Jaroslav Krejzlík

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CONTRIBUTION TO THE THEORY OF PSEUDOCONGRUENCES WITH PROJECTIVE CONNECTION

JAROSLAV KREJZLÍK, Brno

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Using basic ideas and conceptions introduced in [1], [2] and results from [3], the pseudocongruences of $(n-1)$-planes with projective connection are introduced and their projective deformations are studied.

1. Let a special König space $\mathscr{P}^n_{n-1,2n-1}$ be constructed according to [1], p. 71, 72. Using notation of Gejdelman ([4], p. 281), we shall call these spaces $(n-1)$-plane pseudocongruences with projective connection.

Let a $(n-1)$-plane pseudocongruence $\mathscr{L}$ with projective connection be given by the equations

$$(1.1)\qquad \nabla A_i = \sum_{j=1}^{2n} \omega_{ij} A_j,$$

$$\omega_{ij} = \sum_{k=1}^{n} a^k_{ij}(u_1, u_2, \dots, u_n)\,\omega_k; \qquad \sum_{i=1}^{2n} \omega_{ii} = 0,$$

where ω_k $(k = 1, 2, \dots, n)$ are the Pfaff forms in the differentials $du_1, du_2, \dots, du_n$, $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_n \neq 0$. The $(n-1)$-planes of the pseudocongruence $\mathscr{L}$ are $P_{n-1} = (A_1, A_2, \dots, A_n)$. We call the developable varieties $\mathscr{R}_n$ of $\mathscr{L}$ (corresponding to the curves of Ω_n) varieties with developable developments. The equation of developable varieties of the pseudocongruence $\mathscr{L}$ is

$$(1.2)\qquad [A_1, A_2, \dots, A_n, \nabla A_1, \nabla A_2, \dots, \nabla A_n] = 0.$$

The first term of (1.2) is a form of n-th degree in du_i $(i = 1, 2, \dots, n)$. We restrict ourselves to such pseudocongruences whose form mentioned above is the product of n linear forms in du_i $(i = 1, 2, \dots, n)$. Let us denote them $\omega_1, \omega_2, \dots, \omega_n$. The equation (1.2) reduces to

$$\omega_1 \omega_2 \dots \omega_n = 0.$$

If nothing other is mentioned then in all our considerations it will be always

$$(1.3)\qquad s = i+1, i+2, \dots, i+n-1 \qquad (i = 1, 2, \dots, n)$$

and the indices $i, i+1, \ldots, i+2n-1$ are changed according to the scheme

$$(1.4)\qquad \left|\begin{array}{llllllll} i, & 1, & 2, & 3, & \ldots, & n-2, & n-1, & n \\ i+1, & 2, & 3, & 4, & \ldots, & n-1, & n, & 1 \\ i+2, & 3, & 4, & 5, & \ldots, & n, & 1, & 2 \\ \vdots & & & & & & & \\ i+n-1, & n, & 1, & 2, & \ldots, & n-3, & n-2, & n-1 \\ i+n, & n+1, & n+2, & n+3, & \ldots, & 2n-2, & 2n-1, & 2n \\ i+n+1, & n+2, & n+3, & n+4, & \ldots, & 2n-1, & 2n, & n+1 \\ \vdots & & & & & & & \\ i+2n-1, & 2n, & n+1, & n+2, & \ldots, & 2n-3, & 2n-2, & 2n-1 \end{array}\right|$$

We shall deal with such pseudocongruences only where for $\omega_i = 0$, $i = 1, 2, \ldots, n$ (ω_k arbitrary, $k = 1, 2, \ldots, n$, $i \neq k$) there exists just one focus and the n foci considered do not lie in one $(n-2)$-plane. Let us choose these foci to be the points $A_1, A_2, \ldots, A_n$.

A point A_i to be a focus then

$$[(\nabla A_i)_{\omega_i=0}, A_1, A_2, \ldots, A_n] = 0, \qquad (i = 1, 2, \ldots, n),$$

i.e.

$$a^s_{i,i+n} = a^s_{i,i+n-1} = \ldots = a^s_{i+2n-1} = 0,$$

where the indices are changed according to (1.3) and (1.4).

