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OSCILLATIONS OF SUPERLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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The oscillation property of differential equations with deviating arguments has drawn a great deal of attention in the last ten years. An excellent survey of known results on the subject has been done by Mitropol'skiĭ and Ševelo [5]. In particular, for *superlinear* differential equations with deviating arguments, we choose to refer to the papers by Kitamura and Kusano [3], Marushiak [4], Staikos [9], and the same author [8] and to their references. The purpose here is to study the oscillatory and asymptotic behavior of the solutions of strongly superlinear differential equations with general (retarded, advanced or mixed type) deviating arguments. The equations considered involve damping terms and the results obtained extend known fundamental oscillation criteria concerning superlinear equations without damping terms.

The differential equations considered here are of the form

$$(E, \delta)\ [r(t)\, x'(t)]^{(n-1)} + \delta f(t; x[g_1(t)], \dots, x[g_m(t)]) = 0, \qquad t \geqq t_0\ (\delta = \pm 1),$$

where r and g_j ($j = 1, \dots, m$) are continuous real-valued functions on the interval $[t_0, \infty)$ and f is a continuous real-valued function defined at least on $[t_0, \infty) \times (R_+^m \cup R_-^m)$, $R_+ = (0, \infty)$ and $R_- = (-\infty, 0)$. The following assumptions are made:

(i) *r is positive on $[t_0, \infty)$ and such that*

$$\int^{\infty} \frac{dt}{r(t)} = \infty.$$

(ii) *For every $t \geqq t_0$,*

$$f(t; y) \geqq 0 \quad \textit{for all } y \in R_+^m, \qquad f(t; y) \leqq 0 \quad \textit{for all } y \in R_-^m$$

and, moreover, $f(t; y)$ is increasing with respect to y in $R_+^m \cup R_-^m$.

(iii) $\lim\limits_{t \to \infty} g_j(t) = \infty$ ($j = 1, \dots, m$).

Note that the increasing character of real-valued functions defined on subsets of R^m will be considered with respect to the usual order in R^m defined as follows

$$(y_1, \ldots, y_m) \leqq (z_1, \ldots, z_m) \Leftrightarrow y_1 \leqq z_1, \ldots, y_m \leqq z_m.$$

We consider only such solutions $x(t)$ of the equation (E, δ) which are defined for all large t. Sufficient smoothness for the existence of such solutions will be assumed without mention. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function defined on an interval of the form $[T, \infty)$ is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*.

The oscillatory and asymptotic behavior of the bounded solutions of the differential equation (E, δ) is well described by the following theorem, which is a special case of a result given by the author in [6].

Theorem 0. *Let the conditions* (i)–(iii) *and the following one be satisfied:*
(C_0) *For every nonzero constant c either*

$$\int^{\infty} t^{n-2} \, | f(t; c, \ldots, c) | \, dt = \infty$$

or

$$\int^{\infty} \frac{1}{r(t)} \int_t^{\infty} (s - t)^{n-2} \, | f(s; c, \ldots, c) | \, ds \, dt = \infty.$$

Then for n even [*resp. odd*] *all bounded solutions of the differential equation* $(E, +1)$ [*resp. of the equation* $(E, -1)$] *are oscillatory, while for n odd* [*resp. even*] *every bounded solution x of the differential equation* $(E, +1)$ [*resp. of the equation* $(E, -1)$] *is oscillatory or such that x and* $(rx')^{(k-1)}$ $(k = 1, \ldots, n-1)$ *tend monotonically to zero at* ∞.

The purpose here is to study the oscillatory and asymptotic behavior of all solutions of the differential equation (E, δ). For this purpose, we need the following lemma, which is originated in two well-known lemmas due to Kiguradze [1, 2]. This lemma is obtained here as a special case of a lemma given by the author in [7].

Lemma. *Suppose that* (i) *holds and let h be a positive and differentiable function on an interval* $[\tau, \infty)$, $\tau \geqq t_0$, *such that rh' is a* $(n-1)$*-times differentiable function on* $[\tau, \infty)$. *If* $(rh')^{(n-1)}$ *is of constant sign on* $[\tau, \infty)$ *and not identically zero on any interval of the form* $[\tau', \infty)$, $\tau' \geqq \tau$, *then there exist a* $T \geqq \tau$ *and an integer l,* $0 \leqq l \leqq n$, *with* $n + l$ *odd for* $(rh')^{(n-1)}$ *nonpositive or* $n + l$ *even for* $(rh')^{(n-1)}$ *nonnegative so that*

$$\begin{cases} l \leqq n - 1 \Rightarrow (-1)^{l+j} H_j > 0 & \text{on } [T, \infty) \quad (j = l, \ldots, n-1) \\ l > 1 \Rightarrow H_i > 0 & \text{on } [T, \infty) \quad (i = 1, \ldots, l-1), \end{cases}$$

where

$$H_0 = h \quad \textit{and} \quad H_k = (rh')^{(k-1)} \quad (k = 1, \ldots, n-1).$$

