then $\delta\varphi = \delta\psi$, $\partial\varphi = \partial\psi$ on R^∞, too. As a result, the restriction of the diagram (4) to R^∞ makes a sense.

The above requirement may be simplified as follows: The space Φ is generated by contact forms, base forms and functions. Since $\partial\omega_I^j = \mathrm{d}\omega_I^j$ and $\partial\,\mathrm{d}x^i = \mathrm{dd}x^i$ $(= 0)$, it is sufficient to require that the relation $f^\sim = 0$ implies $(\partial f)^\sim = 0$, for every function f. But the condition $f^\sim = 0$ means that f lies in the ideal with generators f_ι (i.e. $f = \Sigma g_\iota f_\iota$, finite sum). Therefore, it is sufficient to require $(\partial f_\iota)^\sim \equiv 0$. Supposing in addition $(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)^\sim \neq 0$ everywhere on R^∞, the last requirement is equivalent with the condition $(\partial_i f_\iota)^\sim \equiv 0$, for all i, ι.

The result admits a nice interpretation in terms of differential equations. Assign to every function $f\,(= f(x^1, \ldots, y_I^j, \ldots))$ the differential operator $\mathscr{F}$,

$$(\mathscr{F}z)(x^1, \ldots) = f(x^1, \ldots, \partial^{|I|}z^j(x^1, \ldots)/\partial x^I, \ldots),$$

acting on m-tuples of functions $z^j(x^1, \ldots, x^n)$ $(j = 1, \ldots, m)$. Then the above conditions may be expressed by saying that the system of partial differential equations $\mathscr{F}_\iota = 0$ (operator $\mathscr{F}_\iota$ corresponds to the function f_ι) is *closed* (*passive, formally integrable*). That means, all equations $\partial\mathscr{F}_\iota z/\partial x^i = 0$ are merely the algebraic consequences of the system $\mathscr{F}_\iota = 0$.

Denoting by $(4)^\sim$ the restriction of the double complex (4) to the set R^∞, we are able to outline the main results of the present part of the paper as follows: *All rows of* $(4)^\sim$ *are locally exact. All columns of* $(4)^\sim$, *with he exceptions of the above mentioned bottoms, are locally exact if we deal with a system of essential order* $\leqq 0$. *All homology classes appearing in the columns of* $(4)^\sim$ *are expressible by differential forms in* $\mathrm{d}x^1, \ldots, \mathrm{d}x^n, \mathrm{d}y^1, \ldots, \mathrm{d}y^m$ *if we deal with an involutive system*. In any case, the behavior of the rows depends only on homology of R^∞, but the columns reflect certain deep algebraic and analytic properties of the system under consideration.

Although these results are very natural and important for certain applications, they are not quite strong for proving the converse statements. We shall postpone this question to the second part of the present paper. Note that the local exactness mentioned above is means in the sheaf sense: A complex $\ldots \to U \xrightarrow{u} V \xrightarrow{v} W \to \ldots$ of sections of certain bundles over a space Y is called *locally exact in the term* V, if for every $\alpha \in V$, $v\alpha = 0$ near a point $y \in Y$, there exists $\beta \in U$, $u\beta = 0$ near y. The remaining concepts used above will be precised in the main text.

NOTES ON HOMOLOGY

3. Tensor spaces. We come out with a finite-dimensional and non-trivial real vector spaces V, T, the tensor power spaces $\overset{k}{\otimes} T$, the spaces of symmetric tensors $\overset{k}{\odot} T$, the spaces of skew-symmetric tensors $\overset{k}{\wedge} T$, and the corresponding dual spaces V^*, T^*, $\otimes T^*$, $\odot T^*$, $\wedge T^*$. All these tensor spaces are defined to be $\mathbf{R}$ for the case $k = 0$, and to consists of the single zero vector if $k < 0$. The bilinear duality form will be allways denoted by $\langle \cdot \mid \cdot \rangle$.

We shall use the operations of tensor multiplication, contraction, symmetrisation $\mathscr{S}$ and alternation $\mathscr{A}$. The spaces of symmetric and skew-symmetric tensors are considered as subspaces of the relevant tensor product space. An illustration by few examples: If $X, Y \in T$, $\xi, \eta \in T^*$, then

$$\mathscr{S}(X \otimes Y) = X \otimes Y + Y \otimes X = X \odot Y \in \overset{2}{\odot} T \subset \overset{2}{\otimes} T,$$

$$\mathscr{A}(X \otimes Y) = X \otimes Y - Y \otimes X = X \wedge Y \in \overset{2}{\wedge} T \subset \overset{2}{\otimes} T,$$

$$\langle X \otimes Y \mid \xi \otimes \eta \rangle = \langle X \mid \xi \rangle \langle Y \mid \eta \rangle,$$

$$\langle X \odot Y \mid \xi \odot \eta \rangle = 2(\langle X \mid \xi \rangle \langle Y \mid \eta \rangle + \langle X \mid \eta \rangle \langle Y \mid \xi \rangle),$$

$$\langle X \wedge Y \mid \xi \wedge \eta \rangle = 2(\langle X \mid \xi \rangle \langle Y \mid \eta \rangle - \langle X \mid \eta \rangle \langle Y \mid \xi \rangle).$$

There is an outstanding tensor, the *identity tensor* $\mathrm{Id} \in T \otimes Z^*$. It corresponds to the *identity mapping* id: $T \to T$, $T^* \to T^*$. If we choose dual bases $\bar{X}_1, \ldots, \bar{X}_n$ in the space T and $\bar{\xi}_1, \ldots, \bar{\xi}_n$ in the space T^*, then $\mathrm{Id} = \Sigma \bar{X}_i \otimes \bar{\xi}_i$. At last, denoting $|T| = \bar{X}_1 \wedge \ldots \wedge \bar{X}_n$, we remind the operator $\lrcorner\, |T| : \overset{k}{\wedge} T^* \to \overset{n-k}{\wedge} T$ defined by $\langle \alpha \lrcorner\, |T| \mid \beta \rangle \equiv \langle |T| \mid \alpha \wedge \beta \rangle$, for all $\beta \in \overset{n-k}{\wedge} T^*$. It depends on very mildly the choice of the basis and possesses the inverse operator $\lrcorner\, |T^*|$.

