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ANOTHER APPROACH TO THE CLASSICAL CALCULUS OF VARIATIONS

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I. Lagrange problem

With a very few exceptions, a prevalent number of recent investigations devoted to geometric theory of the classical calculus of variations deals with the multiple integral problem under the fixed boundary condition. It seems that the true reason for this innatural limitation lies in an inappropriate setting of other classical problems, where the geometric content is suppressed by a hard analysis from the very beginning. We can remind the Lagrange problem as the most instructive example: Using the common approach, one must impose very strong, cumbersome, and for the most part unverifiable assumptions, compare [1].

Our aim is to outline a modified approach to all these topics which is equivalent to the classical one in all favourable cases but operates in full generality as well. Of course, the true difficulties cannot be completely removed, but they appear gradually and acquire the much more simple *linear* character. Roughly speaking, the difference consists in employing the "virtual variations" instead of the "actual" ones, the setting inspired by the famous D'Alambert principle from dynamics. We shall also use the current calculus of differential forms in a large extent together with certain elements of functional analysis and distribution theory on vector bundles.

The main results of the present paper were derived in the year 1979, and were explained in a session of the Seminar on the calculus of variations in Brno, see also [2]. I wish to express my gratitude to D. Krupka, the leader of the Seminar, for his helpful and stimulating criticism, and to I. Kolář for his kind interest.

1. The common approach. Let V be a manifold (we admit an infinite-dimensional Frechet manifold here). A *tangent vector* Z_p of V at a point $p \in V$ may be determined by its action on an arbitrary function $F \in C^\infty(V)$ (the space of all C^∞-smooth functions on V):

$$Z_p F = \lim_{\lambda \to 0} \frac{F(p(\lambda)) - F(p)}{\lambda},$$

where $p(\lambda)$, $-\varepsilon < \lambda < \varepsilon$, $\varepsilon > 0$, is an arbitrary C^∞-smooth one-parameter family of points $p(\lambda) \in V$ with $p(0) = p$. (We shall say that the mentioned family is *related* with the vector Z_p.) Usually, the *tangent cone* of a subset $P \subset V$ at a point $p \in P$ is defined as the set of all such tangent vectors Z_p for which the related family $p(\lambda)$ exists with the property $p(\lambda) \in P$, $-\varepsilon < \lambda < \varepsilon$. And the point $p \in P$ is called *P-critical point* of a function $F \in C^\infty(V)$, if $Z_p F = 0$ is true for all tangent vectors Z_p from this tangent cone.

It might be not easy to operate with the above families $p(\lambda) \in P$, and even to prove their existence. This unhappy state does not change for better even in the nice and frequent case when the subset P is a *level set* of a C^∞-smooth mapping $G : V \to W$, where W is an auxiliary manifold. (That means, the set P consists of all $p \in V$ which satisfy an equation $G(p) = w$, $w \in W$ is a fixed vector.) A typical example follows:

Let $V = C^\infty(P)^m$ be the space of all C^∞-smooth $\mathbf{R}^m$-valued functions $p(x)$, $x = (x^1, \ldots, x^n)$, defined in an open domain $P \subset \mathbf{R}^n$. Endowed by its natural topology, V is a Frechet space and hence a manifold. Let $P \subset V$ be the subset consisting of all C^∞-smooth solutions of a C^∞-smooth system of partial differential equations $G_\iota(x, p, \ldots, \partial^{|I|}p/\partial x^I, \ldots) = 0$ ($\iota = 1, \ldots, c$, $I = (i_1, \ldots, i_k)$, $i_1, \ldots, i_k = 1, \ldots, n$, $k = 0, 1, \ldots$). Clearly, P is a level set of the mapping $G : V \to W = C^\infty(P)^c$, $G = (G_1, \ldots, G_c)$, namely $P = G^{-1}(0)$. However, we cannot say anything more about the existence of any family $p(\lambda) \in P$, $p(\lambda) \not\equiv p$ and, consequently, about the existence of a non-zero vector in the tangent cone of P at a given point $p \in P$.

2. The modified approach. We retain the subset P to be a level set of the above mapping G, and we look for such tangent vectors Z_p at $p \in P$ which surely *do not* touch P. They may be characterized by the inequality $\mathrm{d}G(Z_p) \neq 0$, or, if we wish to eliminate the tangent mapping $\mathrm{d}G$, by the more elementary inequality ($\mathrm{d}G(Z_p) =$)

$$\lim_{\lambda \to 0} \frac{G(p(\lambda)) - G(p)}{\lambda} \neq 0,$$

where $p(\lambda) \in V$ is a family related with Z_p. Now, the other tangent vectors at p will be automatically counted in the new tangent cone. As a result, the *modified* tangent cone of P at the point $p \in P$ is identified with the null-space of the mapping $\mathrm{d}G$ at p. This result suggests the following fundamental

3. Definition. Let $G : V \to W$ be a smooth mapping between manifolds V, W. The point $p \in V$ is called a *G-critical point of a function* $F \in C^\infty(V)$, if $\mathrm{d}F(Z_p) = 0$ for all tangent vectors Z_p of V at p with the property $\mathrm{d}G(Z_p) = 0$. This may be expressed by more elementary terms, as follows: A G-critical point $p \in V$ is characterised by the implication:

$$(1)\qquad \text{If } \lim_{\lambda\to 0}\frac{G(p(\lambda))-G(p)}{\lambda}=0, \quad \text{then} \quad \lim_{\lambda\to 0}\frac{F(p(\lambda))-F(p)}{\lambda}=0,$$

for every C^∞-smooth family of points $p(\lambda)\in \boldsymbol{V}$, $-\varepsilon<\lambda<\varepsilon$, $\varepsilon>0$, with $p(0)=p$.

