Archivum Mathematicum

Karel Svoboda

Remark on one theorem of R. Schneider

Archivum Mathematicum, Vol. 17 (1981), No. 3, 169--172

Persistent URL: http://dml.cz/dmlcz/107107

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 169—172, 1981

REMARK ON ONE THEOREM OF R. SCHNEIDER

KAREL SVOBODA, Brno

(Received May 15, 1980)

The aim of this remark is to generalize one result due to Rolf Schneider [1] concerning the global characterization of the sphere among surfaces in E^3. Thus we give in the following an analogy of the meant Schneider's assertion valid for the 2-dimensional sphere in E^n.

We formulate immediately the result:

Theorem. *Let M be a surface in E^n, $n \geqq 3$, K its Gauss curvature and $S \in E^n$ a fixed point, $S \notin T_m(M)$ for an arbitrary $m \in M$. Let $v_T \in T_m(M)$ be the tangent and $v_N \in N(M)$ the normal component of the vector v defined by $S = m + v$. Let*

(i) $K > 0$ *on* M;

(ii) $\langle v_1 v_1, v_N \rangle \langle v_2 v_2, v_N \rangle - \langle v_1 v_2, v_N \rangle^2 - \langle v_1 v_1 + v_2 v_2, v_N \rangle + 1 \leqq 0$ *on* M, $v_1, v_2 \subset T(M)$ *being tangent orthonormal vector fields on* M;

(iii) $v_T = 0$ *on the boundary* ∂M *of* M.

Then M is a part of a 2-dimensional sphere in E^n with the center S.

Proof. Let M be covered by open domains U_α in such a way that in each U_α there is a field of orthonormal frames $\{m; v_1, v_2, \ldots, v_n\}$ with $v_1, v_2 \in T(M), v_3, \ldots, v_n \in N(M)$, where $T(M)$, $N(M)$ are the tangent and the normal bundles of M, respectively. Then we have

$$\text{(1)} \qquad dm = \sum_{j=1}^{n} \omega^j v_j, \qquad dv_i = \sum_{j=1}^{n} \omega_i^j v_j \qquad (i = 1, 2, \ldots, n),$$

with

$$\text{(2)} \qquad \omega^j = 0 \qquad (j = 3, \ldots, n),$$
$$\omega_i^j + \omega_j^i = 0 \qquad (i, j = 1, 2, \ldots, n)$$

and the structure equations

$$\text{(3)} \qquad d\omega^i = \sum_{j=1}^{n} \omega^j \wedge \omega_j^i, \qquad d\omega_i^j = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j, \qquad (i, j = 1, 2, \ldots, n).$$

We easily get from (2) (see for example [2])

$$\omega_1^i = a_i\omega^1 + b_i\omega^2, \qquad \omega_2^i = b_i\omega^1 + c_i\omega^2 \qquad (i = 3, \ldots, n) \tag{4}$$

and further, differentiating the equations (4) and applying Cartan's lemma, the existence of real-valued functions $\alpha_i, \ldots, \delta_i$ $(i = 3, \ldots, n)$ such that

$$\begin{aligned} \mathrm{d}a_i - 2b_i\omega_1^2 - \sum_{j=3}^{n} a_j\omega_i^j &= \alpha_i\omega^1 + \beta_i\omega^2, \\ \mathrm{d}b_i + (a_i - c_i)\,\omega_1^2 - \sum_{j=3}^{n} b_j\omega_i^j &= \beta_i\omega^1 + \gamma_i\omega^2, \\ \mathrm{d}c_i + 2b_i\omega_1^2 - \sum_{j=3}^{n} c_j\omega_i^j &= \gamma_i\omega^1 + \delta_i\omega^2 \qquad (i = 3, \ldots, n). \end{aligned} \tag{5}$$

Now, let

$$S = m + xv_1 + yv_2 + \sum_{i=3}^{n} p_iv_i \tag{6}$$

be the considered point of E^n. As S is supposed to be fixed, from $\mathrm{d}S = 0$ we obtain using (1) and (4),

$$\begin{aligned} \mathrm{d}x - y\omega_1^2 &= (\sum_{j=3}^{n} a_jp_j - 1)\,\omega^1 + \sum_{j=3}^{n} b_jp_j\omega^2, \\ \mathrm{d}y + x\omega_1^2 &= \sum_{j=3}^{n} b_jp_j\omega^1 + (\sum_{j=3}^{n} c_jp_j - 1)\,\omega^2, \\ \mathrm{d}p_i - \sum_{j=3}^{n} p_j\omega_i^j &= -(a_ix + b_iy)\,\omega^1 - (b_ix + c_iy)\,\omega^2 \qquad (i = 3, \ldots, n). \end{aligned} \tag{7}$$