4. Definitions. We introduce a mapping

$$\Delta'' : V \otimes (\overset{l+1}{\otimes} T^*) \otimes (\overset{s}{\otimes} T^*) \to V \otimes (\overset{l}{\otimes} T^*) \otimes (\overset{s+1}{\otimes} T^*),$$

(l, s are arbitrary integers), the result of the composition

$$\{Z \otimes (\xi_0 \otimes \ldots \otimes \xi_l) \otimes (\eta_1 \otimes \ldots \otimes \eta_s)\} \to \{\ldots\} \otimes \mathrm{Id} \to$$
$$\to \Sigma \langle \bar{X}_i \mid \xi_0 \rangle Z \otimes (\xi_1 \otimes \ldots \otimes \xi_l) \otimes (\bar{\xi}_i \otimes \eta_1 \otimes \ldots \otimes \eta_s).$$

(The first mapping is a tensor multiplication, the second one is a couraction.) The *Spencer differential* ∂'' is defined by the composition

$$\partial'' : V \otimes (\overset{l+1}{\odot} T^*) \otimes (\overset{s}{\wedge} T^*) \subset V \otimes (\overset{l+1}{\otimes} T) \otimes (\overset{s}{\otimes} T) \xrightarrow{\Delta''}$$
$$\xrightarrow{\Delta''} V \otimes (\overset{l}{\otimes} T^*) \otimes (\overset{s+1}{\otimes} T^*) \xrightarrow{C.\mathrm{id} \otimes \mathscr{S} \otimes \mathscr{A}} V \otimes (\overset{l}{\odot} T^*) \otimes (\overset{s+1}{\wedge} T^*),$$

where C is an unimportant constant; we choose $C = 1/l!s!$, for certainty. If certain linear subspaces $E^l_s \subset V \otimes (\overset{l}{\odot} T^*) \otimes (\overset{s}{\wedge} T^*)$ (l, s are arbitrary integers) satisfying $\partial'' E^{l+1}_s \subset E^l_{s+1}$ are given, then the *Spencer homology groups* $H^l_s(E)$ arise from the complexes

$$E(l+s): \ldots \to E^{l+1}_{s-1} \xrightarrow{\partial''} E^l_s \xrightarrow{\partial''} E^{l-1}_{s+1} \to \ldots$$

We introduce a mapping

$$\Delta' \; V^* \otimes (\overset{l}{\otimes} T) \otimes (\overset{s+1}{\otimes} T) \to V^* \otimes (\overset{l+1}{\otimes} T) \otimes (\overset{s}{\otimes} T),$$

the result of the composition

$$\{\zeta \otimes (X_1 \otimes \ldots \otimes X_l) \otimes (Y_0 \otimes \ldots \otimes Y_s)\} \to \{\ldots\} \otimes \mathrm{Id} \to$$
$$\to \Sigma \langle Y_0 \mid \bar{\xi}_i \rangle \, \zeta \otimes (\bar{X}_i \otimes X_1 \otimes \ldots \otimes X_l) \otimes (Y_1 \otimes \ldots \otimes Y_s).$$

The *Koszul differential* ∂' is defined by the composition

$$\partial' \colon V^* \otimes (\overset{l}{\circ} T) \otimes (\overset{s+1}{\wedge} T) \subset V^* \otimes (\overset{l}{\otimes} T) \otimes (\overset{s+1}{\otimes} T) \xrightarrow{\Delta'}$$
$$\xrightarrow{\Delta'} V^* \otimes (\overset{l+1}{\otimes} T) \otimes (\overset{s}{\otimes} T) \xrightarrow{C.\mathrm{id} \otimes \mathscr{S} \otimes \mathscr{A}} V^* \otimes (\overset{l+1}{\circ} T) \otimes (\overset{s}{\wedge} T),$$

where we choose $C = 1/l!\,s!$. If certain linear subspaces $\bar{F}^l_s \subset V^* \otimes (\overset{l}{\circ} T) \otimes (\overset{s}{\wedge} T)$ satisfying $\partial' \bar{F}^l_{s+1} \subset \bar{F}^{l+1}_s$ are given, then ∂' naturally induces a mapping between the factor-spaces $F^l_s = V^* \otimes (\overset{l}{\circ} T) \otimes (\overset{s}{\wedge} T)/\bar{F}^l_s$, again denoted by ∂', and the *Koszul homology groups* $H^l_s(F)$ arise from the complexes

$$F(l+s): \ldots \to F^{l-1}_{s+1} \xrightarrow{\partial'} F^l_s \xrightarrow{\partial'} F^{l+1}_{s-1} \to \ldots$$

We introduce a mapping

$$\Delta \colon V^* \otimes (\overset{l}{\otimes} T) \otimes (\overset{s}{\otimes} T^*) \to V^* \otimes (\overset{l+1}{\otimes} T) \otimes (\overset{s+1}{\otimes} T),$$

the result of the composition

$$\{\zeta \otimes (X_1 \otimes \ldots \otimes X_l) \otimes (\xi_1 \otimes \ldots \otimes \xi_s)\} \to \{\ldots\} \otimes \mathrm{Id} \to$$
$$\to \Sigma \, \zeta \otimes (\bar{X}_i \otimes X_1 \otimes \ldots \otimes X_l) \otimes (\bar{\xi}_i \otimes \xi_1 \otimes \ldots \otimes \xi_s).$$

The *Dedecker differential* ∂ is defined by the composition

$$\partial \colon V^* \otimes (\overset{l}{\circ} T) \otimes (\overset{s}{\wedge} T^*) \subset V^* \otimes (\overset{l}{\otimes} T) \otimes (\overset{s}{\otimes} T^*) \xrightarrow{\Delta}$$
$$\xrightarrow{\Delta} V^* \otimes (\overset{l+1}{\otimes} T) \otimes (\overset{s+1}{\otimes} T^*) \xrightarrow{C.\mathrm{id} \otimes \mathscr{S} \otimes \mathscr{A}} V^* \otimes (\overset{l+1}{\circ} T) \otimes (\overset{s+1}{\wedge} T^*).$$

If certain linear subspaces $\bar{G}^l_s \subset V^* \otimes (\overset{l}{\circ} T) \otimes (\overset{s}{\wedge} T^*)$ satisfying $\partial \bar{G}^l_s \subset \bar{G}^{l+1}_{s+1}$ are given, then ∂ naturally induces a mapping between the factor-spaces $G^l_s =$ $= V^* \otimes (\overset{l}{\circ} T) \otimes (\overset{s}{\wedge} T^*)/\bar{G}^l_r$, again denoted by ∂, and the *Dedecker homology groups* $H^l_s(G)$ arise from the complexes

$$G(l+n-s): \ldots \to G^{l-1}_{s-1} \xrightarrow{\partial} G^l_s \xrightarrow{\partial} G^{l+1}_{s+1} \to \ldots$$

5. A comparison. The Spencer and Koszul homologies are mutually dual. So, Δ'' and Δ' are adjoint mappings, $\langle \cdot \Delta'' \mid \cdot \rangle = \langle \cdot \mid \Delta' \cdot \rangle$, and ∂'', ∂' are adjoint, too. Moreover, if we come out with the subspaces E_s^l relevant to the Spencer homology, then the *orthogonal complements* $F_s^l = E_s^{l\perp}$ (consisting of all $\alpha \in V^* \otimes (\overset{l}{\odot} T) \otimes (\overset{s}{\wedge} T)$ which satisfy $\langle \alpha \mid \beta \rangle \equiv 0$, for all $\beta \in E_s^l$) determine the factor-spaces F_s^l with the dual homology groups: $H_s^l(E) = H_s^l(F)^*$. Reversely, one may start with the Koszul homology as well and determine the dual Spencer homology.