It can be seen that both mentioned approaches are equivalent provided a sufficiently powerfull implicite function theorem for the mapping $\boldsymbol{G}$ is true. We shall not deal, however, with such generalities. Instead of that we intend to consider certain concrete applications of the new approach in the field of the classical calculus of variations. Here, the manifold $\boldsymbol{V}$ appears as an infinite-dimensional manifold of mappings, the point $p\in\boldsymbol{P}$ is realized as a mapping $p: P\to M$ between the given (finite-dimensional) manifolds P, M. However, since a rigorous theory of such manifolds is not simple, we shall avoid any mentioning it and the preceding Sections will be considered as a merely motive for the following development. Fortunately, the crucial concept of the $\boldsymbol{G}$-critical point is quite clear, owing to (1). As far as the terminology is concerned, functions $\boldsymbol{F}\in C^\infty(\boldsymbol{V})$ are called *functionals*, tangent vectors $\boldsymbol{Z}_p$ are known under the name *variations* (of the mapping p), and the critical points p are usually called *extremals.*

4. Setting the problem. We shall operate in real domain and besides the explicitly stated exceptions, with C^∞-smooth maps between C^∞-smooth and finite-dimensional manifolds. We shall also freely employ locally-finite and countable coverings by coordinate systems, partitions of unity, non-vanishing densities, and other nice objects.

Thus, let P be an n-dimensional compact and oriented manifold with boundary Q (which may be empty). We shall denote by $\partial: Q\to P$ the natural inclusion of the boundary. Let M be a manifold, $\alpha^1,\ldots,\alpha^c,\beta^1,\ldots,\beta^d,\varphi,\psi$ be certain exterior differential forms on M. We admit also 0-forms (i.e. functions) and it may be $c=0$ or $d=0$; φ is an n-form, ψ is an $(n-1)$-form.

Let $\boldsymbol{V}$ be the space of all embeddings $p: P\to M$, with the natural topology (of uniform convergence with every finite number of derivatives calculated in appropriate local coordinate systems on P and M), and $q=p\circ\partial: Q\to M$ be the mapping p restricted to the boundary. Let $\boldsymbol{P}$ be the subspace of $\boldsymbol{V}$ consisting of all $p\in\boldsymbol{V}$ which satisfy

$$(2)\qquad p^*\alpha^1=\ldots=p^*\alpha^c=0,$$

$$(3)\qquad q^*\beta^1=\ldots=q^*\beta^d=0.$$

(The asterisks mean the pull-back of forms.) Now, $\boldsymbol{P}$ will be represented as a level set:

Denote by $\mathfrak{A}$ the $C^\infty(M)$-module of all exterior differential n-forms from the ideal of exterior forms generated by $\alpha^1,\ldots,\alpha^c$. (In other words, $\mathfrak{A}$ consists of all n-forms α representable as $\alpha=\Sigma\alpha^j\wedge\gamma^j$, for certain γ^j.) The system (2) is equivalent to the condition $p^*\alpha\equiv 0$ ($\alpha\in\mathfrak{A}$), see [3] p. 26. And, by using the "Fundamental lemma of the calculus of variations", the last condition is equivalent to $\int_P p^*\alpha \equiv 0$ ($\alpha \in \mathfrak{A}$). Analogously, denote by $\mathfrak{B}$ the $C^\infty(M)$-module of all exterior differential $(n-1)$-forms contained in the ideal of exterior forms with generators $\beta^1, \ldots, \beta^d$. Then the system (3) is equivalent to the condition $\int_Q q^*\beta \equiv 0$ ($\beta \in \mathfrak{B}$). Introduce the linear topological space

$$W = \prod_{\alpha \in \mathfrak{A}, \beta \in \mathfrak{B}} \mathbf{R}_{\alpha,\beta},$$

the direct product of real axes $\mathbf{R}$ indexed by all couples (α, β), where $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$. Let $G : V \to W$ be the mapping determined by the components

$$G(p)_{\alpha,\beta} = \int_P p^*\alpha + \int_Q q^*\beta.$$

It is a consequence of the above equivalences that P is a level set of this mapping: It consists of all points $p \in V$ with the property $G(p) = 0$.

At last, let F be the functional on V, where

$$F(p) = \int_P p^*\varphi + \int_Q q^*\psi. \tag{4}$$

Following the line of the Section 3, an embedding $p \in V$ is called a *G-critical point of the functional F*, if the implication (1) is true for every C^∞-smooth family $p(\lambda) \in V$, $-\varepsilon < \lambda < \varepsilon$, $\varepsilon > 0$, with $p(0) = p$.

The *Lagrange problem* consists in investigating the mentioned G-critical points. It is determined by the data $P, M, \mathfrak{A}, \varphi, \mathfrak{B}, \psi$ and will be denoted by $\mathscr{LP}(P, M, \mathfrak{A}, \varphi, \mathfrak{B}, \psi)$, briefly $\mathscr{LP}$.

5. Notice. Another level maps can also be used, instead of G. We mention the most natural example here: Denote by $C^\infty(\wedge T^*P)$, $C^\infty(\wedge T^*Q)$, the space of all exterior differential forms on P, Q, respectively. Let

$$W' = (C^\infty(\wedge T^*P))^c \oplus (C^\infty(\wedge T^*Q))^d,$$

and let the mapping $G' : V \to W'$ be given by

$$G'(p) = (p^*\alpha^1 \oplus \ldots \oplus p^*\alpha^c) \oplus (q^*\beta^1 \oplus \ldots \oplus q^*\beta^d).$$

Then the equations (2), (3) exactly mean that P is a level set of the mapping G'. However, it may be easily proved that a point $p \in V$ is G-critical if and only if it is G'-critical.