Now, we shall formulate our results. For this purpose, we introduce the functions g, R_1 and R_2 defined on $[t_0, \infty)$ as follows

$$g(t) = \min\{t, g_1(t), \ldots, g_m(t)\},$$

$$R_1(t) = \int_{t_0}^{t} \frac{(s-t_0)^{n-2}}{r(s)}\,ds \quad \text{and} \quad R_2(t) = \int_{t_0}^{t} \frac{(t-s)^{n-2}}{r(s)}\,ds.$$

Theorem 1. *Suppose that* (i)–(iii) *hold and let p, φ, ψ be continuous functions subject to the conditions:*

(I) *p is nonnegative on $[t_0, \infty)$.*

(II) *φ is increasing on R-$\{0\}$ and has the sign property*

$$y\varphi(y) > 0 \quad \textit{for all} \quad y \in R - \{0\}.$$

(III) *ψ is positive and increasing on R_+,*

(IV) *$\varphi\psi$ is strongly superlinear in the sense that*

$$\int^{\infty} \frac{dy}{\varphi(y)\,\psi(y)} < \infty \quad \textit{and} \quad \int^{-\infty} \frac{dy}{\varphi(y)\,\psi(y)} < \infty.$$

If

(H) $\quad |f(t; y, \ldots, y)| \geqq p(t)\,|\varphi(y)|$ *for all* $(t, y) \in [t_0, \infty) \times (R - \{0\})$,

(C) $$\int^{\infty} \frac{R_k[g(t)]}{\psi(R_1[g(t)])}\,p(t)\,dt = \infty \quad (k = 1, 2),$$

then we have:

α) *For n even, all solutions of the differential equation $(E, +1)$ are oscillatory.*

β) *For n odd, every solution x of the equation $(E, +1)$ is oscillatory or satisfies*

$$(X_0)\begin{cases} \lim\limits_{t\to\infty} x(t) = 0 & \textit{monotonically} \\ \lim\limits_{t\to\infty} [r(t)\,x'(t)]^{(k-1)} = 0 & \textit{monotonically}\ (k = 1, \ldots, n-1). \end{cases}$$

Theorem 2. *Suppose that* (i)–(iii) *hold and let p, φ, ψ be continuous functions subject to the conditions* (I)–(IV). *If* (H) *and* (C) *are satisfied, then we have:*

α) *For n even, every solution x of the differential equation $(E, -1)$ is oscillatory or satisfies one of (X_0),*

$$(X_\infty) \quad \lim_{t\to\infty} x(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} [r(t)\,x'(t)]^{(k-1)} = \infty \quad (k = 1, \ldots, n-1),$$

$$(X_{-\infty}) \quad \lim_{t\to\infty} x(t) = -\infty \quad \textit{and} \quad \lim_{t\to\infty} [r(t)\,x'(t)]^{(k-1)} = -\infty \quad (k = 1, \ldots, n-1).$$

β) *For n odd, every solution x of the equation $(E, -1)$ is oscillatory or satisfies one of (X_∞), $(X_{-\infty})$.*

Proof of Theorem 1. By (i) and (iii), we have $\lim_{t\to\infty} R_1[g(t)] = \infty$ and consequently

$$R_1[g(t)] \geqq 1 \qquad \text{for all large } t.$$

Therefore, in view of (III), we get

$$\psi(R_1[g(t)]) \geqq \psi(1) > 0 \qquad \text{for all large } t.$$

Thus,

$$\frac{R_2[g(t)]}{\psi(R_1[g(t)])} \leqq \frac{1}{\psi(1)} R_2(t) \qquad \text{for all large } t$$

and so, by virtue of (I) and (C), we have

$$\int^{\infty} R_2(t)\, p(t)\, \mathrm{d}t = \infty.$$

Next, we consider an arbitrary constant $c \neq 0$ and we assume that

$$\int^{\infty} t^{n-2} \,|\, f(t; c, \ldots, c) \,|\, \mathrm{d}t < \infty.$$

By (H), we obtain

$$|\, f(t; c, \ldots, c) \,| \geqq p(t) \,|\, \varphi(c) \,| \qquad \text{for all } t \geqq t_0,$$

where $\varphi(c) \neq 0$. Hence,

$$\int^{\infty} R_2(t) \,|\, f(t; c, \ldots, c) \,|\, \mathrm{d}t = \infty,$$

i.e.

$$\int^{\infty} |\, f(t; c, \ldots, c) \,| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{r(s)} \,\mathrm{d}s\, \mathrm{d}t = \infty.$$

This, after some manipulations, gives

$$\int^{\infty} \frac{1}{r(t)} \int_{t}^{\infty} (s-t)^{n-2} \,|\, f(s; c, \ldots, c) \,|\, \mathrm{d}s\, \mathrm{d}t = \infty.$$

Thus, condition (C) implies (C_0) and hence, by Theorem 0, it suffices to prove that all nonoscillatory solutions of $(E, +1)$ are bounded.

The substitution $w = -x$ transforms $(E, +1)$ into an equation of the same form satisfying the assumptions of the theorem with the function $\hat{\varphi}$ in place of φ, where $\hat{\varphi}(y) = -\varphi(-y)$ for all y in the domain of φ. Hence, with respect to the nonoscillatory solutions of the equation $(E, +1)$ we can confine our discussion only to the positive ones.