The fundamental equations of the pseudocongruence $\mathscr{L}$ are

$$\nabla A_i = \sum_{j=1}^{n} \omega_{ij} A_j + \sum_{j=n+1}^{2n} a^i_{ij} A_j \omega_i \qquad (i = 1, 2, \ldots, n).$$

Using the specialization

$$\sum_{j=n+1}^{2n} a^i_{ij} A_j \to A_{i+n}$$

we obtain the fundamental equations in the form

$$(1.5)\qquad \begin{aligned} \nabla A_i &= \omega_i A_{i+n} + \sum_{j=1}^{n} \omega_{ij} A_j, \\ \nabla A_{i+n} &= \sum_{j=1}^{2n} \omega_{i+n,j} A_j \qquad (i = 1, 2, \ldots, n). \end{aligned}$$

The foci A_i $(i = 1, 2, \ldots, n)$ of the pseudocongruence $\mathscr{L}$ generate n König varieties; let us denote them (A_i) and let us call them focal varieties of the pseudocongruence $\mathscr{L}$. Let A_i be a fixed point of the focal variety (A_i). The developments of all the curves of the focal variety into the local space of A_i are curves with tangents in the n-plane $(A_1, A_2, \ldots, A_n, A_{i+n})$, the s.c. tangent n-plane of the focal variety (A_i). This n-plane is the focal n-plane of the pseudocongruence $\mathscr{L}$.

In the $(n-1)$-plane of the pseudocongruence $\mathscr{L}$ the foci $A_1, A_2, \ldots, A_n$ are vertices of s.c. focal simplex. For any point of a $(k-1)$-plane $[A_{i+1}, A_{i+2}, \ldots, A_{i+k}]$, $k = 2, 3, \ldots; n-1$, s.c. focal $(k-1)$-plane, the focal directions are

$$\omega_{i+1} = \omega_{i+2} = \ldots = \omega_{i+k} = 0.$$

The indices are changed according to (1.4).

For each vertex A_i all the directions satisfying the equation $\omega_i = 0$ $(i = 1, 2, \ldots, n)$ are focal directions. The foci A_i generate varieties of the dimension n. We restrict our consideration to the case when all n focal varieties are of the dimension n.

2. Let $\mathscr{L}$ be a $(n-1)$-plane pseudocongruence with projective connection given by the equations (1.5). Without loss of generality, we may assume

$$\omega_i = \mathrm{d}u_i$$

and we obtain

$$\nabla A_i = \mathrm{d}u_i A_{i+n} + \sum_{j=1}^{n} \omega_{ij} A_j, \tag{2.1}$$

$$\omega_{ij} = \sum_{k=1}^{n} a_{ij}^k \, \mathrm{d}u_k.$$

The variations of parameters and local frames compatible with

$$\omega_{i,i+s} = \mathrm{d}u_i, \qquad \omega_{i,s+n} = 0 \tag{2.2}$$

are given by

$$u_i = u_i(\bar{u}_i) \tag{2.3}$$

$$A_i = \mu_{ii} \bar{A}_i, \qquad A_{i+n} = \sum_{j=1}^{2n} \mu_{i+n,j} \bar{A}_j, \tag{2.4}$$

where

$$\mu_{11}\mu_{22} \ldots \mu_{nn} \det | \mu_{i+n,j} | = 1$$

$(j = n+1, n+2, \ldots, 2n-1)$.

From (2.1) we obtain easily

$$\bar{a}_{i,i+1}^i = \mu_{ii}^{-1}(\mu_{i+1,i+1} a_{i,i+1}^i + \mu_{i+n,i+1}) \frac{\mathrm{d}u_i}{\mathrm{d}\bar{u}_i}, \tag{2.5}$$

$$a_{is}^s = \mu_{ii}^{-1} \mu_{ss} a_{is}^s \frac{\mathrm{d}u_s}{\mathrm{d}\bar{u}_s},$$

and

$$\bar{a}_{i+n,n+s}^j = \mu_{ii}^{-1} \mu_{ss} a_{i+n,n+s}^r \frac{\mathrm{d}u_i \, \mathrm{d}\bar{u}_s \, \mathrm{d}u_j}{\mathrm{d}\bar{u}_i \, \mathrm{d}u_s \, \mathrm{d}\bar{u}_j} \quad \text{for } j \neq s, \tag{2.6}$$

$$\bar{a}_{i+n,n+s}^j = \mu_{ii}^{-1}(\mu_{ss} a_{i+n,n+s}^j - \mu_{i+n,j}) \frac{\mathrm{d}u_i}{\mathrm{d}\bar{u}_i} \quad \text{for } j = s,$$

where $j = i, i+1, \dots, i+n-1$ and the indices $i, i+1, \dots, i+n-1$ are changed according to (1.4).