6. Theorem. *An embedding $p \in V$ is a G-critical point of F if and only if for every vector field Z on M there exist forms $\bar{\alpha} \in \mathfrak{A}$, $\bar{\beta} \in \mathfrak{B}$ satisfying*

$$\int_P p^*Z \lrcorner \, d(\varphi - \bar{\alpha}) + \int_Q q^*Z \lrcorner \, (\varphi - \bar{\alpha} + d(\psi - \bar{\beta})) = 0. \tag{5}$$

Proof: Let $p(\lambda) \in V$ be one of the above families with $p(0) = p$. For every point $t \in P$, we obtain a tangent vector $Z_{p(t)} \in T_{p(t)}M$ (the tangent space of M at the point $p(t)$), where

$$Z'_{p(t)}f \equiv \lim_{\lambda \to 0} \frac{f(p(\lambda)(t)) - f(p(t))}{\lambda} \qquad (f \in C^\infty(M)).$$

Then, using the common technique of partitions of unity, local coordinate systems, and the fact that p is an *embedding* of a *compact* manifold, one can prove the existence of a vector field Z on M with the property $Z'_{p(t)} \equiv Z_{p(t)}$ $(t \in P)$. (In other words, the vector field Z' along the subset $pP \subset M$ is extended into a vector field Z defined on all M.)

Conversely, every vector field Z on M may be obtained from an appropriate family $p(\lambda)$ following the mentioned procedure.

We may employ the *Lie derivative* operator $\mathscr{L}_Z$. Using the well-known identities

$$\begin{aligned} p(\lambda)^* \zeta &= p^*\zeta + \lambda p^*\mathscr{L}_Z\zeta + \text{higher order terms in } \lambda, \\ q(\lambda)^* \zeta &= q^*\zeta + \lambda q^*\mathscr{L}_Z\zeta + \text{higher order terms in } \lambda, \\ \mathscr{L}_Z\zeta &= Z \lrcorner \, d\zeta + dZ \lrcorner \, \zeta, \end{aligned}$$

$$\int_P p^* \, d\zeta = \int_Q q^*\zeta,$$

(ζ is an exterior form on M), one can easily see that

$$\begin{aligned} \lim_{\lambda \to 0} \frac{F(p(\lambda)) - F(p)}{\lambda} &= \int_P p^*\mathscr{L}_Z\varphi + \int_Q q^*\mathscr{L}_Z\psi = \\ &= \int_P p^*Z \lrcorner \, d\varphi + \int_Q q^*Z \lrcorner \, (\varphi + d\psi), \end{aligned}$$

$$\begin{aligned} \lim_{\lambda \to 0} \frac{G(p(\lambda))_{\alpha,\beta} - G(p)_{\alpha,\beta}}{\lambda} &= \int_P p^*\mathscr{L}_Z\alpha + \int_Q q^*\mathscr{L}_Z\beta = \\ &= \int_P p^*Z \lrcorner \, d\alpha + \int_Q q^*Z \lrcorner \, (\alpha + d\beta). \end{aligned}$$

Thus, introducing the brief notation

$$(Z \mid \zeta) = \int_P p^*Z \lrcorner \, \zeta, \qquad [Z \mid \zeta] = \int_Q q^*Z \lrcorner \, \zeta,$$

we may express that p is a G-critical point of F if and only if

$$(Z \mid d\varphi) + [Z \mid \varphi + d\psi] = 0 \tag{6}$$

for all vector fields Z on M, which satisfy

$$(Z \mid d\alpha) + [Z \mid \alpha + d\beta] \equiv 0 \qquad (\alpha \in \mathfrak{A}, \beta \in \mathfrak{B});$$

and the condition (5) may be written as

$$(Z \mid d(\varphi - \bar{\alpha})) + [Z \mid \varphi - \bar{\alpha} + d(\psi - \bar{\beta})] \equiv 0. \tag{7}$$

The rest of the proof is easy: If the forms $\bar{\alpha} \in \mathfrak{A}$, $\bar{\beta} \in \mathfrak{A}$ exist, then p is a G-critical point by (7), (6). Conversely, let p be a G-critical point and Z be a vector field on M. Denote $(Z \mid d\varphi) + [Z \mid \varphi + d\psi] = C$. If $C \neq 0$, then there exist forms $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$ such that $(Z \mid d\alpha) + [Z \mid \alpha + d\beta] = D \neq 0$, and we may choose $\bar{\alpha} = C\alpha/D$, $\bar{\beta} = C\beta/D$. If $C = 0$, then we may choose $\bar{\alpha} = 0$, $\bar{\beta} = 0$.

7. Standard critical points. The Theorem 6 is rather imperfect, and it will be analysed in the Appendix. However, we cannot expect some advanced results for our enormously general case. If we demand a theory comparable with the classical one, we must either restrict the class of problems, or choose a particular type of critical points with certain favourable properties. We shall follow the latter approach.