A close link between Koszul and Dedecker homology follows from the commutative diagram

$$\begin{tikzcd}
V \otimes (\overset{l}{\odot} T) \otimes (\overset{s}{\wedge} T^*) \arrow[r, "\partial"] \arrow[d, "\mathrm{id} \otimes \mathrm{id} \otimes \lrcorner \, |T|"] & V^* \otimes (\overset{l+1}{\odot} T) \otimes (\overset{s+1}{\wedge} T^*) \arrow[d, "(-1)^s \mathrm{id} \otimes \mathrm{id} \otimes \lrcorner \, |T|"] \\
V^* \otimes (\overset{l}{\odot} T) \otimes (\overset{n-s}{\wedge} T) \arrow[r, "\partial'"] & V^* \otimes (\overset{l+1}{\odot} T) \otimes (\overset{n-s-1}{\wedge} T).
\end{tikzcd}$$

The correspondence is determined by $F_s^l = (\mathrm{id} \otimes \mathrm{id} \otimes \lrcorner \mid T|)\, \bar{G}^l_{n-s}$, and then $H_s^l(F) \equiv H^l_{n-s}(G)$.

6. Long exact sequence. Owing to the preceeding section, we shall deal only with the Koszul homology. Then we may consider the short exact sequence (5) of complexes,

$$(5)^{l+s} \qquad \begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
F(l+s)\colon \dots \arrow[r] \arrow[d] & F^{l-1}_{s+1} \arrow[r, "\partial'"] \arrow[d] & F^l_s \arrow[r, "\partial'"] \arrow[d] & F^{l+1}_{s-1} \arrow[r] \arrow[d] & \dots \\
K(l+s)\colon \dots \arrow[r] \arrow[d] & K^{l-1}_{s+1} \arrow[r, "\partial'"] \arrow[d] & K^l_s \arrow[r, "\partial'"] \arrow[d] & K^{l+1}_{s-1} \arrow[r] \arrow[d] & \dots \\
F(l+s)\colon \dots \arrow[r] \arrow[d] & F^{l-1}_{s+1} \arrow[r, "\partial'"] \arrow[d] & F^l_s \arrow[r, "\partial'"] \arrow[d] & F^{l+1}_{s-1} \arrow[r] \arrow[d] & \dots \\
0 & 0 & 0 & 0
\end{tikzcd}$$

where we denote $K_s^l = V^* \otimes (\overset{l}{\odot} T) \otimes (\overset{s}{\wedge} T)$. This gives the long exact sequence of homologies of the row complexes:

$$\dots \to H^{l-1}_{s+1}(K) \to H^{l-1}_{s+1}(F) \to H^l_s(F) \to H^l_s(K) \to \dots$$

The middle row of the diagram $(5)^{l+s}$ $(l + s \neq 0)$ is exact, being a homogeneous component of the famous *Koszul resolution*. That means, $H_0^0(K) = V^*$, $H_s^l(K) \equiv 0$ for other l, s. Then the above long exact sequence gives the isomorphisms $H^{l+1}_{s-1}(F) \equiv H^l_s(F)$ for $l + s \neq 0$, and the exact sequence $0 \to H_0(F) \to V^* \to$ $\to H_0^0(F) \to 0$.

It follows that $H_0^l(F) \equiv 0$ $(l \neq 0)$, $H_0^{l+1}(F) \equiv H_1^l(F)$ $(l \neq -1)$. In particular, $H_1^l(F) = 0$ $(l \neq -1)$ if and only if $\partial' \colon F_1^l \to F_0^{l+1}$ is a surjective mapping.

7. A classical case. We shall suppose that $F_s^l \equiv F_0^l \otimes (\overset{s}{\wedge} T)$, for all l, s. This is equivalent with the isomorphism $F_s^l \equiv F_0^l \otimes (\overset{s}{\wedge} T)$. In this case, the direct sums $F_0 = F_0^0 \oplus F_0^1 \oplus \ldots$, $F_0 = F_0^0 \oplus F_0^1 \oplus \ldots$ are $\otimes T$-modules with the multiplication defined by the differential $\partial' \colon F_0^l \otimes T = F_1^l \to F_0^{l+1}$, $\partial' \colon F_0^l \otimes T = F_1^l \to F_0^{l+1}$. But in fact, we get $\odot T$-modules, owing to the identity $\partial'^2 = 0$. Moreover, $F_0 = \oplus(V^* \otimes (\overset{l}{\odot} T))/F_0$ is a factor-module.

We state here the important result that $H_s^l(F) \equiv 0$ for all sufficiently large l and refer to [4] for a simple proof.

Suppose in addition that $H_1^l(F) \equiv 0$, and remind the result $H_0^l(F) \equiv 0$ $(l \neq 0)$ from Section 6. It follows that

$$0 \to F_0^1 \to F_1^0 \to 0$$

is an exact complex, and we have an isomorphism $F_0^1 = F_1^0$. Using this result,

$$F_2^0 = F_0^0 \otimes T \otimes (\overset{2}{\wedge} T) \to F_1^2 = F_0^0 \otimes T \to F_0^2 \to 0$$

is an exact complex. By comparing with the Koszul resolution $K(2)$, we have an isomorphism $F_0^2 = F_0^0 \otimes (\overset{2}{\odot} T)$. Analogously,

$$F_2^1 = F_0^0 \otimes T \otimes (\overset{2}{\wedge} T) \to F_1^2 = F_0^0 \otimes (\overset{2}{\odot} T) \otimes T \to F_0^3 \to 0$$

is an exact complex, and $K(3)$ gives an isomorphism $F_0^3 = F_0^0 \otimes (\overset{3}{\odot} T)$. Continuing in this way, one can derive $F_0^l \equiv F_0^0 \otimes (\overset{l}{\odot} T)$, hence $F_s^l \equiv F_0^0 \otimes (\overset{l}{\odot} T) \otimes (\overset{s}{\wedge} T)$, $H_0^0(F) = F_0^0$, $H_s^l(F) \equiv 0$ for other l, s.