Let now x be a positive unbounded solution on an interval $[\tau_0, \infty)$, $\tau_0 > \max\{0, t_0\}$, of the equation $(E, +1)$ and let $\tau \geqq \tau_0$ be chosen, by (iii), so that

$$g_j(t) \geqq \tau_0 \qquad \text{for every } t \geqq \tau \qquad (j = 1, \ldots, m).$$

Then, by virtue of (ii), (H), (I) and (II), we obtain that

$$-[r(t)\,x'(t)]^{(n-1)} = f(t;\, x[g_1(t)], \ldots, x[g_m(t)]) \geqq$$
$$\geqq f(t;\, \min_{1 \leqq j \leqq m} x[g_j(t)], \ldots, \min_{1 \leqq j \leqq m} x[g_j(t)]) \geqq p(t)\,\varphi(\min_{1 \leqq j \leqq m} x[g_j(t)]) \geqq 0,$$

for all $t \geqq \tau$. Therefore it follows that the function $(rx')^{(n-1)}$ is nonpositive on $[\tau, \infty)$. Moreover, $(rx')^{(n-1)}$ is not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$, since, because of (C), the same holds for the function p. Thus, by taking into account the fact that x is unbounded and applying the lemma, we conclude that there exist a $T \geqq \tau$ and an integer l, $1 \leqq l \leqq n-1$, with $n + l$ odd so that

$$\begin{cases} (-1)^{l+j}\,[r(t)\,x'(t)]^{(j-1)} > 0 & \text{for every } \ t \geqq T\ (j = l, \ldots, n-1) \\ [r(t)\,x'(t)]^{(i-1)} > 0 & \text{for every } \ t \geqq T\ (i = 1, \ldots, l-1), \text{ when } l > 1. \end{cases}$$

Next, by (iii), we choose a $T_1 \geqq 2T$ such that

$$g_j(t) \geqq T \qquad \text{for every } t \geqq T_1 \ (j = 1, \ldots, m).$$

Then, by taking into account (ii) and the fact that x is increasing on $[T, \infty)$, for $t \geqq T_1$ we get

$$-[r(t)\,x'(t)]^{(n-1)} = f(t;\, x[g_1(t)], \ldots, x[g_m(t)]) \geqq$$
$$\geqq f(t;\, x[g(t)], \ldots, x[g(t)])$$

and so, in view of (H), (I) and (II), we have

$$-[r(t)\,x'(t)]^{(n-1)} \geqq p(t)\,\varphi(x[g(t)]) \geqq 0 \qquad \text{for all } t \geqq T_1.$$

If $l = 1$, by using the Taylor formula with integral remainder, for every t, u with $T_1 \leqq t \leqq u$ we obtain

$$r(t)\,x'(t) = \sum_{j=1}^{n-1} \frac{(t-u)^{j-1}}{(j-1)!}(rx')^{(j-1)}(u) + \frac{1}{(n-2)!}\int_u^t (t-s)^{n-2}(rx')^{(n-1)}(s)\,ds$$
$$= \sum_{j=1}^{n-1} \frac{(u-t)^{j-1}}{(j-1)!}(-1)^{1+j}(rx')^{(j-1)}(u) + \frac{1}{(n-2)!}\int_t^u (s-t)^{n-2}[-(rx')^{(n-1)}(s)]\,ds \geqq$$
$$\geqq \frac{1}{(n-2)!}\int_t^u (s-t)^{n-2}p(s)\,\varphi(x[g(s)])\,ds.$$

Thus, we have

$$r(t)\,x'(t) \geqq \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}p(s)\,\varphi(x[g(s)])\,ds,$$

for all $t \geqq T_1$, provided that $l = 1$. Next, for every t, u with $T_1 \leqq t \leqq u$ we derive

$$(rx')^{(n-2)}(t) \geqq (rx')^{(n-2)}(u) + \int_t^u p(s)\,\varphi(x[g(s)])\,ds \geqq \int_t^u p(s)\,\varphi(x[g(s)])\,ds$$

and consequently

$$(rx')^{(n-2)}(t) \geqq \int_t^\infty p(s)\,\varphi(x[g(s)])\,\mathrm{d}s \qquad \text{for all } t \geqq T_1 .$$

