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BOUNDEDNESS AND OSCILLATORINESS OF SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

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In this paper we shall investigate some properties of solutions of the differential equation

$$(1) \qquad a(t)\, x'' + b(t)\, g(x, x') + f(t, x)\, h(x') = r(t),$$

or

$$(1') \qquad a(t)\, x'' + b(t)\, g(x, x') + [1 + c(t)] f(t, x)\, h(x') = r(t),$$

where $0 \leqq a(t) \in C^1(I_0)$, $0 < b(t) \in C(I_0)$, $c(t) \in C^1(I_0)$, $g(x, y) \in C(R_2)$, $f(t, x) \in \in C(D)$, $\dfrac{\partial f}{\partial t} \in C(D)$, $0 < h(y) \in C(R_1)$, $I_0 = \langle t_0, \infty)$, $t_0 \in R_1 = (-\infty, \infty)$, $R_2 = = R_1 \times R_1$ and $D = I_0 \times R_1$.

In the first part of this paper there are introduced some sufficient conditions for a solution $x(t)$ of equation (1) or (1′), which satisfies in t_0 a certain condition, to be bounded oı bounded together with its first derivative. In the second part of this paper there are introduced theorems, which deal with the oscillatoriness of solutions of the equation (1), where $r(t) \equiv 0$ for every $t \in I_0$. The results introduced in this paper generalize, or complete some results of [1]–[8].

I.

We introduce the following notation:

$$F(t, x) = \int_0^x f(t, s)\, ds, \qquad H(y) = \int_0^y \frac{s}{h(s)}\, ds, \qquad H = \min\{\lim_{y\to\infty} H(y), \lim_{y\to-\infty} H(y)\}$$

and

$$\{\varphi(t)\}_+ = \begin{cases} \varphi(t) & \text{for } \varphi(t) > 0, \\ 0 & \text{for } \varphi(t) \leqq 0. \end{cases}$$

We have

Theorem 1. *Let the following conditions hold:*

1. $a'(t) \leqq 0$ *for every* $t \in I_0$;

2. *there exists a constant* $k > 0$ *such that* $yg(x, y)\, h(y) \geqq ky^2$ *for every* $(x, y) \in R_2$;

3. *for every continuously differentiable function* $u(t)$ *on* $\langle t_0, T)$ *where* $T \leqq \infty$, *which is unbounded for* $t \to T_-$, *there exists a sequence* $\{t_n\}_{n=1}$, $t_n \to T_-$ *for* $n \to \infty$, *such that*

$$\frac{\partial F(t, u(t))}{\partial t} \leqq \frac{F(t, u(t_n))}{\partial t} \qquad \text{for } t_0 \leqq t \leqq t_n$$

and

$$\lim_{n \to \infty} F(t_0, u(t_n)) = F_1,$$

with $F_1 \leqq \infty$ *independent of* $u(t)$.

If in addition

$$\int_{t_0}^{\infty} \frac{r^2(s)}{b(s)}\, \mathrm{d}s = K < \infty, \tag{2}$$

then every solution $x(t)$ *of* (1) *which satisfies the inequality*

$$K_0 = a(t_0)\, H(x'(t_0)) + F(t_0, x(t_0)) + \frac{K}{4k} < F_1, \tag{3}$$

is bounded on its domain.

Proof. Let a solution $x(t)$ of (1) be defined on $\langle t_0, T)$. Suppose that it satisfies the condition (3) and $\limsup_{t \to T_-} |\, x(t)\, | = \infty$. Multiplying (1) by $\dfrac{x'(t)}{h(x'(t))}$ and arranging we get

$$a(t) \frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) + \frac{\mathrm{d}}{\mathrm{d}t} F(t, x(t)) + kb(t) \left[\frac{x'(t)}{h(x'(t))}\right]^2 - r(t) \frac{x'(t)}{h(x'(t))} \leqq \frac{\partial F(t, x(t))}{\partial t}.$$