From (2.5) and (2.6) we obtain

$$\bar{a}^i_{is} - \bar{a}^s_{i+n,n+s} = \mu_{ii}^{-1}[\mu_{ss}(a^i_{is} - a^s_{i+n,n+s}) + 2\mu_{i+n,s}] \frac{du_i}{d\bar{u}_i}.$$

We may specialize the frames in such a way that

$$(2.7) \qquad a^i_{is} - a^s_{i+n,n+s} = 0 \quad \text{and} \quad \mu_{i+n,s} = 0.$$

We obtain

$$(2.8) \qquad A_i = \mu_{ii}\bar{A}_i, \qquad A_{i+n} = \mu_{i+n,i}\bar{A}_i + \mu_{ii}\frac{d\bar{u}_i}{du_i}\bar{A}_{i+n}.$$

Let us introduce the notation

$$(2.9) \qquad h_{is} = a^i_{is} = a^s_{i+n,n+s},$$

$$\nabla\alpha_{is} = \sum_{j=1}^{n} a^j_{is}\, du_j, \quad j \neq i, \qquad \nabla\beta_{is} = \sum_{j=1}^{n} a^j_{i+n,s+n}\, du_j, \quad j \neq s.$$

We may specialize the frames of a pseudocongruence $\mathscr{L}$ with projective connection in such a way that $\mathscr{L}$ is given by the equations

$$(2.10) \qquad \nabla A_i = du_i A_{i+n} + \sum_{j=1}^{n} \omega_{ij} A_j;$$

$$\nabla A_{i+n} = \sum_{r=1}^{n} \omega_{i+n,r} A_r + \sum_{j=1}^{n} \omega_{i+n,n+j} A_{n+j},$$

where for $i \neq j$

$$\omega_{ij} = h_{ij}\, du_i + \nabla\alpha_{ij}, \qquad \omega_{i+n,j+n} = h_{ij}\, du_j + \nabla\beta_{ij}.$$

The most general variation compatible with (2.2) and (2.7) is (2.3) and (2.8). After these variations we obtain

$$(2.11) \qquad \bar{h}_{is} = \mu_{ii}^{-1}\mu_{ss}\frac{du_i}{d\bar{u}_i} h_{is},$$

$$\nabla\bar{\alpha}_{is} = \mu_{ii}^{-1}\mu_{ss}\nabla\alpha_{is}, \qquad \nabla\bar{\beta}_{is} = \mu_{ii}^{-1}\mu_{ss}\frac{du_i\, d\bar{u}_s}{d\bar{u}_i\, du_s}\nabla\beta_{is}.$$

3. The dualization $\mathscr{P}^{r*}_{p,n}$ of the König space $\mathscr{P}^{r}_{p,n}$ is the König space of the type $\mathscr{P}^{r}_{n-p-1,n}$ defined by the construction B ([1], p. 73). The dualization $\mathscr{L}^*$ of the pseudocongruence $\mathscr{L}$ is again a pseudocongruence. Using the dual frames

$$(3.1) \qquad E^i = (-1)^{i+1}[A_i, \dots, A_{i-1}, A_{i+1}, \dots, A_{2n}],$$

the pseudocongruence $\mathscr{L}^*$ is formed by the $(n-1)$-planes $P^*_{n-1} = [E^{n+1}, E^{n+2}, \dots, E^{2n}]$ (P^*_{n-1} being the local centers of $\mathscr{L}^*$) and the connection is given by the equations

$$\text{(3.2)}\qquad \begin{aligned} \nabla E^{i+n} &= -\mathrm{d}u_i E^i - \sum_{j=1}^{n} \omega_{j+n,i+n} E^{j+n}, \\ \nabla E^i &= -\sum_{r=1}^{n} \omega_{r+n,i} E^{r+n} - \sum_{j=1}^{n} \omega_{ji} E^j, \end{aligned}$$

where for $i \neq j$

$$\omega_{ji} = h_{ji}\,\mathrm{d}u_j + \nabla\alpha_{ji}, \qquad \omega_{j+n,i+n} = h_{ji}\,\mathrm{d}u_i + \nabla\beta_{ji}.$$