Furthermore, if $l < n - 1$, by applying again the Taylor formula with integral remainder, for $t \geqq T_1$ we obtain

$$(rx')^{(l-1)}(t/2) = \sum_{j=l}^{n-2} \frac{(t/2 - t)^{j-l}}{(j-l)!}(rx')^{(j-1)}(t) +$$

$$+ \frac{1}{(n-2-l)!} \int_t^{t/2} (t/2 - s)^{n-2-l}(rx')^{(n-2)}(s)\,\mathrm{d}s =$$

$$= \sum_{j=l}^{n-2} \frac{t^{j-l}}{2^{j-l}(j-l)!}(-1)^{l+j}(rx')^{(j-1)}(t) +$$

$$+ \frac{1}{(n-2-l)!} \int_{t/2}^{t} (s - t/2)^{n-2-l}(rx')^{(n-2)}(s)\,\mathrm{d}s \geqq$$

$$\geqq \frac{1}{(n-2-l)!}\Big[\int_{t/2}^{t}(s - t/2)^{n-2-l}\,\mathrm{d}s\Big](rx')^{(n-2)}(t) =$$

$$= \frac{1}{2^{n-2-l}(n-1-l)!}\, t^{n-1-l}(rx')^{(n-2)}(t) \geqq$$

$$\geqq \frac{1}{2^{n-1-l}(n-1-l)!}(t - T_1)^{n-1-l}(rx')^{(n-2)}(t).$$

Hence, we have

$$(rx')^{(l-1)}(t/2) \geqq \frac{1}{2^{n-1-l}(n-1-l)!}(t - T_1)^{n-1-l}\int_t^\infty p(s)\,\varphi(x[g(s)])\,\mathrm{d}s$$

for every $t \geqq T_1$, if $l < n - 1$. But, this inequality holds also in the case where $l = n - 1$. Now, if $l > 1$, by using the Taylor formula with integral remainder, for $t \geqq T_1$ we get

$$(rx')(t/2) = \sum_{i=1}^{l-1} \frac{(t/2 - T)^{i-1}}{(i-1)!}(rx')^{(i-1)}(T) +$$

$$+ \frac{1}{(l-2)!}\int_T^{t/2}(t/2 - s)^{l-2}(rx')^{(l-1)}(s)\,\mathrm{d}s \geqq$$

$$\geqq \frac{1}{(l-2)!}\Big[\int_T^{t/2}(t/2 - s)^{l-2}\,\mathrm{d}s\Big](rx')^{(l-1)}(t/2) =$$

$$= \frac{1}{2^{l-1}(l-1)!}(t - 2T)^{l-1}(rx')^{(l-1)}(t/2) \geqq$$

$$\geqq \frac{1}{2^{l-1}(l-1)!}(t - T_1)^{l-1}(rx')^{(l-1)}(t/2).$$

But, for $l > 1$ the function rx' is increasing on $[T, \infty)$ and so

$$(rx')(t) \geqq (rx')(t/2) \qquad \text{for } t \geqq T_1.$$

Hence, we obtain that for every $t \geqq T_1$

$$r(t)\,x'(t) \geqq \frac{1}{2^{n-2}(l-1)!\,(n-1-l)!}(t-T_1)^{n-2}\int_t^\infty p(s)\,\varphi(x[g(s)])\,\mathrm{d}s,$$

provided that $l > 1$. We have thus proved that for all $t \geqq T_1$

$$(*) \qquad x'(t) \geqq \begin{cases} \dfrac{K}{r(t)}\displaystyle\int_t^\infty (s-t)^{n-2}p(s)\,\varphi(x[g(s)])\,\mathrm{d}s, & \text{if } l = 1 \\ K\dfrac{(t-T_1)^{n-2}}{r(t)}\displaystyle\int_t^\infty p(s)\,\varphi(x[g(s)])\,\mathrm{d}s, & \text{if } l > 1, \end{cases}$$

where

$$K = \frac{1}{(n-2)!} \quad \text{for } l = 1, \qquad K = \frac{1}{2^{n-2}(l-1)!\,(n-1-l)!} \quad \text{for } l > 1.$$

Next, if $n > 2$, by the Taylor formula with integral remainder, we derive that for $t \geqq T_1$

$$\begin{aligned} r(t)\,x'(t) &= \sum_{k=1}^{n-2}\frac{(rx')^{(k-1)}(T_1)}{(k-1)!}(t-T_1)^{k-1} + \frac{1}{(n-3)!}\int_{T_1}^t (t-s)^{n-3}(rx')^{(n-2)}(s)\,\mathrm{d}s \leqq \\ &\leqq \sum_{k=1}^{n-2}\frac{(rx')^{(k-1)}(T_1)}{(k-1)!}(t-T_1)^{k-1} = \frac{(rx')^{(n-2)}(T_1)}{(n-3)!}\int_{T_1}^t (t-s)^{n-3}\,\mathrm{d}s = \\ &= \sum_{k=1}^{n-1}\frac{(rx')^{(k-1)}(T_1)}{(k-1)!}(t-T_1)^{k-1} \end{aligned}$$

and therefore

$$\limsup_{t\to\infty}\frac{r(t)\,x'(t)}{(t-t_0)^{n-2}} \leqq \frac{(rx')^{(n-2)}(T_1)}{(n-2)!}.$$

The last inequality holds also in the case where $n = 2$. Thus, there exists a positive constant α_1 such that

$$x'(t) \leqq \alpha_1\frac{(t-t_0)^{n-2}}{r(t)} \qquad \text{for all } t \geqq T_1.$$

This gives

$$x(t) \leqq x(T_1) + \alpha_1\int_{T_1}^t \frac{(s-t_0)^{n-2}}{r(s)}\,\mathrm{d}s \leqq x(T_1) + \alpha_1 R_1(t), \qquad t \geqq T_1.$$