Using the fact, that for arbitrary real numbers a, b and x, if $a > 0$ then

$$ax^2 + bx \geqq -\frac{b^2}{4a},$$

from the last inequality we have

$$a(t) \frac{\mathrm{d}}{\mathrm{d}t} H(x'(t)) + \frac{\mathrm{d}}{\mathrm{d}t} F(t, x(t)) \leqq \frac{\partial F(t, x(t))}{\partial t} + \frac{1}{4k} \frac{r^2(t)}{b(t)}.$$

Integrating the last inequality from t_0 to $t \in (t_0, T)$ we obtain

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq K_0 + \int_{t_0}^{t} \frac{\partial F(s, x(s))}{\partial s}\, \mathrm{d}s + \int_{t_0}^{t} a'(s)\, H(x'(s))\, \mathrm{d}s. \tag{4}$$

Since $\limsup_{t\to T_-} |x(t)| = \infty$, there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to T_-$ for $n \to \infty$, such that

$$\frac{\partial F(t, x(t))}{\partial t} \leqq \frac{\partial F(t, x(t_n))}{\partial t} \qquad \text{for } t_0 \leqq t \leqq t_n$$

and $\lim_{n\to\infty} F(t_0, x(t_n)) = F_1$. Using the assumptions of theorem from (4) we get

$$F(t_n, x(t_n)) \leqq K_0 + \int_{t_0}^{t_n} \frac{\partial F(s, x(s))}{\partial s} \, ds = K_0 + F(t_n, x(t_n)) - F(t_0, x(t_n)),$$

hence for $n \to \infty$ we have

$$\lim_{n\to\infty} F(t_0, x(t_n)) = F_1 \leqq K_0,$$

which contradicts the assumption (3). This completes the proof.

Theorem 2. *Let the hypotheses of Theorem 1 hold with the exception of assumption 3 instead of which we assume that for any sequences $\{t_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ such that for $n \to \infty$, $t_n \to \infty$, $|x_n| \to \infty$ and*

(5) $$\lim_{n\to\infty} F(t_n, x_n) = F_2.$$

If in addition

(6) $$\frac{\partial F(t, x)}{\partial t} \leqq 0 \qquad \textit{for every} \qquad (t, x) \in D,$$

then every solution $x(t)$ of (1) *which satisfies the inequality*

(7) $$K_0 < F_2,$$

is bounded on its domain.

Furthermore if

(8) $$a(t) \geqq a_0 > 0, \ f(t, x)\, x \geqq 0 \qquad \textit{for every } t \in I_0 \textit{ and } (t, x) \in D,$$

then the first derivative of an arbitrary solution $x(t)$ of (1) *which satisfies the inequality*

$$\frac{K_0}{a_0} < H,$$

is bounded on its domain, too.

Proof. Using assumptions from (4) we get

(9) $$F(t, x(t)) \leqq K_0.$$

Let the solution $x(t)$ of (1) satisfy the condition (7) and $\limsup_{t\to T_-} |x(t)| = \infty$, where $\langle t_0, T)$ is the domain of $x(t)$. Suppose that $T = \infty$. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that, for $n \to \infty$, $t_n \to \infty$ and $\lim_{n\to\infty} |x(t_n)| = \infty$. From (9) $t = t_n$ we have

$$F_2 = \lim_{n\to\infty} F(t_n, x(t_n)) \leqq K_0,$$

contrary to (7).

Now let $T < \infty$; $\{t_n\}_{n=1}^{\infty}$ be a sequence such that for $n \to \infty$, $t_n \to T_-$ and $\lim_{n\to\infty} |x(t_n)| = \infty$. Define a sequence $\{\tau_n\}_{n=1}^{\infty}$ such that for all n $t_n \leqq \tau_n$ and $\lim_{n\to\infty} \tau_n = \infty$.

Using the assumption (6) from (9) we obtain

$$F(\tau_n, x(t_n)) \leqq F(t_n, x(t_n)) \leqq K_0,$$

hence for $n \to \infty$

$$F_2 \leqq K_0,$$

which is again a contradiction.

Furthermore we suppose that (8) holds. Then from (4) we get

$$a(t)\, H(x'(t)) \leqq K_0.$$

If for the solution $x(t)$ in t_0 the inequality $K_0 < a_0 H$ holds and $x'(t)$ is unbounded, then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$H = \lim_{n\to\infty} H(x'(t_n)) \leqq \frac{K_0}{a_0},$$

which is a contradiction. This completes the proof.