The embedding $p \in V$ is called a *standard* G-critical point of F, if there exist such forms $\bar{\alpha} \in \mathfrak{A}$, $\bar{\beta} \in \mathfrak{B}$ that (5) is true for every vector field Z on M. In this case, the condition (5) decomposes into the inner and the boundary component

$$(8)_{i,b} \qquad (Z \mid d(\varphi - \bar{\alpha})) \equiv 0, \qquad [Z \mid -\bar{\alpha} + d(\psi - \bar{\beta})] \equiv 0.$$

Moreover, the integration process may be omitted here:

$$(9)_{i,b} \qquad p^*Z \lrcorner\, d(\varphi \lrcorner\, \bar{\alpha}) \equiv 0, \qquad q^*Z \lrcorner\, (\varphi - \bar{\alpha} + d(\psi - \bar{\beta})) \equiv 0,$$

for all vector fields Z on M.

Without any exaggeration, the whole classical calculus of variations is covert in these simple relations. The equation $(9)_i$ is a concise expression of the fundamental *Euler–Lagrange system*, and $(9)_b$ gives the *tranversality conditions*. It may happen that the form $\varphi - \bar{\alpha}$ is a fixed form on M, *universal* for all G-critical point p. Then we have a far going generalization of the *Poincaré–Cartan–Lepage form*. The form $\psi - \bar{\beta}$ may be considered as a *boundary counterpart* of it. The case $c = 0$, $d = 0$ (hence $\mathfrak{A}$, $\mathfrak{B}$ trivial) is very interesting: All G-critical points are automaticaly standard ones. This case is not as special as it looks, every Lagrange problem can be transferred into it. This process appears as a generalization of the *Hamiltonian theory*.

Usually, the main interest is concentrated only up the inner relations $(8)_i$, $(9)_i$. Then the relevant definition to be used is: A G-critical point $p \in V$ is called an *inner standard point*, if there exists a form $\bar{\alpha} \in \mathfrak{A}$ for which $(9)_i$ is true. Or, following more closely the common terminology, we speak about the *extremal*, if $p^*\alpha \equiv 0$ ($\alpha \in \mathfrak{A}$), and $(9)_i$ is true for certain form $\bar{\alpha} \in \mathfrak{A}$. The space of all extremals depends on the data P, M, $\mathfrak{A}$, $d\varphi$ and will be denoted as $\mathscr{EX}(P, M, \mathfrak{A}, d\varphi)$, briefly as $\mathscr{EX}$. In the case $c = 0$, every G-critical point is an inner standard one. This case will be considered in the following Part of the paper.

As far as the non-standard critical points are concerned, they may be considered as a sort of generalised extremals, for which the Euler–Lagrange equations and transversality conditions hold in certain approximative sense. Usually, they were excluded by imposing strong restriction on the Lagrange problem under consideration.

In dealing with the relations (9), the following remark will be usefull:

8. Lemma. *Let Z be a vector field on M. If $Z_{p(t)} \in dp(T_tP)$ for certain $t \in P$, then $(p^*Z \lrcorner \zeta)_t = 0$ for every $(n + 1)$-form ζ on M. If $Z_{q(t)} \in dq(T_tQ)$ for certain $t \in Q$, then $(q^*Z \lrcorner \zeta)_t = 0$ for every n-form ζ on M.*

Here, $dp(T_tP) - T_{p(t)}M$ is the tangent space to the submanifold pP at the point $p(t)$, the image of the tangent space T_tP of the manifold P at the point t by the tangent map dp. Similarly, $dq(T_tQ) \subset T_{q(t)}M$ is the tangent space of the submanifold qQ. According to the Lemma, the vector fields Z tangent to these submanifolds pP, qQ, are unessential for the relation $(9)_i$, $(9)_b$, respectively: It is sufficient to calculate only with such vector fields Z that the vectors $Z_{p(t)}$, $Z_{q(t)}$, lie in a fixed vector subspace of $T_{p(t)}M$ complementary to $dp(T_tP)$, $dq(T_tQ)$, respectively.

Proof: Let $Z_{p(t)} \in dp(T_tP)$, namely $Z_{p(t)} = dp(Z_0)$ for certain $Z_0 \in T_tP$. Let $Z_1, \ldots, Z_n$ be arbitrary vectors from T_tP. The $(n + 1)$-tuple $Z_0, \ldots, Z_n$ is linearly dependent, hence $(p^*Z \lrcorner \zeta)(Z_1, \ldots, Z_n) = (p^*\zeta)(Z_0, \ldots, Z_n) = 0$. The second statement of the Lemma is similar.

9. Example. We shall consider a simple problem as a mere illustration. Let P be a subdomain of the space R^n with variables t^i $(i = 1, \ldots, n)$, and set $M = R^{n+m+nm}$ with variables x^i, y^j, y^j_i $(i = 1, \ldots, n, j = 1, \ldots, m)$. We shall use the notation

$$dt = dt^1 \wedge \ldots \wedge dt^n, \qquad dt^{(i)} = -(-1)^i dt^1 \wedge \ldots \wedge dt^{i-1} \wedge dt^{i+1} \wedge \ldots \wedge dt^n,$$

and similarly for dx, $dx^{(i)}$.

The Lagrange problem will be determined by the forms $\alpha^j = dy^j - \Sigma y^j_i\, dx^i$ (*contact forms*, $j = 1, \ldots, n = c$), $\beta^k = b^k$ (the functions, $k = 1, \ldots, d$), $\varphi = f\, dx$, $\psi = \Sigma f^i\, dx^{(i)}$ (where we suppose that the functions f^i do not depend on variables y^j_i).