Thus, since $\lim_{t\to\infty} R_1(t) = \infty$, we conclude that for some positive constant α with $\alpha \geqq 1$ we have

$$x(t) \leqq \alpha R_1(t) \qquad \text{for every } t \geqq T_1.$$

For $t \geqq T_1$ we define

$$R_1(t; T_1) = \int_{T_1}^{t} \frac{(s - T_1)^{n-2}}{r(s)} \mathrm{d}s \qquad \text{and} \qquad R_2(t; T_1) = \int_{T_1}^{t} \frac{(t - s)^{n-2}}{r(s)} \mathrm{d}s.$$

Then, by the L'Hospital rule, we derive

$$\lim_{t \to \infty} \frac{R_k(t; T_1)}{R_k(t)} = 1 \qquad (k = 1, 2)$$

and consequently there exist a $\hat{T}_1 > T_1$ and a positive constant β such that

$$R_k(t; T_1) \geqq \beta R_k(t) \qquad \text{for all } t \geqq \hat{T}_1 \qquad (k = 1, 2).$$

Furthermore, we choose a $T_2 \geqq \hat{T}_1$ so that

$$g(t) \geqq \hat{T}_1 \qquad \text{for } t \geqq T_2.$$

Then we have

$$R_k[g(t); T_1] \geqq \beta R_k[g(t)], \qquad t \geqq T_2 \qquad (k = 1, 2).$$

Now, we consider an arbitrary number t^* with $t^* \geqq T_2$. We divide both sides of (*) by $\varphi[x(t)/\alpha]\, \psi[x(t)/\alpha]$, for $T_1 \leqq t \leqq t^*$, and integrate it over $[T_1, t^*]$ obtaining

$$\int_{T_1}^{t^*} \frac{x'(t)}{\varphi[x(t)/\alpha]\, \psi[x(t)/\alpha]} \mathrm{d}t = \alpha \int_{x(T_1)/\alpha}^{x(t^*)/\alpha} \frac{\mathrm{d}y}{\varphi(y)\, \psi(y)} \geqq$$

$$\geqq \begin{cases} K \int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\, \psi[x(t)/\alpha]} \cdot \frac{1}{r(t)} \int_t^{\infty} (s - t)^{n-2} p(s)\, \varphi(x[g(s)])\, \mathrm{d}s\, \mathrm{d}t, & \text{if } l = 1 \\ K \int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\, \psi[x(t)/\alpha]} \cdot \frac{(t - T_1)^{n-2}}{r(t)} \int_t^{\infty} p(s)\, \varphi(x[g(s)])\, \mathrm{d}s\, \mathrm{d}t, & \text{if } l > 1 \end{cases}$$

$$\geqq \begin{cases} K \int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\, \psi[x(t)/\alpha]} \cdot \frac{1}{r(t)} \int_t^{t^*} (s - t)^{n-2} p(s)\, \varphi(x[g(s)])\, \mathrm{d}s\, \mathrm{d}t, & \text{if } l = 1 \\ K \int_{T_1}^{t^*} \frac{1}{\varphi[x(t)/\alpha]\, \psi[x(t)/\alpha]} \cdot \frac{(t - T_1)^{n-2}}{r(t)} \int_t^{t^*} p(s)\, \varphi(x[g(s)])\, \mathrm{d}s\, \mathrm{d}t, & \text{if } l > 1 \end{cases}$$

$$= \begin{cases} K \int_{T_1}^{t^*} p(s) \int_{T_1}^{s} \frac{(s - t)^{n-2}}{r(t)} \cdot \frac{\varphi(x[g(s)])}{\varphi[x(t)/\alpha]} \cdot \frac{1}{\psi[x(t)/\alpha]} \mathrm{d}t\, \mathrm{d}s, & \text{if } l = 1 \\ K \int_{T_1}^{t^*} p(s) \int_{T_1}^{s} \frac{(t - T_1)^{n-2}}{r(t)} \cdot \frac{\varphi(x[g(s)])}{\varphi[x(t)/\alpha]} \cdot \frac{1}{\psi[x(t)/\alpha]} \mathrm{d}t\, \mathrm{d}s, & \text{if } l > 1 \end{cases}$$

$$\geqq \begin{cases} K \int_{T_2}^{t^*} p(s) \int_{T_1}^{g(s)} \frac{[g(s) - t]^{n-2}}{r(t)} \cdot \frac{\varphi(x[g(s)])}{\varphi[x(t)/\alpha]} \cdot \frac{1}{\psi[x(t)/\alpha]} \mathrm{d}t\, \mathrm{d}s, & \text{if } l = 1 \\ K \int_{T_2}^{t^*} p(s) \int_{T_1}^{g(s)} \frac{(t - T_1)^{n-2}}{r(t)} \cdot \frac{\varphi(x[g(s)])}{\varphi[x(t)/\alpha]} \cdot \frac{1}{\psi[x(t)/\alpha]} \mathrm{d}t\, \mathrm{d}s, & \text{if } l > 1 \end{cases}$$