Remark 1. If $r(t) \equiv 0$ and $h(y) = 1$, then from Theorem 1, or Theorem 2 we get Theorem 1, or Theorem 4 in [3]. Furthermore, if $b(t)\, g(x, y) \equiv 0$, then Theorem 1 gives Theorem 1 in [2].

Corollary 1. Let the hypotheses of Theorem 2 hold. If $H = \infty$, then every solution $x(t)$ of (1), which satisfies the inequality (7), is bounded on $\langle t_0, \infty)$ together with its first derivative. If $H < \infty$, then every solution $x(t)$ of (1) which satisfies the inequality

$$K_0 < \min \{a_0 H, F_2\},$$

is bounded on $\langle t_0, \infty)$ together with its first derivative.

Proof. From Theorem 2 it follows that $x(t)$ and $x'(t)$ are bounded on $\langle t_0, T)$. Hence by the theorem of the extension for the solution it follows that $T = \infty$.

Theorem 3. *Let $a(t) > 0$, $a'(t) \geqq 0$, $f(t, x)\, x \geqq 0$ for every $t \in I_0$ and $(t, x) \in D$. Moreover, suppose that the assumption 2 of Theorem 1 and the assumptions (2) and (6) hold. Then the first derivative of an arbitrary solution $x(t)$ of (1) which* *satisfies the inequality*

$$\frac{K_0}{a(t_0)} < H,$$

is bounded on its domain.

If in addition

$$\lim_{t\to\infty} a(t) = a_1 < \infty$$

and (5) *holds, then every solution* $x(t)$ *of* (1) *which satisfies the inequality*

$$\frac{a_1 K_0}{a(t_0)} < F_2,$$

is bounded on its domain.

Proof. From (4) by the assumptions of theorem it follows

$$a(t)\,H(x'(t)) \leqq K_0 + \int_{t_0}^{t} a'(s)\,H(x'(s))\,\mathrm{d}s.$$

By Bellman's lemma the last inequality then yields

$$H(x'(t)) \leqq \frac{K_0}{a(t_0)},$$

from which analogously as in the proof of Theorem 2 the boundedness of $x'(t)$ follows.

Furthermore from (4) it follows

$$F(t, x(t)) \leqq K_0 + \int_{t_0}^{t} a'(s)\,H(x'(s))\,\mathrm{d}s,$$

too; hence

$$F(t, x(t)) \leqq K_0 + \frac{K_0}{a(t_0)}\big(a(t) - a(t_0)\big) \leqq a_1 \frac{K_0}{a(t_0)}.$$

The further process is analogous to that of Theorem 2.

If we assume that $a(t) > 0$ and (6) holds, then the relation (4) can be arranged as follows

$$a(t)\,H(x'(t)) + F(t, x(t)) \leqq K_0 + \int_{t_0}^{t} \{a'(s)\}_+ H(x'(s))\,\mathrm{d}s \leqq$$

$$\leqq K_0 + \int_{t_0}^{t} \frac{\{a'(s)\}_+}{a(s)} \{a(s)\,H(x'(s)) + F(s, x(s))\}\,\mathrm{d}s;$$

hence

$$a(t)\,H(x'(t)) + F(t, x(t)) \leqq K_0 \exp \int_{t_0}^{t} \frac{\{a'(s)\}_+}{a(s)}\,\mathrm{d}s.$$

It is obvious that the following theorem holds.

Theorem 4. *Let the hypotheses of Theorem 2 hold with the exception of the assumption 1 instead of which we assume that $a(t) > 0$ for every $t \in I_0$ and*

$$\int_{t_0}^{\infty} \frac{\{a'(s)\}_+}{a(s)} \, ds = K_1 < \infty .$$

Moreover, suppose that for every $(t, x) \in D$ it is $f(t, x)\, x \geqq 0$. Then every solution $x(t)$ of (1) *which satisfies the inequality*

$$K_0 \exp K_1 < F_2 ,$$

is bounded on its domain.