We shall consider only such G-critical points p which are given by certain equations of the type $x^i \equiv t^i$, $y^j \equiv \bar{y}^j(t^1, \ldots, t^n)$, $y^j_i \equiv \bar{y}^j_i(t^1, \ldots, t^n)$. Then, according to Lemma 8, we may use only special vector fields

$$Z = \Sigma z^j \frac{\partial}{\partial y^j} + \Sigma z^j_i \frac{\partial}{\partial y^j_i}, \tag{10}$$

without $\partial/\partial x^i$ terms. Moreover, it will be proved later on that every G-critical point p of the mentioned kind is an inner standard one. So we may find the form $\bar{\alpha}$ occuring in the relation $(9)_i$. This form may be represented as $\bar{\alpha} = \Sigma \alpha^j \wedge \gamma^j$, for certain forms γ^j, and we obtain

$$T \lrcorner \, d\bar{\alpha} = -\Sigma z^j \wedge d\gamma^j - \Sigma z_i^j \wedge dx^i \wedge \gamma^j + (\dots),$$

$$Z \lrcorner \, d\varphi = \left(\Sigma z^j \frac{\partial f}{\partial y^j} + \Sigma z_i^j \frac{\partial f}{\partial y_i^j}\right) dx + (\dots),$$

where the terms $(\dots)$ involve certain forms α^j or $d\alpha^j$ as a factor. Now, the relation $(9)_i$ looks as follows:

$$p^*\left(\Sigma z^j \left(\frac{\partial f}{\partial y^j} dx + d\gamma^j\right) + \Sigma z_i^j \left(\frac{\partial f}{\partial y_i^j} dx + dx^{(i)} \wedge \gamma^j\right)\right) \equiv 0, \tag{11}$$

for all z^j, z_i^j. Especially, choosing $z^j \equiv 0$, z_i^j arbitrary, we obtain the relation

$$p^*\left(\frac{\partial f}{\partial y_i^j} dx + dx^i \wedge \gamma^j\right) \equiv 0,$$

which is satisfied by $\gamma^j = -\partial f/\partial y_i^j \, dx^{(i)} + \Gamma^j$, where Γ^j is an arbitrary form from the ideal generated by $a^1, \dots, \alpha^n$. Thus we obtain the famous Poincaré–Cartan–Lepage form [4]:

$$\varphi - \bar{\alpha} = f \, dx + \Sigma \frac{\partial f}{\partial y_i^j} \alpha^j \wedge dx^{(i)} + \Sigma \alpha^j \wedge \alpha^{j'} \wedge \Gamma^{jj'}, \tag{12}$$

where the forms $\Gamma^{jj'}$ are arbitrary.

(Note that this is not the most general solution. In fact, Γ^j may be an arbitrary sum of terms involving certain form α^j or $d\alpha^j$ as a factor. Moreover, if the form $\varphi - \bar{\alpha}$ may depend on the critical point p, then we may even choose an arbitrary Γ^j with the property $p^*\Gamma^j = 0$.)

Using the form (12), the relation (11) reduces to

$$p^*\left(\frac{\partial f}{\partial y^j} dx + d\gamma^j\right) \equiv 0.$$

Since $p^* \, d\gamma^j = dp^*\gamma^j = -d(\partial f/\partial y_i^j \circ p) = -\Sigma \, \partial(\partial f/\partial y_i^j \circ p)/\partial t^i$, we get the Euler–Lagrange system

$$\frac{\partial f}{\partial y^j} \circ p - \sum_i \frac{\partial}{\partial t^i}\left(\frac{\partial f}{\partial y_i^j} \circ p\right) \equiv 0 \qquad (j = 1, \dots, m).$$

Turn to the boundary relation $(9)_b$, and suppose f to be nowhere wanishing on M, for simplicity. We will come out from the decomposition

$$\varphi - \bar{\alpha} = \vartheta_1 \wedge \dots \wedge \vartheta_n, \qquad \vartheta_i = f^{1/n-1}\left(f \, dx^{(i)} + \sum_j \frac{\partial f}{\partial y_i^j} \alpha^j\right),$$

of an appropriate Poincaré–Cartan–Lepage form. The forms $p^*\vartheta_1, \dots, p^*\vartheta_n$ are linearly independent at every point of P, so that there exists a 1-form $\alpha_0 =$

$= \Sigma a_i \vartheta_i$ such that $p^*\alpha^0$ nowhere vanishes on P, however $q^*\alpha^0 \equiv 0$. Moreover, we may choose certain forms $\vartheta^1, \dots, \vartheta^{n-1}$ (linear combinations of $\vartheta_1, \dots, \vartheta_n$), for which

$$\varphi - \bar{\alpha} = \alpha^0 \wedge \vartheta^1 \wedge \dots \wedge \vartheta^{n-1}$$

is true. Using the new basis $\vartheta^{i'}, \alpha^{j'}, \mathrm{d}y^j_i$ ($i' = 1, \dots, n-1, j' = 0, \dots, m, i = 1, \dots, n$) the calculations became very simple:

For the form $\bar{\beta} = \Sigma b^k \delta^k \in \mathfrak{B}$, we have the differential $\mathrm{d}\bar{\beta} = \Sigma\, \mathrm{d}b^k \wedge \delta^k + (\dots)$, where the terms $(\dots)$ involve certain function b^k as a factor. Moreover, taking into account the assumptions about the functions f^i, there is a formula of the type $\mathrm{d}\psi = \Sigma g^{j'} \alpha^{j'} \wedge \vartheta + (\dots)$, where $\vartheta = \vartheta^1 \wedge \dots \wedge \vartheta^{n-1}$ and the terms $(\dots)$ involve two or more forms $\alpha^{j'}$ as a factor. It follows:

$$q^*Z \lrcorner (\varphi - \bar{\alpha}) = q^*\alpha^0(Z)\, \vartheta,$$
$$q^*Z \lrcorner\, \mathrm{d}\psi = q^* \Sigma g^{j'} \alpha^{j'}(Z)\, \vartheta,$$
$$q^*Z \lrcorner\, \mathrm{d}\bar{\beta} = q^* \Sigma\, \mathrm{d}b^k(Z)\, \delta^k,$$

and the relation $(9)_b$ is as follows

(12) $$q^*((\alpha^0(Z) + \sum_{j'} g^{j'} \alpha^{j'}(Z))\, \vartheta - \sum_k \mathrm{d}b^k(Z)\, \delta^k) \equiv 0.$$

