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Archivum Mathematicum, Vol. 15 (1979), No. 1, 37--46

Persistent URL: http://dml.cz/dmlcz/107022

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XV: 37—46, 1979

LINEAR INTEGRAL INEQUALITIES

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(Received May 2, 1978)

Since 1918-when Gronwall [4] published an estimate for any continuous non-negative function $x(t)$ defined in $[a, a + h]$ and satisfying the inequality

$$x(t) \leqq \alpha + \int_a^t x(s)\,ds, \qquad \beta > 0$$

a lot of papers have appeared generalizing this result. Many of them have been mentioned or reproved in the monograph by Филатов – Шарова [25].

The inequalities of Gronwall's type are special cases of an inequality for a monotone operator $\mathscr{L}$, $x \leqq f + \mathscr{L}x$ in which the solution of $y = f + \mathscr{L}y$ provides an upper bound for x (Walter [13]). This result has been proved, for Volterra integral inequalities, by Sato – Iwasaki [11]. In the linear case, i.e. if a function $x(t)$ satisfies an inequality

$$x(t) \leqq f(t) + \int_a^t k(t, s)\,x(s)\,ds, \tag{1}$$

an upper bound for $x(t)$ is given by a Neumann series – the solution of the equation – $y(t) = f(t) + \int_a^t k(t, s)\,y(s)\,ds$ – which represents the best upper bound of $x(t)$. It is well known that if $k(t, s) \equiv g(t)\,h(s)$, the solution of this equation can be written in a closed form (see Chu – Metcalf [3]). Putting down the requirement of receiving the best possible bound, we can replace the inequality (1) by another one providing a simple bound in general case, too.

The purpose of this paper is to treat inequalities containing multiple integrals; the results encompass the major part of papers quoted below.

There is an elegant generalization of Gronwall's inequality to systems of n linear inequalities in m variables published by Chandra – Davies [2].

For a better orientation we establish a survey of typical results concerning the linear integral inequalities for functions of one variable. In this survey, all functions are supposed to be continuous and nonnegative on the interval $I = [a, T)$, $T \leqq \infty$

or on the Cartesian product $I x \dots x I$; all constants are supposed to be nonnegative.

Gronwall (1918) [4]

$$x(t) \leqq \int_a^t [\alpha x(s) + \beta]\, ds, \qquad t \in [a, a + h] \Rightarrow x(t) \leqq \beta h \exp\{dh\}, \quad t \in [a, a + h].$$

Reid (1930) [10]

$$x(t) \leqq f(t) + \int_a^t [h(s)\, x(s) + g(s)]\, ds,$$

$$f(t) \leqq M \Rightarrow x(t) \leqq [\int_a^t g(s)\, ds + M] \exp \int_a^t h(s)\, ds$$

Харламов (1955) [26]

$$x(t) \leqq f(t) + g(t) \int_a^t h(s)\, x(s)\, ds \Rightarrow x(t) \leqq f(t) + $$

$$+ g(t) \int_a^t f(s)\, h(s) \exp \int_s^t g(r)\, h(r)\, dr\, ds$$

Сарбарлы (1965) [23]

$$x(t) \leqq f(t) + g(t) \int_t^\infty h(s)\, x(s)\, ds;$$

$$\int^\infty f h, \int^\infty g h < \infty \Rightarrow x(t) \leqq f(t) + g(t) \int_t^\infty f(s)\, h(s) \exp \int_s^t g(r)\, h(r)\, dr\, ds$$

Willet (1965) [14]

$$x(t) \leqq f(t) + \sum_{i=1}^n g_i(t) \int_a^t h_i(s)\, x(s)\, ds \Rightarrow x(t) \leqq E_n f(t),$$

$$E_i = D_i D_{i-1} \dots D_0, \qquad D_0 f = f, \qquad D_j f = f + (E_{j-1} g_j)(\exp \int_a^t h_j E_{j-1} g_j) \int_a^t h_j f.$$

Мовлянкулов – Филатов (1972) [21]

$$x(t) \leqq f(t) + g(t) \int_a^t k(t, s)\, x(s)\, ds \Rightarrow x(t) \leqq F(t) \exp\{G(t) \int_a^t K(t, s)\, ds\},$$

$$F(t) = \sup_{a \leqq r \leqq t} f(r), \qquad G(t) = \sup_{a \leqq r \leqq t} g(r), \qquad K(t, s) = \sup_{a \leqq r \leqq t} k(r, s)$$

Филатов – Шарова (1976) [25]

$$x(t) \leqq c + \int_a^t k(t, s)\, x(s)\, ds, \qquad \frac{\partial k}{\partial t} \in C^0 \Rightarrow x(t) \leqq c \exp \int_a^t \left(k(s, s) + \int_a^s \frac{\partial k}{\partial s}(s, \sigma)\, d\sigma\right) ds$$

Pachpatte (1975) [7]

$$x(t) \leqq f(t) + \int_a^t g(s)\, x(s)\, ds + \int_a^t g(s) \int_a^s h(r)\, x(r)\, dr\, ds,$$

$$f(t) \text{ nondecreasing} \Rightarrow x(t) \leqq f(t)\, [1 + \int_a^t g(s) \exp \int_a^s [g(r) + h(r)]\, dr\, ds$$