If in addition (8) *holds and*

$$\frac{K_0}{a_0} \exp K_1 < H ,$$

then $x'(t)$ is bounded on its domain, too.

Remark 2. The assumption (6) in the previous theorems can be replaced by the assumption:

there exists a continuous function $\varphi(t)$ such that

$$\frac{\partial F(t, x)}{\partial t} \leqq \varphi(t)\, F(t, x) \qquad \text{for every } t \in I_0 \qquad \text{and} \qquad (t, x) \in D,$$

with

$$\int_{t_0}^{\infty} \{\varphi(s)\}_+ \, ds = K_2 < \infty . \tag{10}$$

From (4) we have

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq$$
$$\leqq K_0 + \int_{t_0}^{t} \left[\frac{\{a'(s)\}_+}{a(s)} + \{\varphi(s)\}_+ \right] \left[a(s)\, H(x'(s)) + F(s, x(s)) \right] ds$$

i.e.

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq K_0 \exp \int_{t_0}^{t} \left[\frac{\{a'(s)\}_+}{a(s)} + \{\varphi(s)\}_+ \right] ds .$$

If the solution $x(t)$ of (1) is defined on $\langle t_0, \infty)$, then we can easily prove e.g. the following theorem.

Theorem 4'. *Let the hypotheses of Theorem 4 and* (10) *hold. Then every solution $x(t)$ of* (1) *which satisfies the inequality*

$$K_0 \exp (K_1 + K_2) < F_2 ,$$

is bounded on $\langle t_0, \infty)$.

If in addition (8) *holds and*

$$\frac{K_0}{a_0}\exp(K_1 + K_2) < H,$$

then $x'(t)$ *is bounded on* $\langle t_0, \infty)$, *too.*

Remark 3. Evidently, if $F_1 = \infty$, then under the assumptions of Theorem 1 every solution $x(t)$ of (1) is bounded. Analogously if $F_2 = H = \infty$, then under the assumptions of Theorems 2–4′ every solution $x(t)$ of (1) is bounded or is bounded together with its first derivative.

Analogously as for the equation (1) it can be easily shown that for equation (1′) it holds:

$$a(t)\,H(x'(t)) + [1 + c(t)]\,F(t, x(t)) \leqq K_0' + \int_{t_0}^{t}[1 + c(s)]\,\frac{\partial F(s, x(s))}{\partial s}\,\mathrm{d}s + \\ + \int_{t_0}^{t} c'(s)\,F(s, x(s))\,\mathrm{d}s + \int_{t_0}^{t} a'(s)\,H(x'(s))\,\mathrm{d}s,$$

where

$$K_0' = a(t_0)\,H(x'(t_0)) + [1 + c(t_0)]\,F(t_0, x(t_0)) + \frac{K}{4k}\,.$$

If we assume that $1 + c(t) > 0$ for every $t \in I_0$ and that (6) holds, then from the last inequality it follows:

$$a(t)\,H(x'(t)) + [1 + c(t)]\,F(t, x(t)) \leqq \\ \leqq K_0' \exp \int_{t_0}^{t}\left[\frac{\{c'(s)\}_+}{1 + c(s)} + \frac{\{a'(s)\}_+}{a(s)}\right]\mathrm{d}s.$$

By the last inequality it can be easily proved e.g. the following theorem.

Theorem 5. *Let the hypotheses of Theorem 4 be fulfilled and suppose that* $1 + c(t) \geqq \geqq k_1 > 0$ *for every* $t \in I_0$, *where* k_1 *is a constant. If* $F_2 = H = \infty$ *and*

$$\int_{t_0}^{\infty}\frac{\{c'(s)\}_+}{1 + c(s)}\,\mathrm{d}s = K_3 < \infty,$$

then every solution $x(t)$ *of* (1′) *is bounded on* $\langle t_0, \infty)$ *together with its first derivative.*

Remark 4. Theorem 5 generalizes Theorem 8 and Theorem 21 in [3].