NOTES ON JET SPACES

8. Jets of sections. A manifold M is a *fibered manifold* with *base* B and *projection* π, if $\pi : M \to B$ is a surjective mapping with tangent mappings also surjective at every point of M. The submanifolds $\pi^{-1}(t)$, $t \in B$, are *fibers*, the mappings $\sigma : B \to M$ with the property $\pi \circ \sigma = \mathrm{id}$ are *sections*.

We shall consider the fibered manifolds J^l $(l = 0, 1, \ldots)$ of *l-jets* of sections with base B and projections $\pi^l : J^l \to B$; we identify $J^0 = M$, $\pi^0 = \pi$. There are mappings $\pi_p^l : J^l \to J^p$, $l > p$, possessing the obvious transitivity properties. This permits to introduce the space $J^\infty = \varprojlim J^l$ (the *inverse limit*) of *∞-jets* of sections, and the mappings $\pi_p^\infty = \varprojlim \pi_p^l : J^\infty \to J^p$.

The relationship between sections and jets can be briefly expressed by using *adapted* local coordinates $x^1, \ldots, x^n, y^1, \ldots, y^m$ on the space M and $t^1, \ldots, t^n$ on B, for which $\pi^* t^i \equiv x^i$. Then there exist certain local coordinates

$$x^i, y^j_I (i = 1, \ldots, n, I = i_1 \ldots i_s, i_1 \leqq \ldots \leqq i_s, s = 0, \ldots, l)$$

on the space J^l, and similar coordinates on J^∞ with s unlimited from above. The mappings $\pi^l, \pi^l_p, \pi^\infty_p$ arise by a simple omitting of relevant coordinates. Every section σ is locally described by certain relations

$$x^i \equiv t^i, y^j \equiv \bar{y}^j(t^1, \ldots, t^n),$$

and the prolonged sections $j^l\sigma, j^\infty\sigma$ are determined by

$$y^j_I = \partial^{|I|}\bar{y}^j(t^1, \ldots, t^n)/\partial t^I;$$

we use the convention of Section 1.

All functions and differential forms will depend only on a finite number of the above variables. That means, $C^\infty(T^* J^\infty) = \varinjlim C^\infty(\pi_l^{\infty *} T^* J^l)$, *direct limit,* where $C^\infty, T^*, \pi_l^\infty$, is the functor of taking sections, cotangent bundle, and pull-back, respectively. On the contrary, the vector fields on the space J^∞ may depend on all variables: $C^\infty(TJ^\infty) = \varprojlim C^\infty(\pi_l^{\infty *} TJ^l)$. Such a vector field Z admits a local expression

$$Z = \Sigma z_i \frac{\partial}{\partial x^i} + \Sigma z^j \frac{\partial}{\partial y^j} + \Sigma z^j_{i_1} \frac{\partial}{\partial y^j_{i_1}} + \ldots, \tag{6}$$

in terms of formal infinite series; z_i, z^j_I are arbitrary functions.

A *contact form* $\omega \in C^\infty(T^* J^\infty)$ is defined by the property $(j^\infty\sigma)^* \omega \equiv 0$, for all sections σ. These forms are locally generated by the forms ω^j_I of Section 1. Following Section 1, we also introduce the spaces $\Phi_{r,s}$ of *r-contact* $(r + s)$*-forms* and the *differentials* δ, ∂. A form $\varphi \in \Phi_{r,s}$ looks like (1) in every adapted local coordinate system, and the differentials satisfy (2), being locally defined by the formula stated in Section 1.

9. Filtrations. The *first filtration*

$$\Phi_{r,s} = \cup \Phi^l_{r,s} \supset \ldots \supset \Phi^{l+1}_{r,s} \supset \Phi^l_{r,s} \supset \ldots \supset \Phi^0_{r,s} \supset \Phi^{-1}_{r,s} = 0 \supset \ldots$$

is determined by the spaces $\Phi^l_{r,s}$ consisting of all forms φ with $f^{J,I}_{\mathcal{J}} \equiv 0$ $(|\mathcal{J}| > l)$ in local expression (1). The *second filtration*

$$\Phi_{r,s} = \cup \Phi^{(l)}_{r,s} \supset \ldots \supset \Phi^{(l+1)}_{r,s} \supset \Phi^{(l)}_{r,s} \supset \ldots \supset \Phi^{(0)}_{r,s} \supset \Phi^{(-1)}_{r,s} = 0 \supset \ldots$$

is given by the spaces $\Phi^{(l)}_{r,s}$ of forms φ which coefficients $f^{J,I}_{\mathcal{J}}$ in the formulae (1) depend only on the variables $x^i, y^j_{I'}$ $(|I'| \leqq l - |\mathcal{J}|$; we put $f^{J,I}_{\mathcal{J}} = 0$ if $l < |\mathcal{J}|)$. In the *third filtration*

$$\Phi_{r,s} = \cup \Phi^{[l]}_{r,s} \supset \ldots \supset \Phi^{[l+1]}_{r,s} \supset \Phi^{[l]}_{r,s} \supset \ldots \supset \Phi^{[0]}_{r,s} \supset \Phi^{[-1]}_{r,s} = 0 \supset \ldots$$

appear spaces $\Phi^{[l]}_{r,s}$ of all forms φ with $f_J^{J,I}$ dependent only on the variables $x^i, y^j_{I'}$ $| I' | \leqq l$) and expressible only by the forms $dx^i, \omega^j_{I'}$ ($| I' | \leqq l$).

The differential ∂ induces certain maps $\Phi^l_{r,s} \to \Phi^{l+1}_{r,s+1}$, $\Phi^{(l)}_{r,s} \to \Phi^{(l+1)}_{r,s+1}$, again denoted by ∂; the differential δ induces certain maps $\Phi^{[l]}_{r,s} \to \Phi^{[l]}_{r+1,s}$. Beside this, there are another induced operators, but they behave too badly for our aims. Note that the above filtrations possess an invariant meaning, cf. Appendix.