The foci of the dualization are

$$E^{i+n} \qquad (i = 1, 2, \ldots, n).$$

As a consequence of passing to the dualization, we obtain the following substitution

$$\text{(3.3)}\qquad \left\downarrow \begin{matrix} \mathscr{L} & A_i & E^i & \mathrm{d}u_i & h_{is} & \nabla\alpha_{is} & \nabla\beta_{is} & \omega_{ij} \\ \mathscr{L}^* & E^{i+n} & A_{i+n} & -\mathrm{d}u_i & h_{si} & -\nabla\beta_{si} & -\nabla\alpha_{si} & -\omega_{j+n,i+n} \end{matrix} \right\downarrow$$

where the indices are changed according to the scheme

$$\left\downarrow \begin{matrix} i,j: & 1 & 2 & \ldots\ n & n+1 & n+2\ \ldots & 2n \\ i+n, j+n: & n+1 & n+2 & \ldots\ 2n & 1 & 2 & \ldots\ n \end{matrix} \right\downarrow.$$

The natural correspondence $\mathscr{L} \to \mathscr{L}^*$ is hence developable.

4. From (2.1) we obtain the following invariant forms of the pseudocongruence.

(4.1) Point forms

$$\varphi_{i_1, i_2, \ldots, i_k} = \nabla\alpha_{i_1, i_2} \nabla\alpha_{i_2, i_3} \cdots \nabla\alpha_{i_{k-1}, i_k} \nabla\alpha_{i_k, i_1},$$

(4.2) Hyperplanar forms

$$\varphi^*_{i_1, i_2, \ldots, i_k} = \nabla\beta_{i_1, i_2} \nabla\beta_{i_2, i_3} \cdots \nabla\beta_{i_{k-1}, i_k} \nabla\beta_{i_k, i_1},$$

where k is the order of the form.

The indices of these forms are generated by all the permutations of numbers $1, 2, \ldots, n$ taken k at a time. The forms having the same cyclic order of indices are equal. The number of each of these forms is $\dfrac{n!}{k(n-k)!}$.

It can be shown that

$$\varphi_{i_1 \ldots i_{j-1} i_j i_{j+1} \ldots i_k} = \frac{\varphi_{i_1 \ldots i_{j-1} i_j},\ \varphi_{i_1 i_j \ldots i_k}}{\varphi_{i_1 i_j}}$$
$$(3 \leqq j \leqq k - 1).$$

Similar relations are true for the hyperplanar forms. With respect to these relations it is sufficient to consider the forms of second and third order only.

The set of point forms of second and third order will be called the point element of the pseudocongruence and the set of hyperplanar forms of second and third order will be called the hyperplanar element of the pseudocongruence.

(4.3) Focal forms

$$F_{is} = \nabla\alpha_{is}\,\nabla\beta_{si}\frac{du_s}{du_i}.$$

(4.4) Pseudoasymptotic forms

$$G_{is} = \frac{\nabla\alpha_{is}\,du_s}{\nabla\beta_{is}\,du_i}.$$

For the study of projective deformations we must introduce the forms

$$\psi_{is} = (a^s_{i+n,i+n} - a^s_{ii})\,du_s, \tag{4.5}$$

The substitution (3.3) will be completed by

$$\left|\begin{matrix} \mathscr{L}: & \varphi_{i_1\dots i_k} & F_{is} & G_{is} & \psi_{is} \\ \mathscr{L}^*: & \varphi^*_{i_1\dots i_k} & F_{is} & 1/G_{is} & \psi_{is}\end{matrix}\right| \tag{4.6}$$

5. Let $\mathscr{L}$ be a $(n-1)$-plane pseudocongruence with projective connection given by (2.10). Let $\tilde{\mathscr{L}}$ be another pseudocongruence; we denote all expressions connected with $\tilde{\mathscr{L}}$ by a tilde. Let the frames associated with $\tilde{\mathscr{L}}$ be specialized in the same way as those associated with $\mathscr{L}$.