Theorem 6. *Suppose that the following assumptions are fulfilled;*

1. $b(t) \geqq 0$ *and* $g(x, y)\,y \geqq 0$ *or* $b(t) \leqq 0$ *and* $g(x, y)\,y \leqq 0$ *for every* $t \in I_0$ *and* $(x, y) \in R_2$;

2. *there exist non-negative numbers m and M such that*

$$\frac{|y|}{h(y)} = m + MH(y)$$

for every $y \in R_1$;

3. $M \mid r(t) \mid + a'(t) \leqq 0$ *for every $t \in I_0$.*

If

$$\int_{t_0}^{\infty} | r(s) | \, \mathrm{d}s = K_4 < \infty \tag{11}$$

and the assumption 3 of Theorem 1 holds, then every solution $x(t)$ of (1) *which satisfies the inequality*

$$K_0^* = a(t_0)\, H(x'(t_0)) + F(t_0, x(t_0)) + mK_4 < F_1,$$

is bounded on its domain.

Proof. From (1) we get

$$a(t)\frac{\mathrm{d}}{\mathrm{d}t} H(x'(t)) + \frac{\mathrm{d}}{\mathrm{d}t} F(t, x(t)) + \frac{b(t)\, g(x(t), x'(t))\, x'(t)}{h(x'(t))} =$$
$$= \frac{\partial F(t, x(t))}{\partial t} + r(t)\frac{x'(t)}{h(x'(t))},$$

hence by the assumptions of theorem, integrating from t_0, to $t \in (t_0 T)$, we get

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq K_0^* + \int_{t_0}^{t} \frac{\partial F(s, x(s))}{\partial s} \mathrm{d}s + \tag{12}$$
$$+ M \int_{t_0}^{t} | r(s) | \, H(x'(s))\, \mathrm{d}s + \int_{t_0}^{t} a'(s)\, H(x'(s))\, \mathrm{d}s,$$

i.e.

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq K_0^* + \int_{t_0}^{t} \frac{\partial F(s, x(s))}{\partial s} \mathrm{d}s.$$

The further process is analogous to that in the proof of Theorem 1.

By analogy with Theorem 2 we can prove the following theorem.

Theorem 7. *Let the hypotheses of Theorem 6 hold with the exception of the assumption 3 of Theorem 1 instead of which we assume that* (5) *holds.*

If (6) *holds, then every solution $x(t)$ of* (1) *which satisfies the inequality*

$$K_0^* < F_2,$$

is bounded on its domain.

If in addition (8) *holds, then even the first derivative of an arbitrary solution* $x(t)$ *of* (1) *for which*

$$\frac{K_0^*}{a_0} < H,$$

is bounded on its domain.

Theorem 8. *Let* (5), (6), (11) *and the assumptions 1 and 2 of Theorem 6 hold. Moreover, suppose that for every* $(t, x) \in D$ *it is* $f(t, x)\, x \geqq 0$.

If

$$\int_{t_0}^{\infty} \frac{M\,|\,r(s)\,| + \{a'(s\}_+}{a(s)}\, ds = K_5 < \infty,$$

then every solution $x(t)$ *of* (1) *which satisfies the inequality*

$$K_0^* \exp K_5 < F_2$$

is bounded on its domain.

If in addition (8) *holds then even the first derivative of an arbitrary solution* $x(t)$ *of* (1) *for which*

$$\frac{K_0^*}{a_0} \exp K_5 < H,$$

is bounded on its domain.

Proof. From (12) it follows

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq K_0^* +$$
$$+ \int_{t_0}^{t} \frac{M\,|\,r(s)\,| + \{a'(s)\}_+}{a(s)} [a(s)\, H(x'(s)) + F(s, x(s))]\, ds,^{1}$$

i.e.

$$a(t)\, H(x'(t)) + F(t, x(t)) \leqq K_0^* \exp K_5 .$$

Now, the proof can be completed exactly as the proof of the previous theorems.

Remark 5. If we replace (1) by (1'), Theorems 6-8 remain to be valid.