10. Direct decompositions. The tangent bundle $TJ^\infty = \varprojlim \pi_l^\infty{}^* TJ^l$ admits the direct decomposition $TJ^\infty = HJ^\infty \otimes VJ^\infty$ into the horizontal bundle HJ^∞ and the vertical bundle VJ^∞. Here, HJ^∞ consists of all vectors Z satisfying $\omega(Z) \equiv 0$ for every contact form ω. The natural projection $\pi_*^\infty : HJ^\infty \to TB$ ($\pi_*^\infty = \varprojlim \pi_*^l$, $\pi_*^l : TJ^l \to TB$) is an isomorphism at every point of J^∞. Consequently, HJ^∞ may be identified with the pull-back $\pi^{\infty *} TB$. Locally, HJ^∞ is generated by the vector fields (2) and the above isomorphism is determined by the correspondence $\partial_i \Leftrightarrow \Leftrightarrow \partial/\partial t^i$. The bundle VJ^∞ consists of all vectors Z tangent to the fibers. They look like (6) with $z_i \equiv 0$.

The cotangent bundle $T^*J^\infty = \varinjlim \pi_l^\infty T^*J^l$ decomposes into the dual direct sum $T^*J^\infty = H^*J^\infty \otimes V^*J^\infty$. Here, the space H^*J^∞ may be identified with $\pi^{\infty *} T^*B$, and $V^*J^\infty = \varinjlim \pi_l^\infty{}^* V^*J^l$, where V^*J^l are the bundles of covectors locally expressible by the forms ω^j_I ($| I | \leqq l$) restricted to the fibers. We shall denote these restrictions by ζ^j_I, for clarity.

At this place, we must remind the well-known exact sequence

$$0 \to \pi_{l-1}^{\infty *} V^*J^{l-1} \xrightarrow{\mathrm{i}} \pi_l^\infty{}^* V^*J^l \xrightarrow{\mathrm{j}} C^\infty((\pi_0^\infty{}^* V^*M) \otimes (\pi^{\infty *} \overset{l}{\odot} TB)) \to 0,$$

where we put $V^*J^l \equiv 0$ ($l < 0$), and the projection j is locally determined by $\mathrm{j}\zeta^j_I = \zeta^j \otimes (\odot\, \partial^{|I|}/\partial t^I)$. After tensoring by $\pi^{\infty *} \overset{r}{\wedge} T^*B$ and applying the functor C^∞, we obtain the exact sequence

$$0 \to \Phi^{l-1}_{1,s} \to \Phi^l_{1,s} \to C^\infty(\pi_0^\infty{}^* V^*M \otimes \pi^{\infty *}((\overset{l}{\odot} TB) \otimes (\overset{s}{\wedge} T^*B)) \to 0$$

appearing in the exact commutative diagram

$$(7) \qquad \begin{array}{ccccccc} 0 \to & \Phi^{l-1}_{1,s} & \to & \Phi^l_{1,s} & \xrightarrow{\mathrm{j}} & C^\infty(V^*M \otimes (\overset{l}{\odot} TB) \otimes (\overset{s}{\wedge} T^*B)) & \to 0, \\ & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow & \\ 0 \to & \Phi^l_{1,s-1} & \to & \Phi^{l+1}_{1,s+1} & \xrightarrow{\mathrm{j}} & C^\infty(V^*M \otimes (\overset{l+1}{\odot} TB) \otimes (\overset{s+1}{\wedge} T^*B)) & \to 0, \end{array}$$

where we simplify the notation by omitting certain pull-backs which express only the simple fact that all bundles are considered over the base J^∞. Using the above formulae for the projection j, one can easily verify that the induced operator in the third column arises from certain vector bundle mapping

$$V^*M \otimes (\overset{l}{\odot} TB) \otimes (\overset{s}{\wedge} T^*B) \to V^*M \otimes (\overset{l+1}{\odot} TB) \otimes (\overset{s+1}{\wedge} T^*B);$$

the simplified notation does not express the fact that these bundles are taken over the base J^∞. Denoting by $V = (\mathrm{V}^*M)_y$, $T = (\mathrm{T}B)_y$, $T^* = (\mathrm{T}^*B)_y$ the fibers over a point $y \in J^\infty$, the above vector bundle mapping when restricted to these fibers coincides (up to sign) with the *Dedecker differential.*

NOTES ON DIFFERENTIAL EQUATIONS

11. Regular and closed systems. A *system of partial differential equations* consists of certain conditions $(j^\infty\sigma)^* f_\iota \equiv 0$; f_ι are given functions, ι varies in certain index set. A section σ satisfying these condition is a *solution.* An intrinsic approach employs the subset $R^\infty \subset J^\infty$ consisting of all points in which the functions f_ι vanish; a solution σ is characterised by $(j^\infty\sigma)(t) \in R^\infty$, for all $t \in B$. We shall deal with *regular systems,* that means, we *shall* suppose that every set $R^l = \pi_l^\infty R^\infty$ is a fibered manifold over the base B (projection π^l) and also over the base $R^p = \pi_p^\infty R^\infty$ ($l \geq p$; projection π_p^l). Moreover, the condition $(j^\infty\sigma)^* f = 0$ implies

$$0 = \mathrm{d}(j^\infty\sigma)^* f = (j^\infty\sigma)^* \,\mathrm{d}f = (j^\infty\sigma)^* \,\partial f = \Sigma (j^\infty\sigma)^* \,\partial_i f \,.\, \mathrm{d}t^i,$$

so we may (and *shall*) suppose that together with every function f vanishing on R^∞, also the functions $\partial_I f$ vanish on R^∞. These are the *closed* systems.

Using the regularity assumption, we may restrict the differential forms from the space Φ to the subset R^∞. Namely, there are exact sequences

$$0 \to (\overline{\Psi}{}^l_{r,s})_y \to (\Phi^l_{r,s})_y \xrightarrow{\mathrm{j}} (\Psi^l_{r,s})_y \to 0$$

over every point $y \in R^\infty$, where $\overline{\Psi}{}^l_{r,s}$ is the subspace of all forms from $\Phi^l_{r,s}$ vanishing on R^∞ and $\Psi^l_{r,s}$ is the corresponding factor-space. It is exactly the last factor-space which may be considered as the space of all differential forms from $\Phi^l_{r,s}$ restricted to R^∞. We may also consider the spaces $\Psi^{(l)}_{r,s} = \mathrm{j}\Phi^{(l)}_{r,s}$, $\Psi^{[l]}_{r,s} = \mathrm{j}\Phi^{[l]}_{r,s}$. These spaces determine certain filtrations of the space $\Phi_{r,s}$. Then, reasonings similar to that of the Section 2 show that the closedness is equivalent with the fact that δ, ∂ are intrinsic operators on R^∞. By another words, ∂ induces certain mappings

$$\Psi_{r,s} \to \Psi_{r,s+1}, \; \Psi^l_{r,s} \to \Psi^{l+1}_{r,s+1}, \; \Psi^{(l)}_{r,s} \to \Psi^{(l+1)}_{r,s+1}$$

again denoted by ∂; δ behaves analogously.

