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A FUNCTIONAL CHARACTERIZATION OF PARALLELOGRAM SPACES

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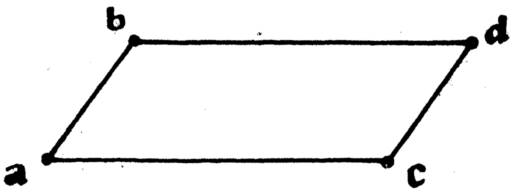
This note may be considered as a contribution to work done by B. Csákány [1, 2], H. Peter Gumm [4] and F. Ostermann – J. Schmidt [6, 7]. It is shown that a Mal'cev function commuting with itself is closely related with the geometrical structure introduced by F. Ostermann and J. Schmidt [6] under the name parallelogram space (Theorem 1). Further, similar investigations are realized for other well-known ternary functions, i.e. for a Pixley function and for a majority function (Theorem 2).

Firstly, let us recall some basic notions and notations using here:

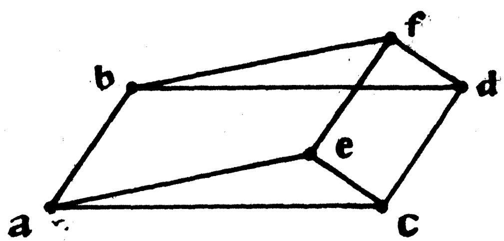
(i) A *parallelogram space* (A, P) is a nonvoid set A with a 4-ary relation P on A such that the following four conditions are satisfied

(P1) $(a, b, c, d) \in P$ implies $(a, c, b, d) \in P$;

(P2) $(a, b, c, d) \in P$ implies $(c, d, a, b) \in P$;



(P3) $(a, b, c, d) \in P$ and $(c, d, e, f) \in P$ imply $(a, b, e, f) \in P$;



(P4) For any $a, b, c \in A$ there is exactly one element $d \in A$ such that $(a, b, c, d) \in P$.

(ii) A *Mal'cev function* p on a set A is a function $p : A^3 \to A$ satisfying $x = p(x, y, y) = p(y, y, x)$;

(iii) A *Pixley function* t on a set A is a function $t : A^3 \to A$ satisfying $x = t(x, y, y) = t(x, y, x) = t(y, y, x)$;

(iv) A *majority function* m on a set A is a function $m : A^3 \to A$ satisfying $x = m(x, x, y) = m(x, y, x) = m(y, x, x)$.

The notation p, t and m will be reserved for a Mal'cev function, a Pixley function and a majority function, respectively, in this paper.

(v) Functions $r : A^m \to A$ and $s : A^n \to A$ are called commutative if

$$r(s(a_{11}, \ldots, a_{1n}), \ldots, s(a_{m1}, \ldots, a_{mn})) = s(r(a_{11}, \ldots, a_{m1}), \ldots, r(a_{1n}, \ldots, a_{mn})) \quad \text{holds}$$

for every elements $a_{ij} \in A$, $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$ (see [5] for this concept).

Now, we are ready to state the main result of this paper.

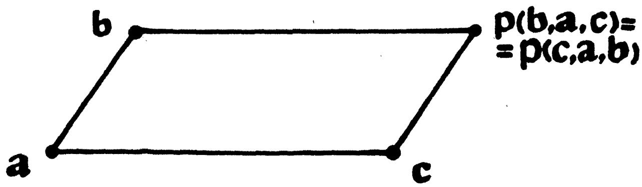
Theorem 1. *Let A be a nonvoid set. The following conditions are equivalent:*

(1) *There is a Mal'cev function p on A commuting with itself;*

(2) *There is a 4-ary relation P on A such that (A, P) is a parallelogram space;*

(3) *There is an abelian group $\langle A, +_a, -_a, a\rangle$ with arbitrary chosen neutral element $a \in A$.*

Proof. (1) $\Rightarrow$ (2): Denote by P the 4-ary relation $\{(a, b, c, p(b, a, c)); a, b, c \in A\}$ on the set A. We claim that (A, P) is a parallelogram space.