We see that if $p \in V$ is a G-critical point of F, then

(13) $$(\alpha^0(Z) + \sum_{j'} g^{j'} \alpha^{j'}(Z)) \circ q = 0 \quad \text{for all vector fields } Z, \text{ which satisfy } \mathrm{d}b^k(Z) \equiv 0 \ (k = 1, \dots, d);$$

this is the famous transversality condition.

At the same time, (13) serves as the compatibility condition for the linear system (12) with the unknowns $q^*\delta^k$ ($k = 1, \dots, d$). (We shall express these conditions explicitly supposing the differentials $\mathrm{d}b^1, \dots, \mathrm{d}b^d$ to be linearly independent at every point of the subset qQ of M: There exist vector fields $Z^1, \dots, Z^d$ on M with $\det(\mathrm{d}b^k(Z^{k'})) \neq 0$ on the subset qQ, not necessarily continuous. Then, owing to (13), the system (12) is equivalent to

$$\sum_k (\mathrm{d}b^k(Z^{k'}) \circ q)\, q^*\delta^k \equiv q^*((\alpha^0(Z^{k'}) + \sum_{j'} g^{j'} \alpha^{j'}(Z^{k'})),$$

($k' = 1, \dots, d$); this system may be solved by the Cramer rule.) Determining the forms $q^*\delta^k$, one can easily find the forms δ^k and, as a final result, we have proved the existence of the boundary counterpart of the Poincaré–Cartan–Lepage form $\psi - \bar{\beta} = \Sigma f^i\, \mathrm{d}x^{(i)} \to \Sigma b^k \delta^k$.

At the end of this example we briefly sketch the proof that *every* G-critical point $p \in V$ given by the above equations $x^i \equiv t^i$, $y^j \equiv \bar{y}^j, y^j_i \equiv \bar{y}^j_i$ is an inner standard one. To this end, it suffices to prove an existence of a form $\bar{\alpha} \in \mathfrak{A}$ such that $(9)_i$ is satisfied only for the vector fields Z of the type (10) (a consequence of Lemma 8) vanishing on the subset qQ (continuity argument). Every such a vector field may be represented as $Z = V + W$, where

$$V = \Sigma z^j \frac{\partial}{\partial y^j} + \Sigma \frac{\partial z^j}{\partial x^i} \frac{\partial}{\partial y_i^j}, \qquad W = \Sigma \left(z_i^j - \frac{\partial z^j}{\partial x^i} \right) \frac{\partial}{\partial y_i^j}.$$

Here, the component V preserves the contact forms, $\mathscr{L}_V \alpha^j \equiv 0$, hence $\mathscr{L}_V \alpha \equiv 0$ ($\alpha \in \mathfrak{A}$). (This decomposition is widely used in [4], in a slightly more general situation.) The sought form $\bar{\alpha}$ would satisfy $(8)_i$, i.e.

$$0 = (Z \mid \mathrm{d}(\varphi - \bar{\alpha})) = (V \mid \mathrm{d}(\varphi - \bar{\alpha})) + (W \mid \mathrm{d}(\varphi - \bar{\alpha})).$$

The first summand does not depend on $\bar{\alpha}$: $(V \mid \mathrm{d}(\varphi - \bar{\alpha})) = \int_P p^* V \lrcorner\, \mathrm{d}(\varphi - \bar{\alpha}) = = \int_P p^* \mathscr{L}_V(\varphi - \bar{\alpha}) = \int_P p^* \mathscr{L}_V \varphi$, and hence it must vanish separately. (Formal proof: Theorem 6 applied on the case $Z = V$ gives certain forms $\bar{\alpha} \in \mathfrak{A}$, $\bar{\beta} \in \mathfrak{B}$ for which (7) is true. However (7) is reduced to $(V \mid \mathrm{d}(\varphi - \bar{\alpha})) = 0\ (= \int_P p^* \mathscr{L}_V \varphi)$.) The second summand $0 = (W \mid \mathrm{d}(\varphi - \bar{\alpha}))$ remains, and this is satisfied by the Poincaré–Cartan–Lepage form, which follows from the main text of this example.

APPENDIX

10. Sections of a vector bundle. At the end of this Part, we aim to outline a preliminary analysis of Theorem 6. For this purpose, we need some elementary facts from the distribution theory ([5] Chapter IX, [6] pages 302–303, [7] pages 246–248, [8]).

Let E be a vector bundle over M. We introduce the spaces $\mathrm{C}^\infty(E)$, $\mathrm{C}^k(E)$, $\mathrm{H}^k(E)$, $\mathrm{C}^\infty(E)^*$ (the *conjugate space*), consisting of sections of the bundle E and a related bundle $E^* \otimes |\wedge| M$ over M. For this purpose, cover M by a countable number of coordinate systems and take a partition of unity, $\Sigma f_i = 1$, subordinate to this covering. We may also suppose that the bundle E is trivial over the support of every function f_i of the partition. And choosing a trivialisation of E over this support, we may operate with the derivatives $\partial^{|I|} s / \partial x^I$ of a section s in the corresponding coordinate system $x = (x^1, \ldots, x^m)$ on M. (Here, we use the multi-index notation: $I = (i_1, \ldots, i_k)$, $|I| = i_1 + \ldots + i_k$, $\partial x^I = \partial x^{i_1} \ldots \partial x^{i_k}$.)