Let C: $\mathscr{L} \to \tilde{\mathscr{L}}$ be a correspondence between $\mathscr{L}$ and $\tilde{\mathscr{L}}$ given by the equations

$$d\tilde{u}_i = \sum_{j=1}^{n} m_{ij}\,du_j, \tag{5.1}$$

where

$$\det |m_{ij}| \neq 0.$$

The correspondence associates to a $(n-1)$-plane $P_{n-1} \in \mathscr{L}$ a $(n-1)$-plane $\tilde{P}_{n-1} \in \tilde{\mathscr{L}}$

$$CP_{n-1} = \tilde{P}_{n-1}.$$

The correspondence C: $\mathscr{L} \to \tilde{\mathscr{L}}$ is called the projective deformation of order k if for each $(n-1)$-plane P_{n-1} of the pseudocongruence $\mathscr{L}$ there exists a collineation K: $P_{2n-1} \to \tilde{P}_{2n-1}$ such that the pseudocongruences $K\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the analytic contact of order k along the $(n-1)$-plane $\tilde{P}_{n-1} = CP_{n-1}$. We say that K realizes the projective deformation C of order k.

The conditions for the correspondence C to be a projective deformation of the first order consist in the existence of the collineation

$$K\tilde{A}_j = \sum_{r=1}^{2n} c_{jr}A_r \qquad (j = 1, 2, \dots, 2n) \tag{5.2}$$

and such a form ϑ_1 that it holds

$$\begin{aligned} K[\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n] &= [A_1, A_2, \dots, A_n] \\ K\nabla[\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n] &= \nabla[A_1, A_2, \dots, A_n] + \vartheta_1[A_1, A_2, \dots, A_n]. \end{aligned} \tag{5.3}$$

From these equations we get

$$K\tilde{A}_i = \sum_{r=1}^{n} c_{ir} A_r, \qquad \det | c_{ir} | = 1$$

and further

$$c_{is} = c_{i+n,s+n} = 0; \qquad du_i = c_{i+1,i+1} c_{i+2,i+2} \dots c_{i+n,i+n} \, d\tilde{u}_i$$

and (5.1) may be reduced to

$$du_i = d\tilde{u}_i.$$

Proposition 1. The correspondence $C\colon \mathscr{L} \to \tilde{\mathscr{L}}$ is the projective deformation of the first order if and only if C is developable. The collineation realizing this deformation transforms the focal formations of the pseudocongruence $\mathscr{L}$ into the corresponding focal formations of the pseudocongruence $\tilde{\mathscr{L}}$.

The tangent collineation K is of the form

$$\text{(5.4)} \qquad K\tilde{A}_i = \varrho_i A_i, \qquad K\tilde{A}_{i+n} = \varrho_i A_{i+n} + \sum_{r=1}^{n} c_{i+n,r} A_r,$$

where

$$\varrho_1 \varrho_2 \dots \varrho_n = 1$$

and

$$\text{(5.5)} \qquad \tau_{ij} = \tilde{\omega}_{ij} - \omega_{ij}, \qquad \vartheta_1 = \sum_{i=1}^{n} (\tau_{ii} - \varrho_i^{-1} c_{i+n,i} \, du_i).$$

The dual collineation $K^*\colon P^*_{2n-1} \to \tilde{P}^*_{2n-1}$ is given by

$$\text{(5.6)} \qquad K^* \tilde{E}^{i+n} = \varrho_i^{-1} E^{i+n},$$

$$K^* \tilde{E}^i = \varrho_i^{-1} E^i - \sum_{r=1}^{n} \varrho_i^{-1} \varrho_r^{-1} c_{n+r,i} E^{n+r}.$$

This collineation is tangent to the correspondence $C\colon \mathscr{L} \to \mathscr{L}^*$.

The correspondence C is a projective deformation of the second order if and only if there exists (for each $(n-1)$-plane $P_{n-1} \in \mathscr{L}$) a tangent collineation K satisfying (5.3), (5.5) and

$$\text{(5.7)} \qquad K\nabla^2[\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n] =$$

$$= \nabla^2[A_1, A_2, \dots, A_n] + 2\vartheta_1 \nabla[A_1, A_2, \dots, A_n] + (.) [A_1, A_2, \dots, A_n].$$