But, for every t, s with $T_1 \leqq t \leqq g(s)$, $T_2 \leqq s \leqq t^*$ we have

$$x[g(s)] \geqq x(t) \geqq x(t)/\alpha, \qquad x(t)/\alpha \leqq R_1(t) \leqq R_1[g(s)]$$

and consequently, by (II) and (III), we obtain

$$\frac{\varphi(x[g(s)])}{\varphi[x(t)/\alpha]} \geqq 1, \qquad \frac{1}{\psi[x(t)/\alpha]} \geqq \frac{1}{\psi(R_1[g(s)])}.$$

Thus, we derive

$$\alpha \int_{x(T_1)/\alpha}^{x(t^*)/\alpha} \frac{\mathrm{d}y}{\varphi(y)\,\psi(y)} \geqq \begin{cases} K \int_{T_2}^{t^*} \frac{1}{\psi(R_1[g(s)])} p(s) \int_{T_1}^{g(s)} \frac{[g(s)-t]^{n-2}}{r(t)} \mathrm{d}t\, \mathrm{d}s, & \text{if } l = 1 \\ K \int_{T_2}^{t^*} \frac{1}{\psi(R_1[g(s)])} p(s) \int_{T_1}^{g(s)} \frac{(t-T_1)^{n-2}}{r(t)} \mathrm{d}t\, \mathrm{d}s, & \text{if } l > 1 \end{cases}$$

$$= \begin{cases} K \int_{T_2}^{t^*} \frac{R_2[g(s);\, T_1]}{\psi(R_1[g(s)])} p(s)\, \mathrm{d}s, & \text{if } l = 1, \\ K \int_{T_2}^{t^*} \frac{R_1[g(s);\, T_1]}{\psi(R_1[g(s)])} p(s)\, \mathrm{d}s, & \text{if } l > 1. \end{cases}$$

Hence, we have

$$\alpha \int_{x(T_1)/\alpha}^{x(t^*)/\alpha} \frac{\mathrm{d}y}{\varphi(y)\,\psi(y)} \geqq K\beta \int_{T_2}^{t^*} \frac{R[g(s)]}{\psi(R_1[g(s)])} p(s)\, \mathrm{d}s,$$

where $R = R_2$ for $l = 1$, $R = R_1$ for $l > 1$. Therefore it follows that

$$\alpha \int_{x(T_1)/\alpha}^{\infty} \frac{\mathrm{d}y}{\varphi(y)\,\psi(y)} \geqq K\beta \int_{T_2}^{\infty} \frac{R[g(t)]}{\psi(R_1[g(t)])} p(t)\, \mathrm{d}t,$$

which contradicts the conditions (IV) and (C). Hence, only bounded nonoscillatory solutions of $(E, +1)$ may exist and the proof of the theorem is complete.

Proof of Theorem 2. Condition (C) implies (C_0) and hence, by Theorem 0, it is enough to prove that every unbounded nonoscillatory solution x of the equation $(E, -1)$ satisfies one of (X_∞), $(X_{-\infty})$. Furthermore, with respect to the nonoscillatory solutions of $(E, -1)$ we can restrict our attention only to the positive ones.

Let x be a positive unbounded solution on an interval $[\tau_0, \infty)$, $\tau_0 > \max\{t_0, 0\}$, of the equation $(E, -1)$ and let $\tau \geqq \tau_0$ be chosen so that

$$g_j(t) \geqq \tau_0 \qquad \text{for every } t \geqq \tau \qquad (j = 1, \ldots, m).$$

Then, as in the proof of Theorem 1, we can conclude that $(rx')^{(n-1)}$ is nonnegative on $[\tau, \infty)$ and not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$. Thus, by the lemma, there exist a $T \geqq \tau$ and an integer l, $1 \leqq l \leqq n$, with $n + l$ even (therefore $l \neq n - 1$) so that

$$\begin{cases} l < n - 1 \Rightarrow (-1)^{l+j}\,(rx')^{(j-1)} > 0 & \text{on } [\tau, \infty) \quad (j = l, \ldots, n-1), \\ l > 1 \Rightarrow (rx')^{(i-1)} > 0 & \text{on } [\tau, \infty) \quad (i = 1, \ldots, l-1). \end{cases}$$

We consider next the following two cases.

Case 1. $1 \leqq l < n - 1$. The same arguments as in the proof of Theorem 1 lead to a contradiction.