Further, consider the 1-form

$$\begin{aligned} \omega &= x\,\mathrm{d}y - y\,\mathrm{d}x + (x^2 + y^2)\,\omega_1^2 = \\ &= [x\sum_{i=3}^{n} b_ip_i - y(\sum_{i=3}^{n} a_ip_i - 1)]\,\omega^1 + [x(\sum_{i=3}^{n} c_ip_i - 1) - y\sum_{i=3}^{n} b_ip_i]\,\omega^2. \end{aligned} \tag{8}$$

According to (3) and (7), we get from (8) by an easy calculation

$$\mathrm{d}\omega = 2\left\{J_\omega - \frac{1}{2}(x^2 + y^2)\,K\right\}\omega^1 \wedge \omega^2, \tag{9}$$

where

$$J_\omega = (\sum_{i=3}^{n} a_ip_i - 1)\,(\sum_{i=3}^{n} c_ip_i - 1) - (\sum_{i=3}^{n} b_ip_i)^2.$$

From (1), we have

$$v_1v_1 = \sum_{i=3}^{n} a_iv_i (\mathrm{mod}\ v_2), \qquad v_1v_2 = \sum_{i=3}^{n} b_iv_i (\mathrm{mod}\ v_1),$$

$$v_2 v_1 = \sum_{i=3}^{n} b_i v_i (\text{mod } v_2), \qquad v_2 v_2 = \sum_{i=3}^{n} c_i v_i (\text{mod } v_1)$$

and J_ω can be written in the form

$$J_\omega = \langle v_1 v_1, v_N \rangle \langle v_2 v_2, v_N \rangle - \langle v_1 v_2, v_N \rangle^2 - - \langle v_1 v_1 + v_2 v_2, v_N \rangle + 1. \tag{10}$$

It is not difficult to show that the expression (10) is invariant on M.

From (8) and the assumption (iii) it follows immediately that $\omega = 0$ on ∂M and thus, applying Stokes theorem, we have

$$\int_M \{2J_\omega - (x^2 + y^2) K\} \omega^1 \wedge \omega^2 = 0. \tag{11}$$

Hence, from (11), according to the suppositions (i) and (ii), $x = y = 0$, i.e. $v_T = 0$ on M.

This being proved, we have, according to (7) and (2),

$$\mathrm{d} \langle v, v \rangle = \mathrm{d}(\sum_{i=3}^{n} p_i^2) = 0.$$

From hence it follows that the length of v is constant and thus M is a part of a 2-dimensional sphere with the center S.

Corollary. *Let M be a surface in E^3, $S = m + v$, $m \in M$, a fixed point of E^3 and v_T the tangent component of v. Let*

(i) $K > 0$ *on* M;

(ii) $(pk_1 - 1)(pk_2 - 1) \leqq 0$ *on* M, k_1, k_2 *being the principal curvatures of* M, p *the support function*;

(iii) $v_T = 0$ *on the boundary* ∂M.

Then M is a part of a sphere in E^3 with the center S.

Proof. Let $n = 3$ in the proof of our Theorem. From (10), when omitting the index 3, we have immediately

$$J_\omega = (ac - b^2) p^2 - (a + c) p + 1 = Kp^2 - 2Hp + 1,$$

where p is the support function. Thus, k_1, k_2 being the principal curvatures of M,

$$J_\omega = (k_1 p - 1)(k_2 p - 1)$$

and the suppositions (i)–(iii) of Corollary yield the assertion.

Similar result for ovaloids of the class C^2 has been proved in R. Schneider's paper [1].

REFERENCES

[1] R. Schneider: *Eine Kennzeichnung der Kugel,* Archiv der Math., 16 (1965), 235—240.
[2] K. Svoboda: *On the 2-dimensional sphere in* E^n, Knižnice VUT, A-18 (1978), 283—297

K. Svoboda
602 00 Brno, Gorkého 13
Czechoslovakia