Then, $\mathrm{C}^\infty(E)$ is the linear topological space of all C^∞-smooth sections over M, endowed by the seminorms $|s|_{i,I} = \max |f_i \partial^{|I|} s / \partial x^I|$. And a ***bounded subset*** of $\mathrm{C}^\infty(E)$ is given by the inequalities $|s|_{i,I} \leqq C_{i,I}$, with arbitrary constants on the right-hand side. Note that $\mathrm{C}^\infty(E)$ is a reflexive Frechet space.

The following linear topological space $\mathrm{C}^k(E)$ consists of all C^k-smooth sections over M; it is endowed by the seminorms $|s|_{i,I}$, where $|I| \leq k$. Unfortunately, but this is not a reflexive space, so we must introduce the space $H^k(E)$.

Choose a nowhere vanishing density form $|\omega|$ on M. The space $H^k(E)$ is the completion of the previous spaces with respect to the norm

$$|s|_k = (\int_M \sum_{i,|I| \leq k} |f_i(\partial^{|I|}s/\partial x^I)^2|\ |\omega|)^{1/2}.$$

We get a Hilbert space.

The last space $C^\infty(E)^*$ is related to the space $C^\infty(E^*)$ of sections of the dual vector bundle E^*. Denote the duality form between fibers of E, E^* as the ordinary multiplication of numbers. Then, $s\,.\,\vartheta$ is a C^∞-smooth function on M for every $s \in C^\infty(E)$, $\vartheta \in C^\infty(E^*)$. Let $C_c^\infty(E^*)$ be the space of all sections $\vartheta \in C^\infty(E^*)$ with *compact support* (i.e. vanishing outside a compact subset of M). Choosing a density form $|\omega|$ on M, we get a well-defined integral $\int_M s\,.\,\vartheta\,|\omega|$ for every $s \in C^\infty(E)$, $\vartheta \in C_c^\infty(E^*)$. This fact suggests to introduce the linear bundle $|\wedge|\,M$ of densities on M. Then, using the natural pairing $C^\infty(E) \times C^\infty(E^* \otimes |\wedge|\,M) \to |\wedge|\,M$, we have the bilinear form

$$\langle s \mid \varphi \rangle = \int_M s\,.\,\varphi \qquad (s \in C^\infty(E), \varphi \in C_c^\infty(E^* \otimes |\wedge|\,M)).$$

Owing to this form, the space $C_c^\infty(E^* \otimes |\wedge|\,M)$ consists of certain continuous linear functionals on the space $C^\infty(E)$, but not necessarily of all of them. This defect may be removed by a completion process applied to the space $C_c^\infty(E^* \otimes |\wedge|\,M)$. The resulting space is exactly the sought space $C^\infty(E)^*$. It is a very wide space containing the space $C_c^\infty(E^* \otimes |\wedge|\,M)$ as a dense subset; it may be denoted $C^\infty(E)^* = C_c^{-\infty}(E^* \otimes |\wedge|\,M)$, and its vectors are called *distributional sections* with compact support of the bundle $E^* \otimes |\wedge|\,M$.

There are two topologies used in the space $C^\infty(E)^*$, for both it appears as a complete space: We identify

$$C^\infty(E)^* = \bigcup_K C_K^\infty(E)^* \qquad (K \text{ are compact subsets of } M),$$

where $\varphi \in C_K^\infty(E)^*$ if and only if $\langle s \mid \varphi \rangle \equiv 0$ for all sections $s \in C^\infty(E)$ which vanish identically outside K. And the topology on $C_K^\infty(E)^*$ is given by the seminorms

$$|\sigma|_S = \sup_{s \in S} |\langle s \mid \sigma \rangle|,$$

where S is either an arbitrary finite subset of $C^\infty(E)$ (the *weak topology*), or an arbitrary bounded subset of $C^\infty(E)$ (the *strong topology*).

After this necessary digression, we return to the main topic. We begin with a purely algebraic result:

11. Theorem. *An embedding $p \in V$ is a G-critical point of F if and only if for every finite set of vector fields $Z_1, \ldots, Z_k$ on M there exist forms $\bar{\alpha} \in \mathfrak{A}$, $\bar{\beta} \in \mathfrak{B}$ satisfying the condition* (5) *for $Z = Z_1, \ldots, Z = Z_k$ simultaneously.*

Proof: Let p be a G-critical point. Denote $C = (C_1, \ldots, C_k) \in \mathbf{R}^k$, $Z = C_1 Z_1 + \ldots + C_k Z_k$, $l(C) = (Z \mid \mathrm{d}\varphi) + [Z \mid \varphi + \mathrm{d}\psi]$, $l_{\alpha,\beta}(C) = (Z \mid \mathrm{d}\alpha) + [Z \mid \alpha + \mathrm{d}\beta]$ $(\alpha \in \mathfrak{A},\ \beta \in \mathfrak{B})$. The condition (6) means that $l(C) = 0$, if $l_{\alpha,\beta}(C) \equiv 0 (\alpha \in \mathfrak{A},\ \beta \in \mathfrak{B})$. It follows that l is a finite linear combination:

$$l = \Sigma c_i l_{\alpha_i,\beta_i} = l_{\Sigma c_i \alpha_i, \Sigma c_i \beta_i}.$$

We may choose $\bar{\alpha} = \Sigma c_i \alpha_i$, $\bar{\beta} = \Sigma c_i \beta_i$.