Case 2. $l = n$. If $n > 2$, by using the Taylor formula with integral remainder, for every $t \geqq T$ we obtain

$$r(t)\,x'(t) = \sum_{k=1}^{n-2} \frac{(t-T)^{k-1}}{(k-1)!}(rx')^{(k-1)}(T) + \frac{1}{(n-3)!}\int_T^t (t-s)^{n-3}(rx')^{(n-2)}(s)\,ds \geqq$$
$$\geqq \frac{(rx')^{(n-2)}(T)}{(n-3)!}\int_T^t (t-s)^{n-3}\,ds = \frac{(rx')^{(n-2)}(T)}{(n-2)!}(t-T)^{n-2}$$

and consequently

$$x'(t) \geqq \gamma_1 \frac{(t-T)^{n-2}}{r(t)} \qquad \text{for all } t \geqq T,$$

where $\gamma_1 = (rx')^{(n-2)}(T)/(n-2)!$ The last inequality holds also if $n = 2$. By an integration, for $t \geqq T$ we get

$$x(t) \geqq x(T) + \gamma_1 \int_T^t \frac{(s-T)^{n-2}}{r(s)}\,ds.$$

Therefore it follows that there exist a $\tau_1 > T$ and a positive constant γ with $\gamma \leqq 1$ such that

$$x(t) \geqq \gamma R_1(t) \qquad \text{for all } t \geqq \tau_1.$$

Furthermore, we choose a $\tau_2 \geqq \tau_1$ such that

$$g_j(t) \geqq \tau_1 \qquad \text{for every } t \geqq \tau_2 \qquad (j = 1, \dots, m).$$

Then, by taking into account (ii) and the fact that x is increasing on $[T, \infty)$, for $t \geqq \tau_2$ we obtain

$$[r(t)\,x'(t)]^{(n-1)} = f(t; x[g_1(t)], \dots, x[g_m(t)]) \geqq$$
$$\geqq f(t; x[g(t)], \dots, x[g(t)]) \geqq f(t; \gamma R_1[g(t)], \dots, \gamma R_1[g(t)]).$$

Therefore, by (H), it follows that

$$[r(t)\,x'(t)]^{(n-1)} \geqq p(t)\,\varphi(\gamma R_1[g(t)]), \qquad t \geqq \tau_2.$$

But, (IV) ensures that

$$\lim_{y\to\infty} \frac{\varphi(y)\,\psi(y)}{y} = \infty$$

and hence

$$\varphi(y) \geqq y/\psi(y) \qquad \text{for all large } y.$$

Thus, for some $\tau_3 \geqq \tau_2$ and every $t \geqq \tau_3$ we have

$$\varphi(\gamma R_1[g(t)]) \geqq \frac{\gamma R_1[g(t)]}{\psi(\gamma R_1[g(t)])}$$

and so, because of (I) and (III), we obtain

$$[r(t)\,x'(t)]^{(n-1)} \geqq \gamma \frac{R_1[g(t)]}{\psi(R_1[g(t)])}\, p(t), \qquad t \geqq \tau_3.$$

By an integration, the last inequality gives

$$[r(t)\,x'(t)]^{(n-2)} \geqq (rx')^{(n-2)}(\tau_3) + \gamma \int_{\tau_3}^{t} \frac{R_1[g(s)]}{\psi(R_1[g(s)])}\, p(s)\, \mathrm{d}s$$

for every $t \geqq \tau_3$. Thus, in view of (C), we get

$$\lim_{t\to\infty} [r(t)\,x'(t)]^{(n-2)} = \infty$$

and consequently the solution x satisfies (X_∞).

Now, let us consider the special case where $r = 1$, i.e. the case of the differential equation

$$(\tilde{E}, \delta) \qquad x^{(n)}(t) + \delta f(t; x[g_1(t)], \ldots, x[g_m(t)]) = 0, \qquad t \geqq t_0,$$

where there is no loss of generality to suppose that $t_0 \geqq 0$. Then

$$R_1(t) = R_2(t) = \frac{1}{n-1}(t - t_0)^{n-1}, \qquad t \geqq t_0$$

and consequently for some constant μ we have

$$0 < \mu t^{n-1} \leqq R_1(t) = R_2(t) \leqq t^{n-1} \qquad \text{for all large } t.$$

Thus, because of (iii) and (III), we obtain that for all large t

$$\frac{R_k[g(t)]}{\psi(R_1[g(t)])} \geqq \mu \frac{[g(t)]^{n-1}}{\psi([g(t)]^{n-1})} \qquad (k = 1, 2).$$

Hence, in the considered special case the condition (C) follows from the following one

$$(\tilde{C}) \qquad \int^{\infty} \frac{[g(t)]^{n-1}}{\psi([g(t)]^{n-1})}\, p(t)\, \mathrm{d}t = \infty,$$

provided that (I) holds. From Theorem 1, by applying it for the differential equation $(\tilde{E}, \delta)$, we obtain a recent result of Kitamura and Kusano [3]. The method used here in proving Theorems 1 and 2 is originated in that of Kitamura and Kusano [3].

Next, we turn our attention to differential equations of the form

$$(D) \qquad [r(t)\,x'(t)]^{(n-1)} + a(t)\,\Phi(x[g_1(t)], \ldots, x[g_m(t)]) = 0,$$

where a is a continuous real-valued function on the interval $[t_0, \infty)$ and Φ is a continuous real-valued defined at least on $R^m_+ \cup R^m_-$. The functions a and Φ are subject to the following conditions.

(iv) *a is of constant sign on* $[t_0, \infty)$.