There is

$$\text{(5.8)} \quad \nabla[A_1, A_2, \dots, A_n] = \sum_{i=1}^{n} \{\omega_{ii}[A_1, A_2, \dots, A_n] + du_i[A_{i+1}, A_{i+2}, \dots, A_{i+n}]\},$$

$$\nabla[A_{i+1}, A_{i+2}, \dots, A_{i+n}] = \sum_{r=1}^{n} \omega_{i+r,i+r}[A_{i+1}, A_{i+2}, \dots, A_{i+n}] +$$

$$+ \sum_{r=1}^{n-1} \{(h_{i,i+r} du_{i+r} + \nabla\beta_{i,i+r}) [A_{i+1}, A_{i+2}, \dots, A_{i+n-1}, A_{i+n+r}] +$$

$$+ (h_{i+r,i} du_{i+r} + \nabla\alpha_{i+r,i}) [A_i, A_{i+r+1}, A_{i+r+2}, \dots, A_{i+r+n-1}] -$$

$$- \omega_{i+n,i}[A_i, A_{i+1}, \dots, A_{i+n-1}] + (-1)^{r(n-1)} du_{i+r}[A_{i+r+1}, A_{i+r+2}, \dots, A_{i+r+n}]\}.$$

The indices are changed according to (1.4).
Consequently

(5.9)
$$\nabla^2[A_1, A_2, \dots, A_n] = (.)\,[A_1, A_2, \dots, A_n] + $$
$$+ \sum_{i=1}^{n} \sum_{r=1}^{n} \{\omega_{ii}\, du_r [A_{r+1}, A_{r+2}, \dots, A_{i+r-1}] + $$
$$+ (d^2u_i + du_i \omega_{i+r,i+r})\,[A_{i+1}, A_{i+2}, \dots, A_{i+n}]\} + $$
$$+ \sum_{i=1}^{n} \sum_{r=1}^{n-1} \{(-1)^{r(n-1)}\, du_i\, du_{i+r} [A_{i+r+1}, A_{i+r+2}, \dots, A_{i+r+n}] + $$
$$+ (\nabla\alpha_{i+r,i}\, du_i - \nabla\beta_{i+r,i}\, du_{i+r})\,[A_i, A_{i+r+1}, \dots, A_{i+r+n-1}]\}.$$

The indices are changed according to (1.4).

From (5.9), an analogous equation for $\nabla^2[\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n]$ and (5.5) we obtain

(5.10)
$$K\nabla^2[\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n] = \nabla^2[A_1, A_2, \dots, A_n] + $$
$$+ 2\vartheta_1\, \nabla[A_1, A_2, \dots, A_n] + (.)\,[A_1, A_2, \dots, A_n] + $$
$$+ \sum_{r=1}^{i+n-1} \sum_{i=1}^{n} \Phi^r_{i+1,i+2,\dots,i+n-1} [A_{i+1}, A_{i+2}, \dots, A_n],$$

where

(5.11)
$$\Phi^i_{i+1,i+2,+,i+n-1} = (\tau_{i+s,i+s} - \tau_{ii})\, du_i - 2\varrho_i^{-1} c_{i+n,i}\, du_i^2,$$
$$\Phi^s_{i+1,i+2,+,i+n-1} = $$
$$= \nabla\alpha_{is}\, du_s - \nabla\beta_{is}\, du_i - \varrho_s \varrho_i^{-1}(\nabla\tilde{\alpha}_{is}\, du_s - \nabla\tilde{\beta}_{is}\, du_i) - 2\varrho_i^{-1} c_{i+n,s}\, du_i\, du_s.$$

If C is a projective deformation of the second order then there exist such functions $c_{i+n,i}$, $c_{i+n,s}$ that

(5.12)
$$\Phi^i_{i+1,i+2,+,i+n-1} = \Phi^s_{i+1,i+2,\dots,i+n-1} = 0.$$

From (5.10) and (5.11) it follows

(5.13)
$$c_{i+n,s} = 0, \qquad c_{i+n,i} = \frac{1}{2}(\tilde{a}^i_{i+n,i+n} - \tilde{a}^i_{ii} - a^i_{i+n,i+n} + a^i_{ii})\,\varrho_i,$$

(5.14)
$$\varrho_i\, \nabla\alpha_{is} = \varrho_s\, \nabla\tilde{\alpha}_{is}, \qquad \varrho_i\, \nabla\tilde{\beta}_{is} = \varrho_s\, \nabla\beta_{is}$$