By restricting (7) to R^∞, we obtain a diagram of the type

$$\begin{array}{ccccccccc} 0 & \to & \Psi^{l-1}_{1,s} & \to & \Psi^l_{1,s} & \to & \Gamma^l_s & \to & 0 \\ & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow & & \\ 0 & \to & \Psi^l_{1,s} & \to & \Psi^{l+1}_{1,s+1} & \to & \Gamma^l_{s+1} & \to & 0, \end{array} \tag{8}$$

where the spaces in the third column admit a better interpretation then a mere factor-spaces as they appear in (8). (These spaces *cannot* be obtained by a formal application of the projection j on the third column of (7).) Namely, these spaces appear in the commutative and exact diagram (9)

$$
(9)\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bar{\Psi}^{l-1}_{1,s} \arrow[r] \arrow[d] & \Phi^{l-1}_{1,s} \arrow[r] \arrow[d] & \Psi^{l-1}_{1,s} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{\Psi}^{l}_{1,s} \arrow[r] \arrow[d, "\mathrm{j}"] & \Phi^{l}_{1,s} \arrow[r] \arrow[d] & \Psi^{l}_{1,s} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{\Gamma}^{l}_{s} \arrow[r] \arrow[d] & C^{\infty}(V^*M \otimes (\overset{l}{\odot} TB) \otimes (\overset{s}{\wedge} T^*B)) \arrow[r] \arrow[d] & \Gamma^{l}_{s} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

taken over R^∞. Here, the spaces $\bar{\Psi}^l_{1,s}$ (consisting of certain forms vanishing on the tangent spaces of R^∞) are generated by the forms

$$(10)\qquad \alpha = \delta f \wedge dx^I = \Sigma\, \partial f/\partial y^j_{I'} \,.\, \omega^j_{I'} \wedge dx^I,$$

where f is an arbitrary function of the variables $x^i, y^j_{I'}$ ($|I'| \leqq l$) which vanish on R^∞. It follows that the space $\bar{\Gamma}^l_s$ is generated by the projections

$$(11)\qquad \mathrm{j}\alpha = \Sigma\, \partial f/\partial y^j_{I'} \,.\, \zeta^j \otimes (\odot\, \partial^l/\partial t^{I'}) \wedge (dt^I) \quad (|I'| = l).$$

Owing to the regularity assumptions, $\bar{\Gamma}^l_s = C^\infty(\bar{G}^l_s)$, where $\bar{G}^l_s$ is a vector bundle, subbundle of the bundle $V^*M \otimes (\overset{l}{\odot} TB) \otimes (\overset{s}{\wedge} T^*B)$, locally generated by the tensor fields (11). Consequently, $\Gamma^l_s = C^\infty(G^l_s)$, where $G^l_s = V^*M \otimes (\overset{l}{\odot} TB) \otimes (\overset{s}{\wedge} T^*B)/\bar{G}^l_s$ is a factor-bundle.

Note that beside the clarification of the spaces Γ^l_s we have verified that the fibers of the bundles $\bar{G}^l_s$, G^l_s (considered as a vector spaces dependent on the parameter $y \in R^\infty$) are exactly the spaces of the classical type of Section 7.

12. Homology of systems. The diagram (8) is a constituent part of the short exact sequence (12) of complexes.

$$
(12)^{l+n}\qquad
\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
\Psi(l+n-1){:}\,0 \arrow[d] & & \Psi^{l-1}_{1,0} \arrow[r, "\partial"] \arrow[d] & \Psi^{l}_{1,1} \arrow[r, "\partial"] \arrow[d] & \cdots \arrow[r, "\partial"] & \Psi^{l+n-1}_{1,n} \arrow[r] \arrow[d] & 0 \\
\Psi(l+n){:} \arrow[d] & 0 \arrow[r] & \Psi^{l}_{1,0} \arrow[r, "\partial"] \arrow[d, "\mathrm{j}"] & \Psi^{l+1}_{1,1} \arrow[r, "\partial"] \arrow[d, "\mathrm{j}"] & \cdots \arrow[r, "\partial"] & \Psi^{l+n}_{1,n} \arrow[r] \arrow[d, "\mathrm{j}"] & 0 \\
\Gamma(l+n){:} \arrow[d] & 0 \arrow[r] & \Gamma^{l}_{0} \arrow[r] \arrow[d] & \Gamma^{l+1}_{1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Gamma^{l+n}_{n} \arrow[r] \arrow[d] & 0 \\
0 & & 0 & 0 & & 0 &
\end{tikzcd}
$$

We already know that $\Gamma^l_s = C^\infty(G^l_s)$ and that the mappings in the lowest row of (12) are induced by Dedecker differentials operating between the fibers of the bundles G^l_s with base R^∞. By another words, $\Gamma(l + n) = C^\infty(G(l + n))$, where $G(l + n)$ is the Dedecker complex dependent on the parameter point $y \in R^\infty$. The homology of the complexes $\Gamma(l + n)$, $G(l + n)$ are intimately related and will be the main object of our interest.

The homology $H^l_s(G)$ of the above mentioned complex $G(l + n)$ of vector bundles is called the *Dedecker homology of the system* R^∞; it depends on $y \in R^\infty$ as a parameter. This is the only difference in regard of the purely algebraic theory of Section 4. Owing to the Section 5, we get also the *Spencer* ($H^l_s(E)$) and *Koszul* ($H^l_s(F)$) *homology of the system* R^∞; these concepts are known for a long time.