(v) Φ *is increasing on* $R_+^m \cup R_-^m$ *and has the sign property*

$$\Phi(y) > 0 \qquad \textit{for all } y \in R_+^m,\ \Phi(y) < 0 \qquad \textit{for all } y \in R_-^m.$$

Under condition (iv), the equation (D) is of the form (E, δ) with $\delta = +1$ for $a \geqq 0$ or $\delta = -1$ for $a \leqq 0$, and $f(t; y) = |a(t)|\, \Phi(y)$ for all $(t; y) \in [t_0, \infty) \times \times \operatorname{dom} \Phi$. By applying Theorems 1 and 2 with

$$p(t) = |a(t)|,\ t \geqq t_0;\ \varphi(y) = \Phi(y, \ldots, y),\ y \neq 0;\ \psi(y) = 1,\ y > 0$$

for the differential equation (D), we derive the following corollary.

Corollary. *Suppose that* (i), (iii), (iv) *and* (v) *hold and let the differential equation* (D) *be strongly superlinear in the sense that*

$$\text{(vi)} \qquad \int^{\infty} \frac{\mathrm{d}y}{\Phi(y, \ldots, y)} < \infty \qquad \textit{and} \qquad \int^{-\infty} \frac{\mathrm{d}y}{\Phi(y, \ldots, y)} < \infty.$$

Then, under the condition

$$\text{(A)} \qquad \int^{\infty} R_k[g(t)]\, |a(t)|\, \mathrm{d}t = \infty \qquad (k = 1, 2),$$

we have the following:

α_1) *For a nonnegative and n even, all solutions of* (D) *are oscillatory.*

β_1) *For a nonnegative and n odd, every solution x of* (D) *is oscillatory or satisfies* (X_0).

α_2) *For a nonpositive and n even, every solution x of* (D) *is oscillatory or satisfies one of* (X_0), (X_∞), $(X_{-\infty})$.

β_2) *For a nonpositive and n odd, every solution x of* (D) *is oscillatory or satisfies one of* (X_∞), $(X_{-\infty})$.

Finally, we remark that in the case where (D) is an ordinary or advanced differential equation the condition (A) becomes

$$\text{(A*)} \qquad \int^{\infty} R_k(t)\, |a(t)|\, \mathrm{d}t = \infty \qquad (k = 1, 2).$$

Our corollary ceases, in general, to hold if (A) is replaced by (A*). This is illustrated by the following four examples of retarded equations, which fail to satisfy (A). However, they satisfy the rest of the assumptions of Corollary and the condition (A*).

Example 1. The equation

$$[t^{1/3}x'(t)]' + \frac{1}{12} t^{-5/3} x^3(t^{1/3}) = 0, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{1/2}$, a contradiction to conclusion α_1) of Corollary.

Example 2. The equation

$$[t^{1/3}x'(t)]'' + \frac{5}{24}t^{-8/3}x^3(t^{1/3}) = 0, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{3/2}$ with $\lim_{t\to\infty} x(t) = \infty$, a contradiction to conclusion β_1) of Corollary.

Example 3. The equation

$$[tx'(t)]''' - \frac{3}{16}t^{-3}x^3(t^{1/3}) = 0, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{1/2}$ for which we have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} tx'(t) = \infty$ while $\lim_{t\to\infty} [tx'(t)]' = \lim_{t\to\infty} [tx'(t)]'' = 0$, a contradiction to conclusion α_2) of Corollary.

Example 4. The equation

$$[t^{1/3}x'(t)]'' \quad \frac{7}{72}t^{-8/3}x^3(t^{1/3}) = 0, \qquad t \geqq 1$$

has the nonoscillatory solution $x(t) = t^{1/2}$ for which we have $\lim_{t\to\infty} x(t) = \infty$ while $\lim_{t\to\infty} tx'(t) = \lim_{t\to\infty} [tx'(t)]' = 0$, a contradiction to conclusion β_2) of Corollary.

Final Remark. We have already noted that in the special case where $r = 1$ Theorem 1 leads to a recent result of Kitamura and Kusano [3]. We now notice that our corollary extends and improves a recent result of Staikos [9] concerning also the special case $r = 1$.

The same author [8], by using a similar method as in this paper, has proved Theorems 1 and 2 for the differential equation

$$[r(t)\,x^{(n-1)}(t)]' + \delta f(t; x[g_1(t)], \ldots, x[g_m(t)]) = 0. \tag{E$'$, δ}$$

It remains an open question if Theorems 1 and 2 can be extended for more general differential equations of the form

$$[r(t)\,x^{(n-N)}(t)]^{(N)} + \delta f(t; x[g_1(t)], \ldots, x[g_m(t)]) = 0,$$

where $1 \leqq N \leqq n - 1$, or of the general form

$$[r_{n-1}(t)\,[r_{n-2}(t)\,[\ldots\,[r_1(t)\,x'(t)]'\ldots]']']' + \delta f(t; x[g_1(t)], \ldots, x[g_m(t)]) = 0,$$

where r_i $(i = 1, \ldots, n-1)$ are positive continuous functions on the interval $[t_0, \infty)$ with $\int^{\infty} [1/r_i(t)]\,\mathrm{d}t = \infty$ $(i = 1, \ldots, n-1)$.

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