(P1) We show that $p(b, a, c) = p(c, a, b)$:

$$p(b, a, c) = p(p(a, a, b), p(a, c, c), p(c, c, c)) =$$
$$= p(p(a, a, c), p(a, c, c), p(b, c, c)) = p(c, a, b);$$

(P2) We have to prove that $b = p(p(b, a, c), c, a)$:

$$b = p(b, a, a) = p(p(b, b, b), p(a, b, b), p(c, c, a)) =$$

$$= p(p(b, a, c), p(b, b, c), p(b, b, a)) = p(p(b, a, c), c, a);$$

(P3) Assume $d = p(b, a, c)$ and $f = p(d, c, e)$. Then

$$f = p(d, c, e) = p(p(b, a, c), p(a, a, c), p(a, a, e)) =$$
$$= p(p(b, a, a), p(a, a, a), p(c, c, e)) = p(b, a, e),$$

i.e. $(a, b, e, f) \in P$ which is to be proved.

Finally, (P4) follows immediately from the definition of the relation P.

(2) $\Rightarrow$ (3): The proof of this part is a matter of the F. Ostermann's and J. Schmidt's paper [6], so we refer the reader to this material.

(3) $\Rightarrow$ (1): Let $\langle A, +_a, -_a, a\rangle$ be an abelian group. Then it is a routine to verify that the ternary function $p(x, y, z) = x -_a y +_a z$ is a Mal'cev function on A commuting with itself.

Remark. The relationship between abelian groups and Mal'cev functions was investigated by H. Peter Gumm [4] in a more general situation and—as we noted above—the connection between parallelogram spaces and abelian groups is also well-known. However, the existence of neutral element of an abelian group needs the introduction of so-called parallelogram space with centrum, see [6]. Obviously the application of a Mal'cev function easily removes this defect.

Simultaneously, we get that a Mal'cev function commuting with itself is characterizable by identities derived from the axioms (P1), (P2) and (P3):

$$\begin{aligned} p(b, a, c) &= p(c, a, b) \\ p(p(b, a, c), c, a) &= b \\ p(p(b, a, c), c, e) &= p(b, a, e). \end{aligned}$$

So, a Mal'cev function commuting with itself is sufficiently described and a natural question raises: Are there similar results for a Pixley function or for a majority function? The following theorem answers this question in the negative.

Theorem 2. *Let r and s be arbitrary functions from the set $\{p, t, m\}$. Excepting the case $r = s = p$, the following two conditions are equivalent for any nonvoid set A:*

(1) *r commutes with s on A;*

(2) *A is trivial, i.e. $|A| = 1$.*

Proof. (i) A Pixley function commuting with itself:

$$x = t(x, y, y) = t(t(x, x, x), t(y, x, x), t(x, x, y)) =$$
$$= t(t(x, y, x), t(x, x, x), t(x, x, y)) = t(x, x, y) = y;$$

(ii) A majority function commuting with itself:

$$x = m(x, x, y) = m(m(y, x, x), m(x, x, y), m(y, y, y)) =$$
$$= m(m(y, x, y), m(x, x, y), m(x, y, y)) = m(y, x, y) = y;$$

(iii) A majority function commuting with a Mal'cev function:

$$x = m(y, x, x) = m(p(x, x, y), p(x, y, y), p(x, x, x)) = \\ = p(m(x, x, x), m(x, y, x), m(y, y, x)) = p(x, x, y) = y;$$

(iv) A Pixley function commuting with a majority function:

$$x = m(x, x, y) = m(t(y, y, x), t(x, x, x), t(y, x, y)) = \\ = t(m(y, x, y), m(y, x, x), m(x, x, y)) = t(y, x, x) = y;$$

(v) A Pixley function commuting with a Mal'cev function:

$$x = t(x, y, y) = t(p(x, x, x), p(y, x, x), p(x, x, y)) = \\ = p(t(x, y, x), t(x, x, x), t(x, x, y)) = p(x, x, y) = y.$$

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