We introduce the following definition: A system R^∞ is called of *essential order* $\leqq 0$, if $H^l_{n-1}(G) \equiv 0$ for all l. (A motivation for this terminology will be explained in Appendix.) After Section 5, this property is equivalent with $H^l_1(F) \equiv 0$, which implies $H^l_s(F) = 0$ ($s \geqq 1$), cf. Section 7. We see that $H^l_s(G) \equiv 0$ ($s \leqq n - 1$) for every system R^∞ of essential order $\leqq 0$. A system R^∞ is called *k-involutive* if $H^l_s(G) \equiv 0$ ($l \geqq k$). We shall deal only with 1-*involutive* (in short: *involutive*) systems; in this case $H^l_s(G) \equiv 0$ ($l \neq 0$). (That means, $H^l_s(F) \equiv 0$ ($l \neq 0$), which is in agreement with the common terminology, cf. [4].) Observe that a system of essential order $\leqq 0$ is allways involutive. This follows from the above result $H^l_s(G) \equiv 0$ ($s \leqq n - 1$) and from the identity $H^l_n(G) \equiv 0$ ($l \neq 0$) valid for every system, cf. Section 6.

If R^∞ is an involutive system, then the families of vector spaces $H^l_s(G)$ are in fact vector bundles over the base R^∞. One can easily prove this by considering the dimensions of kernels and images of the Dedecker differential in the complex $G(l + n)$. Consequently, we have the isomorphisms

$$H^l_s(\Gamma) \equiv C^\infty(H^l_s(G)), \tag{13}$$

which essentially simplify all investigations.

DIRECT THEOREMS

13. Main lemma. *Consider the following complexes*:

$$\begin{aligned}
&\Psi_1: && 0 \to \Psi_{1,0} \to \Psi_{1,1} \to \dots \to \Psi_{1,n} \to 0,\\
&\Psi(l+n): && 0 \to \Psi^l_{1,0} \to \Psi^{l+1}_{1,1} \to \dots \to \Psi^{l+n}_{1,n} \to 0,\\
&\Psi((k+n)): && 0 \to \Psi^{(k)}_{1,0} \to \Psi^{(k+1)}_{1,1} \to \dots \to \Psi^{(k+n)}_{1,n} \to 0,
\end{aligned}$$

$$\Psi(l+n) \cap \Psi((k+n)) : 0 \to \Psi^l_{1,0} \cap \Psi^{(k)}_{1,0} \to$$
$$\to \Psi^{l+1}_{1,1} \cap \Psi^{(k+1)}_{1,1} \to \dots \to \Psi^{l+n}_{1,n} \cap \Psi^{(k+n)}_{1,n} \to 0.$$

If R^∞ is of essential order $\leqq 0$, then the complexes $\Psi_1, \ldots, \Psi(l+n) \cap \Psi(k+n)$ are exact (consequently: *locally exact*) *with the only exception of the last but one term $\Psi_{1,n}, \ldots, \Psi^{l+n}_{1,n} \cap \Psi^{(k+n)}_{1,n}$, respectively. If R^∞ is an involutive system, then all homology classes* (consequently: *all local homology classes*) *of s-forms appearing in the complexes $\Psi_1, \ldots, \Psi(l+n) \cap \Psi(k+n)$ are representable by certain forms from the subspaces $\Psi^0_{1,s} \subset \Psi_{1,s}$, $\Psi^0_{1,s} \subset \Psi^l_{1,s}$, $\Psi^0_{1,s} \cap \Psi^{(k+s)}_{1,s} \subset \Psi^{(k+s)}_{1,s}$, $\Psi^0_{1,s} \cap \cap \Psi^{l+s}_{1,s} \cap \Psi^{(k+s)}_{1,s} \subset \Psi^{l+s}_{1,s} \cap \Psi^{(k+s)}_{1,s}$, respectively.*

Proof: The assertion concerning the last complex is the strongest one, the other complexes arise by $l \to \infty$ or (and) $k \to \infty$. But the case of the first two complexes is such simple that it will be treated separately as follows:

The short exact sequence (12) of complexes gives the long exact sequence of homologies

$$\ldots \to H^l_{s-1}(\Gamma) \to H^l_s(\Psi) \to H^{l+1}_s(\Psi) \to H^{l+1}_s(\Gamma) \to \ldots \tag{18}$$

($H^l_s(\Psi)$ is the homology of the complex $\Psi(l+n-s)$ in the term $\Psi^l_{1,s}$). Owing to (13) and $H^l_s(\Gamma) \equiv 0$ ($l \neq 0$), the sequence (18) reduces to the isomorphisms $H^l_s(\Psi) \equiv H^{l+1}_s(\Psi)$ ($l \neq 0$) and the exact sequence

$$0 \to H^0_s(\Psi) \to H^0_s(\Gamma) \to H^0_{s+1}(\Psi) \to H^1_{s+1}(\Psi) \to 0. \tag{19}$$

So we have the isomorphisms $H^1_s(\Psi) = H^2_s(\Psi) = \ldots$ and the surjection $H^0_{s+1}(\Psi) \to \to H^1_{s+1}(\Psi)$, which concludes the proof for the involutive case. (Note at this place, that *we know the kernel* of the above surjection. It consists of all $\partial\gamma \in H^0_{s+1}(\Psi)$, where $\gamma \in \Psi^0_{1,s}$ satisfies $\mathrm{j}\,\partial\gamma = 0$, as follows from the definition of the connecting homomorphisms of (18).) If we deal with a system of essential order $\leqq 0$, then $H^0_s(\Gamma) \equiv 0$ ($s \neq n$) and the above exact sequence gives $H^0_s(\Psi) \equiv 0$ ($s \neq n$), the expected results.

Turn to the complex $\Psi(l+n) \cap \Psi((k+n))$. Then the assertion may be proved by a more carefull diagram chasing in (12). We will proceed by induction on l; assume the assertion of the theorem being true for all $l < L$, and verify the case $l = L$.

Choose $\varphi \in \Psi^{L+s}_{1,s} \cap \Psi^{(k+s)}_{1,s}$ satisfying $\partial\varphi = 0$. We may also suppose that $s \neq n$ ($L + s > 0$) if we treat the case of a system of essential order $\leqq 0$ (an involutive system). Then $\mathrm{j}\varphi \in \Gamma^{L+s}_s$, $\partial(\mathrm{j}\varphi) = 0$, and the coefficients of $\mathrm{j}\varphi$ calculated in a local coordinate system depend only on the variables x^i, y^j_I ($|I| \leqq k - L$). Using a partion of unity in *these variables* and exactness of $\Gamma(L+n)$, one can prove that there exists a section $\gamma \in \Gamma^{L+s-1}_{s-1}$ with $\partial\gamma = \mathrm{j}\varphi$ and a form $\chi \in \Psi^{L-1+s}_{s-1}$ satisfying $\mathrm{j}\chi = \gamma$. Then, $\mathrm{j}(\varphi - \partial\chi) = \mathrm{j}\varphi - \partial\mathrm{j}\chi = \mathrm{j}\varphi - \partial\gamma = 0$. Hence, $\psi = \varphi - \partial\chi \in \in \Psi^{L-1+s}_{1,s} \cap \Psi^{(k+s)}_{1,s}$. Moreover, $\partial\psi = \partial(\varphi - \partial\chi) = 0$. We replace the form φ by the form $\psi = \varphi - \partial\chi$ lying in the same homology class and use the induction assumption.