The following theorems deal with the unboundedness of solutions $x(t)$ of (1). We have

Theorem 9. *Let for the equation* (1) $a(t) > 0$, $b(t) < 0$ *for every* $t \in I_0$ (*instead of* $b(t) > 0$) *and let the assumption 2 of Theorem 1 be fulfilled.*

If for every $t \in I_0$, $x \in R_1$ *and* $(t, x) \in D$ *it is*

$$a'(t) \geqq 0, \qquad f(t, x)\, x \leqq 0, \qquad \frac{\partial F(t, x)}{\partial t} \geqq 0,$$

then every solution $x(t)$ of (1) *defined on* $\langle t_0, \infty)$ *such that*

$$a(t_0)\,H(x'(t_0)) + F(t_0, x(t_0)) + \frac{1}{4k}\int_{t_0}^{\infty} \frac{r^2(s)}{b(s)}\,\mathrm{d}s = K_0 > 0,$$

is unbounded for $t \to \infty$.

Proof. Multiplying (1) by $\dfrac{x'(t)}{h(x'(t))}$ and arranging we get

$$a(t)\frac{\mathrm{d}}{\mathrm{d}t} H(x'(t)) + \frac{\mathrm{d}}{\mathrm{d}t} F(t, x(t)) + kb(t)\left[\frac{x'(t)}{h(x'(t))}\right]^2 - \\ - r(t)\,\frac{x'(t)}{h(x'(t))} \geqq \frac{\partial F(t, x(t))}{\partial t}.$$

Using the fact, that for arbitrary real numbers a, b and x, $a < 0$ implies

$$ax^2 + bx \leqq -\frac{b^2}{4a},$$

the last inequality yields, integrating from t_0 to $t \in (t_0, \infty)$,

$$a(t)\,H(x'(t)) + F(t, x(t)) \geqq a(t_0)\,H(x'(t_0)) + F(t_0, x(t_0)) + \\ + \frac{1}{4k}\int_{t_0}^{\infty} \frac{r^2(s)}{b(s)}\,\mathrm{d}s + \int_{t_0}^{t} a'(s)\,H(x'(s))\,\mathrm{d}s.$$

Therefore for every $t \in \langle t_0, \infty)$ it is

$$a(t)\,H(x'(t)) \geqq K_0 + \int_{t_0}^{t} \frac{a'(s)}{a(s)}\,a(s)\,H(x'(s))\,\mathrm{d}s.$$

By Bellman's lemma the last inequality then yields

$$a(t)\,H(x'(t)) \geqq K_0 \exp\left[\ln a(t) - \ln a(t_0)\right] = K_0\,\frac{a(t)}{a(t_0)}.$$

Therefore for every $t \in (t_0, \infty)$ it is

$$H(x'(t)) \geqq \frac{K_0}{a(t_0)} > 0. \tag{13}$$

Since $H(y) \in C(R_1)$, there exists x_0', $\operatorname{sgn} x_0' = \operatorname{sgn} x'(t_0)$, such that $H(x_0') = \dfrac{K_0}{a(t_0)}$. From (13) it follows, that for every $t \in \langle t_0, \infty)$ it is $x'(t) \neq 0$, i.e. $x'(t)$ does not change its sign. Let $x'(t_0) > 0$, then it is $x'(t) > 0$ for every $t \in \langle t_0, \infty)$. Therefore it is

$$\frac{\mathrm{d}}{\mathrm{d}y} H(y) = \frac{y}{h(y)} > 0$$

for $y = x'(t)$, i.e. $H(x'(t)) \geqq H(x'(t_0))$. Hence $x'(t) \geqq x_0'$, from where $\lim_{t\to\infty} x(t) = \infty$ follows.

Now let $x_0' < 0$. Then it is $x'(t) < 0$ for every $t \in \langle t_0, \infty)$ and therefore $\frac{\mathrm{d}H(y)}{\mathrm{d}y} <$ < 0 for $y = x'(t)$. From $H(x'(t)) \geqq H(x'(t_0))$ it follows that $x'(t) \leqq x_0'$. This means that $\lim_{t\to\infty} x(t) = -\infty$. This completes the proof.

Theorem 10. *Let the assumption 2 of Theorem 6 be fulfilled. Moreover, suppose that the following assumptions hold:*

1. $b(t) \geqq 0,\ g(x, y)\, y \leqq 0$

   or

   $b(t) \leqq 0,\ g(x, y)\, y \geqq 0$

   for every $t \in I_0$ *and* $(x, y) \in R_2$;

2. $a'(t) - M\,|\,r(t)\,| \geqq 0,\ f(t, x)\, x \leqq 0,\ \frac{\partial F(t, x)}{\partial t} \geqq 0,\ a(t) > 0$

   for every $t \in I_0$ *and* $(t, x) \in D$.

If (11) *holds, then every solution* $x(t)$ *of* (1) *defined on* $\langle t_0, \infty)$ *such that*

$$K_0^* = a(t_0)\, H(x'(t_0)) + F(t_0, x(t_0)) - mK_4 > 0,$$

is unbounded for $t \to \infty$.

Proof. Analogously as in the proof of Theorem 9 from the assumptions of Theorem we get

$$a(t)\, H(x'(t)) \geqq K_0^* + \int_{t_0}^{t} \frac{a'(s) - M\,|\,r(s)\,|}{a(s)}\, a(s)\, H(x'(s))\, \mathrm{d}s.$$

Therefore for $t \in \langle t_0, \infty)$ it is

$$H(x'(t)) \geqq \frac{K_0^*}{a(t_0)} \exp\left(-\frac{MK_4}{a(t_0)}\right) > 0.$$

The further process is analogous to that in the proof of Theorem 9.

II.

In this part we shall investigate the oscillatory properties of solutions of a nonlinear differential equation (1), where $r(t) \equiv 0$ for every $t \in I_0$, i.e. of the equation

$$a(t)\, x'' + b(t)\, g(x, x') + f(t, x)\, h(x') = 0. \tag{14}$$

We shall assume that $a(t) > 0$, $b(t) \geqq 0$ for every $t \in I_0$. We have

Theorem 11. *Let* $g(x, y)\, x > 0$ *for* $x \neq 0$, $f(t, x)\, x \geqq 0$ *for every* $x \in R_1$, $(t, x) \in D$ *and* $(x, y) \in R_2$. *Moreover, suppose that* $g(x, y)$ *is increasing in* x *for every* y.

If

$$\int^{\infty} \frac{b(s)}{a(s)} \mathrm{d}s = +\infty, \tag{15}$$

then the solution $x(t)$ of (14) *which is defined on $\langle t_0, \infty)$ is oscillatory.*

Proof. Suppose that a solution $x(t)$ of (14) is defined on $\langle t_0, \infty)$ and let it is not oscillatory. Let e.g. $x(t) > 0$ for every $t \in \langle t_1, \infty)$, $t_1 \geqq t_0$. We can easily show that $x'(t) \geqq 0$ and $x''(t) \leqq 0$ for $t \in \langle t_1, \infty)$.

From (14) we get

$$a(t)\, x''(t) \leqq -b(t)\, g(x(t), x'(t)) \leqq -b(t)\, g(x(t_1, x'(t)) \leqq -b(t) \min_{0 \leqq y \leqq x'(t_1)} g(x(t_1), y).$$

By (15) from the last inequality it follows that $x'(t) \to -\infty$ for $t \to \infty$ which is a contradiction.

If we assume that $x(t) < 0$ for $t \in \langle t_1, \infty)$, then the proof is analogous as in the case $x(t) > 0$.

Theorem 12. *Let*

$$f(t, x)\, x > 0 \qquad \text{for } x \neq 0, \qquad g(x, y)\, x \geqq 0 \tag{16}$$

for every $x \in R_1$, $(t, x) \in D$ and $(x, y) \in R_2$. Moreover, suppose that $f(t, x)$ is increasing for every $t \in I_0$ on R_1.

If for every $\delta \neq 0$ it is

$$\operatorname{sgn} \delta \int^{\infty} \frac{f(s, \delta)}{a(s)} \mathrm{d}s = \infty, \tag{17}$$

then the solution $x(t)$ of (14) *which is defined on $\langle t_0, \infty)$ is oscillatory.*

Proof. Let a solution $x(t)$ of (14) is defined on $\langle t_0, \infty)$ and let it is not oscillatory. Suppose, e.g. that $x(t) > 0$ for every $t \in \langle t_1, \infty)$, $t_1 \geqq t_0$, i.e. that $x(t) > 0$, $x'(t) \geqq 0$ and $x''(t) \leqq 0$ for every $t \in \langle t_1, \infty)$.