The converse statement of the theorem is evident.

12. Theorem. *An embedding $p \in V$ is a G-critical point of F if and only if for every bounded subset Z of the space $\mathrm{C}^\infty(TM)$ there exist sequences $\alpha_1, \alpha_2, \ldots \in \mathfrak{A}$, $\beta_1, \beta_2, \ldots \in \mathfrak{B}$ satisfying*

$$(14) \qquad \lim_{l \to \infty} \Big(\int_P p^* Z \lrcorner\, \mathrm{d}(\varphi - \alpha_l) + \int_Q q^* Z \lrcorner\, (\varphi - \alpha_l + \mathrm{d}(\psi - \beta_l)\Big) = 0,$$

for every vector field $Z \in Z$.

Proof: Let $p \in V$ be a G-critical point of F. Denote by R the linear subspace of the space $\mathrm{C}^\infty(TM)^*$ consisting of all functionals

$$\zeta_{\alpha,\beta} = (. \mid \mathrm{d}\alpha) + [. \mid \alpha + \mathrm{d}\beta] \qquad (\alpha \in \mathfrak{A},\ \beta \in \mathfrak{B}).$$

The condition (6) means that the null-space of the functional

$$\zeta_0 = (. \mid \mathrm{d}\varphi) + [. \mid \varphi + \mathrm{d}\psi]$$

contains the intersection of null-spaces of all functionals from R. This means exactly that ζ_0 lies in the weak closure of R, and hence also in the strong closure: In every neighbourhood of ζ_0 there exists a functional $\zeta_{\alpha,\beta}$.

Such a neighbourhood may be determined by a bounded subset Z of the space $\mathrm{C}^\infty(TM)$, as the set of all $\zeta \in \mathrm{C}^\infty(TM)^*$ satisfying $|\langle Z \mid \zeta - \zeta_0 \rangle| < \frac{1}{l}$ $(Z \in Z)$.

There exists a functional l_{α_l,β_l} in the above neighbourhood, and we obtain

$$\frac{1}{l} > |\langle Z \mid \zeta_{\alpha_l,\beta_l} - \zeta_0 \rangle| = |(Z \mid \mathrm{d}(\varphi - \alpha_l) + [Z \mid \varphi - \alpha_l + \mathrm{d}(\psi - \beta_l)]|,$$

which is the desired relation (14).

The converse statement of the theorem is evident.

13. Theorem. *Let $p \in V$, then the following statements are equivalent; i) There exist sequences $\alpha_1, \alpha_2, \ldots \in \mathfrak{A}$, $\beta_1, \beta_2, \ldots \in \mathfrak{B}$ satisfying* (14) *for every $Z \in \mathrm{C}^k(TM)$*

and every sufficiently large k. ii) There exist the above sequences satisfying (14) *for every* $Z \in C^\infty(TM)$. *iii) There exists such k* ($k = 0, 1, \ldots$) *that* (6) *is true for all* $Z \in C^k(TM)$.

Proof: i) ⇒ ii) is trivial.

ii) ⇒ iii): We retain the notation of the preceding proof, and one can observe that the functionals $\zeta_{\alpha,\beta}$, ζ_0 are well defined on every space $C^k(TM)$. Now, ii) implies that the set of values $\langle Z \mid \zeta_{\alpha_l,\beta_l} \rangle$ ($l = 1, 2, \ldots$) is bounded for every fixed $Z \in C^\infty(TM)$. We deal with the Frechet space, therefore it is uniformly bounded for Z varying in an appropriate neighbourhood of the zero vector in the space $C^\infty(TM)$. Such a neighbourhood is given by a finite number of inequalities $| Z |_{i_\iota, I_\iota} < C$, and denoting $k = \max_\iota (| I_\iota |)$ (where ι varies in a finite set), we get the result that the sequence of functionals $\zeta_{\alpha_1,\beta_1}, \zeta_{\alpha_2,\beta_2}, \ldots$ is uniformly bounded (and hence also uniformly continuous) on a neighbourhood of the zero vector in the space $C^k(TM)$. This sequence tends to ζ_0 on a dense subset $C^\infty(TM)$ of $C^k(TM)$, therefore, it converges to ζ_0 everywhere in the space $C^k(TM)$. This is exactly iii).

iii) ⇒ i): There is $C^k(ZM) - H^{k'}(TM)$, for an appropriate k' (*Sobolev embedding theorem*). Therefore, (6) is true for every $Z \in H^{k'}(TM)$. The latter space is a reflexive one, and we may follow the preceding proof: The functional ζ_0 (considered on the space $H^{k'}(TM)$) lies in the closure of the subset R of the space $H^{k'}(TM)$, and there exists a sequence from R which converges to ζ_0. This is i).

14. Notice. At this stage, it is not difficult to observe that the standardness property may be expressed as a certain closeness property of the subspace R in a space of sections. Namely, we may expect that the limit of the sequence $\zeta_{\alpha_1,\beta_1}, \zeta_{\alpha_2,\beta_2}, \ldots$ is exactly $\zeta_{\bar{\alpha};\bar{\beta}}$ with the forms $\bar{\alpha}, \bar{\beta}$ required in (8). We shall not follow these lines, because the used notions (the forms (|), [|], ⟨|⟩, and the bundles over M) are not adequate for the problem: They contain unnecessary ingredients, the whole problem is in the fact concentrated along the subset pP of M. We postpone these questions to other place.

(The part II Hamiltonian theory follows.)

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