14. Exterior powers of complexes. By example, the complex

$$\Psi_r\colon 0 \to \Psi_{r,0} \to \Psi_{r,1} \to \dots \to \Psi_{r,n} \to 0$$

is the *r-fold exterior power* of the complex Ψ_1, $\Psi_{r,s} = \Sigma \overset{r}{\wedge} \Psi_{1,s}$ $(\Sigma s_i = s)$. Hence, the homology of Ψ_r may be determined by using the adapted Künneth formulae $H(\Psi_r) = H(\Psi_1) \wedge \dots \wedge H(\Psi_1)$. Note that the usual Künneth formulae for complexes $\mathscr{C}_1, \dots, \mathscr{C}_r$ of vector spaces reduces to the isomorphism $H(\mathscr{C}_1 \otimes \dots \otimes \mathscr{C}_r) =$ $= H(\mathscr{C}_1) \otimes \dots \otimes H(\mathscr{C}_r)$, because the functor Ext identically vanishes; the above adapted case may be obtained by alternation.

After the results of the preceeding Section 13, $H^l_s(\Psi_r) \equiv 0$ $(s \neq n)$ for the case of a system of essential order $\leqq 0$, and all homology classes (local homology classes) from $H^l_s(\Psi_r)$ may be represented by certain forms lying in the space $\Psi^0_{r,s}$.

Similar results for the exterior powers of the complexes $\Psi(l + n)$, $\Psi((k + n))$, $\Psi(l + n) \cap \Psi((k + n))$ can be also obtained, but we shall not deal with them.

15. Theorem. *Denote by* (4)˜ *the double complex arising from* (4), *if the terms* $\Phi_{r,s}$ *are replaced by* $\Psi_{r,s}$ *and the result is considered only at the points of* R^∞. *All rows of* (4)˜ *are locally exact. If* R^∞ *is of essential order* $\leqq 0$, *then all columns of* (4)˜ *are locally exact with the exception of the term* $\Psi_{r,n}$. *If* R^∞ *is an involutory system, then all homology classes appearing in the columns of* (4)˜ *at the term* $\Psi_{r,s}$ *may be represented by certain forms from the space* $\Psi^0_{r,s}$.

Proof: We may follow rather closely [1], [2] omitting the easy details: (1) The rows of (4)˜ may be represented as direct sums of certain de Rham complexes, hence they are locally exact. (2) The first column is de Rham complex, hence is locally exact. (3) The assertion for the second column may be proved by a simple diagram chasing, provided the other columns are locally exact. (4) The main lemma solves the case of the third column. (5) The Section 14 solves the case of the subsequent columns.

Turn to the only remaining point (1). Consider the complex

$$0 \to \Xi_s \to \Psi^{[l]}_{0,} \xrightarrow{\delta} \Psi^{[l]}_{1,s} \xrightarrow{\delta} \dots$$

Here is

$$\delta\varphi = \Sigma\, \partial f^{\mathscr{y},I}_{\mathscr{J}} / \partial y^j_{I'} \,.\, \omega^{j\mathscr{y}}_{I'\mathscr{J}} \wedge \mathrm{d}x^I$$

for a form φ expressed as (1) in local coordinates. Choose an index I. The spaces of coefficients of $\mathrm{d}x^I$ of the above complex form an autonomous complex of the type

$$0 \to \{c^I\} \to \{f^I\} \xrightarrow{\mathrm{d}} \{\Sigma f^{j_1,I}_{I_1} \omega^{j_1}_{I_1}\} \xrightarrow{\mathrm{d}} \{\Sigma f^{j_1 j_2}_{I_1,I_2} \omega^{j_1 j_2}_{I_1,I_2}\} \xrightarrow{\mathrm{d}} \dots,$$

where the differentials d act as follows:

$$\mathrm{d} f^{J,I}_{\mathscr{J}} \omega^J_I = \Sigma\, \partial f^{J,I}_{\mathscr{J}} / \partial y^j_{I'} \,.\, \omega^{jJ}_{I'\mathscr{J}}.$$

Using the change $\omega^j_{\mathcal{J}} \leftrightarrow \mathrm{d}y^j_{\mathcal{J}}$, the above differential d exactly coincides with the exterior differential in the variables y^j_I ($|I| \leqq l$); the variables x^i are mere parameters. We have de Rham complex dependent on parameters. It is locally exact, also after the restriction on the manifold R^l. The local exactness of the complex

$$0 \to \Xi_s \to \Psi_{0,s} \to \Psi_{1,s} \to \dots \tag{20}$$

follows by taking $l \to \infty$.

16. Note. The above proof shows that the homology classes appearing in the s^{th} row of (4) may be represented by the direct sum of $\binom{n}{s}$ copies of the homology space of the intersection of R^∞ with a fiber. Consequently, the rows are exact if these intersections are homologically trivial; this is the case of a linear system. If in addition the base B is also a homologically trivial space, and we deal with a system of essential order $\leqq 0$, then $(4)^\sim$ is an exact double complex, with the exception of the bottoms of the rows. This appears as a generalisation of the Tulczyjew result.

REFERENCES

[1] Tulczyjew, W. M.: *The Euler-Lagrange resolution,* Internat. Coll. on Differential Geometric Methods in Mathematical Physics, Aix-en-Provence 1979, Lecture Notes in Mathematics 836, Springer, 22—48.

[2] Haine, L.: *La suite spectrale d'Euler-Lagrange,* Séminaire de Math. Pure, Institute Catholique de Louvain, 1-er semestre 1980.

[3] Spencer, D. C.: *Overdetermined systems of linear partial differential equations,* Bull. Amer. Math. Soc., 75 (1969), 179—236.

[4] Serre, J. P.: *A letter in the paper by Guillemin, V. W. and Sternberg, S.,* Bull. Amer. Math. Soc. 70 (1964), 42—46.

[5] Cartan, H. and Eilenberg, S.: *Homological algebra,* Princeton University, 1956.

[6] Dedecker, P.: *On the generalisation of symplectic geometry to multiple integrals in the calculus of variations,* Lecture Notes in Mathematics 570, Springer, 395—456.

J. Chrastina
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia