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TERM FUNCTIONS ON NON-ABELIAN GROUPS OF ORDER pq

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The purpose of this note is to determine the number of all the different term functions in n variables over non-abelian groups of order pq, p, q being distinct primes. In her paper [1], Coufalová proved a formula for the case $n = 2$, $p = 3$, $q = 2$, by a more or less explicit enumeration of the term functions. By a term function on a group G in n variables we mean a function t: $G^n \to G$ of the form $(g_1, \ldots, g_n) \mapsto w(g_1, \ldots, g_n)$, $g_i \in G$, where $w(x_1, \ldots, x_n)$ is an element of the free group freely generated by $x_1, \ldots, x_n$. Coufalová also considered the case $n = 3$, $p = 3$, $q = 2$ and, for the same values of p and q, provided a formula for any n. Her method consisted essentially of solving a system of congruences which led to elaborate calculations. As an alternative this paper is to offer a different approach based on Schreier's formula arising from his subgroup theorem (see e.g. [3]) which has the advantage of being structural and possibly open to further generalization, and moreover helps to explain Coufalová's formula. It should also be noted that B. H. Neumann [2] in 1937 gave an upper bound for the number of term functions on S_3 in two variables, namely $6^3 \cdot 3^4$, which is only 18 times the actual value.

Let us first observe that a law in a group G is a word $w(x_1, \ldots, x_n)$ of some free group F having $\{x_1, \ldots, x_n\}$ as a subset of its free generating set such that the term function t: $(g_1, \ldots, g_n) \mapsto w(g_1, \ldots, g_n)$ sends every n-tuple of elements $(g_1, \ldots, g_n) \in G^n$ to the identity of G. Consequently, the number of term functions in n variables on G is just the order of the relatively free group $F_n(\text{var } G)$ of rank n of the variety var G generated by G. For the remainder of this note, let G be the non-abelian group of order pq, p, q being distinct primes; we observe that $q/p - 1$ and that every extension of an elementary abelian p-ground by an elementary abelian q-group belongs to var G, by virtue of being a subdirect product of groups isomorphic to either $G \times C_q \times \ldots \times C_q$ or $C_p \times C_q \times \ldots \times C_q$.

Theorem. *There are exactly $q^n p^{(n-1)q^n+1}$ different term functions in n variables over the group G.*

Proof. We have to show that $|F_n(\text{var } G)| = q^n p^{(n-1)q^n+1}$.

Let F_n be the free group of rank n, $W \lhd F_n$ such that F_n/W is elementary abelian of order q^n, and R the least normal subgroup of W such that W/R is an elementary abelian p-group. Then F_n/R is an n-generator group in var G, and every n-generator group in var G is a homomorphic image of F_n/R, thus $F_n/R \cong F_n(\text{var } G)$. By the Schreier subgroup theorem, W is free of rank $(n-1)\, q^{n+1}$, hence W/R is elementary abelian of order $p^{(n-1q^n+1}$. Therefore $|F_n(\text{var } G)| = |F_n/W|\;|W/R| = q^n p^{(n-1)q^n+1}$, Q.E.D.

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