(5.15)
$$\tilde{a}^s_{i+n,i+n} - \tilde{a}^s_{ii} = a^s_{i+n,i+n} - a^s_{ii}.$$

Eliminating ϱ_i from (5.14), we get

(5.16)
$$\nabla\tilde{\alpha}_{is}\, \nabla\tilde{\alpha}_{si} = \nabla\alpha_{is}\, \nabla\alpha_{si}, \qquad \nabla\tilde{\alpha}_{ij}\, \nabla\tilde{\alpha}_{jr}\, \nabla\tilde{\alpha}_{ri} = \nabla\alpha_{ij}\, \nabla\alpha_{jr}\, \nabla\alpha_{ri}$$

(5.17)
$$\nabla\tilde{\beta}_{is}\, \nabla\tilde{\beta}_{si} = \nabla\beta_{is}\, \nabla\beta_{si}, \qquad \nabla\tilde{\beta}_{ij}\, \nabla\tilde{\beta}_{jr}\, \nabla\tilde{\beta}_{ri} = \nabla\beta_{ij}\, \nabla\beta_{jr}\, \nabla\beta_{ri}$$

(5.18)
$$\nabla\tilde{\alpha}_{is}\, \nabla\tilde{\beta}_{is} = \nabla\alpha_{is}\, \nabla\beta_{is}$$

(5.19)
$$\frac{\nabla\tilde{\alpha}_{is}}{\nabla\tilde{\beta}_{is}} = \frac{\nabla\alpha_{is}}{\nabla\beta_{is}}.$$

The indices i, j, r are generated by all the permutations of numbers $1, 2, \ldots, n$ taken 3 at a time. The indices having the same cyclic order are taken once only.

With respect to (4.1)–(4.5) we have

$$\tilde{\varphi}_{is} = \varphi_{is}, \quad \tilde{\varphi}_{ijr} = \varphi_{ijr}, \quad \tilde{\varphi}^*_{is} = \varphi^*_{is}, \quad \tilde{\varphi}^*_{ijr} = \varphi^*_{ijr}$$
$$\tilde{F}_{is} = F_{is}, \quad \tilde{G}_{is} = G_{is}, \quad \tilde{\psi}_{is} = \psi_{is}. \tag{5.20}$$

We can prove easily that these conditions are necessary and sufficient.

Proposition 2. Let $C\colon \mathscr{L} \to \tilde{\mathscr{L}}$ be a developable correspondence. The correspondence C is a projective deformation of the second order if and only if pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the same point and hyperplanar element, the same focal and pseudoasymptotic forms and the same forms ψ_{is}.

Substitution (3.3) and (4.6) yields

Proposition 3. Let $C\colon \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. The correspondence $C\colon \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is also a projective deformation of the second order.

6. Let $C\colon \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. According to (5.4) and (5.13) the osculating collineation realizing this deformation

$$K\tilde{A}_i = \varrho_i A_i, \qquad K\tilde{A}_{i+n} = c_{i+n,i} A_i + \varrho_i A_{i+n} \tag{6.1}$$

where $c_{i+n,i}$ is determined by (5.13).

The dualization $C\colon \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is also a projective deformation of the second order and the osculating collineation realizing this deformation is

$$K\tilde{E}^{i+n} = \varrho_i^{-1} E^{i+n}, \qquad K\tilde{E}^i = -\varrho_i^{-2} c_{i+n,i} E^{i+n} + \varrho_i^{-1} E^i, \tag{6.2}$$

where $c_{i+n,i}$ are determined by (5.13).

If expressed in terms of points, relations (6.2) give (6.1).

Proposition 4. Let $C\colon \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order and (6.1) be its osculating collineation. The projective deformation $C\colon \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is realized by the same osculating collineation.

Let $C\colon \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. Suppose that (5.13) and (5.14) hold. The osculating collineation is (6.1). We shall say that C is weakly singular, (singular, strongly singular) if $C_i : (A_i) \to (\tilde{A}_i)$ is a projective deformation of order one, (two, three) and it is possible to realize the deformations C_i by the same collineation.

There is

$$K\tilde{A}_i = \varrho_i A_i, \qquad K\,\nabla\tilde{A}_i = \varrho_i\,\nabla A_i + (\varrho_i \tau_{ii} + c_{i+n,i}\,\mathrm{d}u_i)\,A_i +$$
$$+ \sum_{r=1}^{n-1} \mathrm{d}u_i(\varrho_{i+r} h_{i,i+r} - \varrho_i h_{i,i+r})\,A_{i+r},$$
$$i = 1, 2, \ldots, n.$$

The correspondence C to be weakly singular we obtain

$$\varrho_s \tilde{h}_{is} = \varrho_i h_{is}. \tag{6.3}$$

Proposition 5. Let $C: \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. C is weakly singular if and only if (6.3) holds. Further we have

$$\begin{aligned} K\nabla^2 \tilde{A}_i = {} & \varrho_i \nabla^2 A_i + 2(\varrho_i \tau_{ii} + c_{i+n,i} \,\mathrm{d}u_i) A_i + [\varrho_i \,\mathrm{d}\tau_{ii} + \varrho_i \,\mathrm{d}u_i \tau_{i+n,i} + \\ & + \varrho_i \tau_{ii}^2 + (\tau_{ii} + \tilde{\omega}_{i+n,i+n} - \omega_{ii}) \,\mathrm{d}u_i c_{i+n,i} + \mathrm{d}^2 u_i c_{i+n,i}] A_i + \\ & + \sum_{r=1}^{n-1} \{\omega_{i,i+r} \left[\varrho_{i+r} \mathrm{d}\left(\frac{\varrho_i}{\varrho_{i+r}}\right) + \varrho_i(\tau_{i+n,i+n} - \tau_{i+r,i+r})\right] + \\ & + \varrho_{i+r} \,\mathrm{d}u_i \tilde{\omega}_{i+n,i+r} - \varrho_i \,\mathrm{d}u_i \omega_{i+n,i+r}\} A_{i+r}. \end{aligned}$$

The correspondence C to be singular then we get (6.3) and

$$\varrho_s \tilde{\omega}_{i+n,s} = \varrho_i \omega_{i+n,s}, \qquad \tau_{i+n,i+n} - \tau_{ss} = \varrho_i^{-1} \varrho_s \frac{\mathrm{d}\varrho_i}{\mathrm{d}\varrho_s}. \tag{6.4}$$

Proposition 6. Let $C: \mathscr{L} \to \tilde{\mathscr{L}}$ be a weakly singular projective deformation. C is singular if and only if (6.4) holds.

Carrying out similar consideration for the correspondence $C: \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ and using substitution (3.3) we get.

Proposition 7. Let $C: \mathscr{L} \to \tilde{\mathscr{L}}$ be a weakly singular (singular) projective deformation. Then the correspondence $C: \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is also weakly singular (singular).

To solve the problem of the strongly singular projective deformation let us simplify at first the osculating collineation. By a suitable choice of local frames we obtain

$$K\tilde{A}_i = A_i, \qquad i = 1, 2, \ldots, 2n.$$

In this case we have

$$\varrho = 1, \qquad \tilde{a}^i_{i+n,i+n} - \tilde{a}^i_{ii} = a^i_{i+n,i+n} - a^i_{ii}$$

and the equations (5.14) and (5.15). If C is a singular projective deformation then

$$\tau_{is} = \tau_{i+n,s} = \tau_{ii} = \tau_{i+n,i+n} = \tau_{i+n,s+n} = 0.$$

Further

$$K\tilde{A}_i = A_i, \qquad K\nabla\tilde{A}_i = \nabla A_i, \qquad K\nabla^2\tilde{A}_i = \nabla^2 A_i + \mathrm{d}u_i \tau_{i+n,i} A_i,$$

$$K\nabla^3\tilde{A}_i = \nabla^3 A_i + 3\,\mathrm{d}u_i \tau_{i+n,i} \nabla A_i + (.)\, A_i +$$

$$+ \sum_{r=1}^{n-1} \{-2\,\mathrm{d}u_i \omega_{i,i+r} \tau_{i+n,i} + (\omega_{i,i+r} \,\mathrm{d}u_{i+r} + \omega_{i+n,i+n+r} \,\mathrm{d}u_i) \tau_{i+n+r}\} A_{i+r}.$$

As the equations $\omega_{i,i+r} \,\mathrm{d}u_{i+r} + \omega_{i+n,i+n+r} \,\mathrm{d}u_i = 0$ are not satisfied identically all the forms $\tau_{ij} = 0$ $(i, j = 1, 2, \ldots, 2n)$.

Proposition 8. If $C: \mathscr{L} \to \tilde{\mathscr{L}}$ is a strongly singular projective deformation, pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are identical.

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J. Krejzlík
602 00 Brno, Bayerova 33
Czechoslovakia