Since $x'(t)$ is bounded and $h(y)$ is continuous, there exists $m > 0$ such that

$$m = \min_{0 \leqq y \leqq x'(t_1)} h(y) \leqq h(x'(t))$$

for every $t \in \langle t_1, \infty)$. Therefore from (14) it follows:

$$a(t)\, x''(t) \leqq -f(t, x(t))\, h(x'(t)) \leqq -m f(t, x(t_1)),$$

hence by (17) we get

$$x'(t) \leqq x'(t_1) - m \int_{t_1}^{t} \frac{f(s, x(t_1))}{a(s)} \mathrm{d}s \to -\infty \qquad \text{for } t \to \infty,$$

which is a contradiction.

If we assume that $x(t) < 0$ for every $t \in \langle t_1, \infty)$, then the proof is analogous as for $x(t) > 0$.

Remark 6. Theorem 12 is a generalization of Theorem 23 of [3].

Theorem 13. *Let* (16) *hold and suppose that for every* $\delta > 0$, $\alpha < 1$ *it is*

$$\int^{\infty} \frac{s^{\alpha}}{a(s)} \inf_{\delta \leqq |x| < \infty} \frac{f(s, x)}{x} \, ds = \infty.$$

Then the solution $x(t)$ *of* (14) *which is defined on* $\langle t_0, \infty)$ *is oscillatory.*

Proof. We can show analogously as in the previous Theorems the following: for a non-oscillatory solution $x(t)$ of (14) there exists $t_1 \in I_0$ such that $x(t) > 0$, $x'(t) \geqq 0$ and $x''(t) \leqq 0$ or $x(t) < 0$, $x'(t) \leqq 0$ and $x''(t) \geqq 0$ for every $t \in \langle t_1, \infty)$. Let $x(t) > 0$, $x'(t) \geqq 0$ and $x''(t) \leqq 0$ (for $x(t) < 0$, $x'(t) \leqq 0$ and $x''(t) > 0$ the proof is quite analogous). Define a function

$$v(t) = t^{\alpha} \frac{x'(t)}{x(t)} \qquad \text{on } \langle t_1, \infty).$$

Then by (14) we get

$$v'(t) + \frac{1}{t^{\alpha}} v^2(t) - \frac{\alpha}{t} v(t) = \tag{18}$$

$$= -\frac{t^{\alpha}}{x(t)} \frac{b(t)}{a(t)} g(x(t), x'(t)) - \frac{t^{\alpha}}{x(t)} \frac{f(t, x(t))}{a(t)} h(x'(t)).$$

Using the fact that for arbitrary real numbers a, b and x, $a > 0$ implies

$$ax^2 + bx \geqq -\frac{b^2}{4a},$$

from (18) by the assumptions of theorem we obtain

$$v'(t) \leqq \frac{\alpha^2}{4} \frac{1}{t^{2-\alpha}} - m \frac{t^{\alpha}}{a(t)} \inf_{x(t_1) \leqq x < \infty} \frac{f(t, x)}{x} \tag{19}$$

for every $t \in \langle t_1, \infty)$, where $t_1 > 0$ and $m = \min_{0 \leqq y \leqq x'(t_1)} h(y)$. Integrating (19) from t_1 to $t \in \langle t_1, \infty)$ we get

$$v(t) \leqq v(t_1) + \frac{\alpha^2}{4} \int_{t_1}^{t} \frac{1}{s^{2-\alpha}} \, ds - m \int_{t_1}^{t} \frac{s^{\alpha}}{a(s)} \inf_{x(t_1) < x < \infty} \frac{f(s, x)}{x} \, ds,$$

hence for $t \to \infty$ we have $v(t) \to -\infty$ (because for $\alpha < 1$ the first integral is finite), which is a contradiction. This completes the proof.

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