Ведь (1965) [18]

$$x(t) \leqq c_1 + g(t)\{c_2 + c_3 \int_a^t [h(s)\,x(s) + \int_a^s k(s, r)\,x(r)\,\mathrm{d}r]\,\mathrm{d}s\} \Rightarrow$$

$$\Rightarrow x(t) \leqq c_1 + g(t)[c_2 \exp\{c_3 \int_a^t (g(s)\,h(s) + \int_a^s k(s, r)\,g(r)\,\mathrm{d}r)\,\mathrm{d}s\} +$$

$$+ c_1 c_3 \int_a^t (h(s) + \int_a^s k(s, r)\,\mathrm{d}r \exp\{c_3 \int_s^t (h(r)\,g(r) + \int_a^r k(r, u)\,g(u)\,\mathrm{d}u)\,\mathrm{d}r\}\,\mathrm{d}s]$$

Vencková (1977) [12]

$$x(t) \leqq f(t) + \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\,x(t_0)\,\mathrm{d}t_0 \dots \mathrm{d}t_{2n-1} \Rightarrow$$

$$\Rightarrow x(t) \leqq f(t) \exp \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\,\mathrm{d}t_0 \dots \mathrm{d}t_{2n-1}.$$

In the sequel let R^n denote the real n-dimensional, euclidian space of elements $(t_1, \dots, t_n)$. As usual, we shall use R instead of R^1. Let E be a set in R^{n+1}. We shal mean by $C(E, R^m)$ the class of continuous mappings from E to R^m. If f is a member of this class, one writes $f \in C(E, R^m)$. Another accepted significations: $R^+ = [0, \infty)$, $R^- = (-\infty, 0]$, $J = [t_0, T)$, $T \leqq \infty$. Finally let us define $D_{k+1} = \{(t, t_1, \dots, t_k) \in \in R^{k+1}, t_0 \leqq t_k \leqq \dots \leqq t_1 \leqq t < T\}$.

Theorem 1. *Let* $v_k(t, t_1, \dots, t_k)$, $\dfrac{\partial v_k}{\partial t}(t, t_1, \dots, t_k) \in C(D^{k+1}, R^+)$, $k = 1, \dots, n$. *For* $u(t) \in C(J, R^+)$ *let us define*

$$R_2[u](t) = \int_{t_0}^t v_2(t, t, t_2)\,u(t_2)\,\mathrm{d}t_2,$$

$$R_k[u](t) = \int_{t_0}^t \int_{t_0}^{t_2} \dots \int_{t_0}^{t_{k-1}} v_k(t, t, t_2, \dots, t_k)\,u(t_k)\,\mathrm{d}t_k \dots \mathrm{d}t_2, \qquad k = 3, \dots, n$$

$$R[u](t) = v_1(t, t)\,u(t) + \sum_{k=2}^n R_k[u](t),$$

$$S_1[u](t) = \int_{t_0}^t \frac{\partial v_1}{\partial t}(t, t_1)\,u(t_1)\,\mathrm{d}t_1,$$

$$S_k[u](t) = \int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{k-1}} \frac{\partial v_k}{\partial t}(t, t_1, \dots, t_k)\,u(t_k)\,\mathrm{d}t_k \dots \mathrm{d}t_1, \qquad k = 2, \dots, n$$

$$S[u](t) = \sum_{k=1}^n S_k[u](t).$$

Suppose $a(t), b(t), x(t) \in C(J, R^+)$ and

(2)

$$x(t) \leqq a(t) + b(t) \Big[\int_a^t v_1(t, t_1)\, x(t_1)\, dt_1 + \ldots + \int_{t_0}^t \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{n-1}} v_n(t, t_1, \ldots, t_n)\, x(t_n)\, dt_n \ldots dt_1 .$$

Then

$$x(t) \leqq a(t) + b(t) \int_{t_0}^t (R[a] + S[a])(s) \exp \int_s^t (R[b] + S[b])(r)\, dr\, ds. \qquad (3)$$

Proof. Denote

$$v(t) = \int_{t_0}^t v_1(t, t_1)\, x(t_1)\, dt_1 + \ldots + \int_{t_0}^t \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{n-1}} v_n(t, t_1, \ldots, t_n)\, x(t_n)\, dt_n \ldots dt_1 .$$

Then $x(t) \leqq a(t) + b(t)\, v(t)$ and $v(t)$ is nonnegative and nondecreasing. Since $u_1(t)$, $u_2(t) \in C(J, R^+)$, $u_1(t) \leqq u_2(t)$ imply $R[u_1](t) \leqq R[u_2](t)$ and $S[u_1](t) \leqq S[u_2](t)$, we obtain the following bound for the derivative $v'(t)$:

$$v'(t) = R[x](t) + S[x](t) \leqq R[a + bv](t) + S[a + bv](t) =$$
$$= (R + S)[a](t) + (R + S)[bv](t) \leqq (R + S)[b](t)\, v(t) + (R + S)[a](t).$$

Multiplying this inequality by $\exp \{ -\int_{t_0}^t (R + S)[b](s)\, ds \}$, it may be written in the form

$$\Big[v(t) \exp \{ -\int_{t_0}^t (R + S)[b](s)\, ds \} \Big]' \leqq (R + S)[a](t) \exp \{ -\int_{t_0}^t (R + S)[b](s)\, ds \}.$$

Integrating from t_0 to t we receive with respect to $v(t_0) = 0$

$$v(t) \exp \{ -\int_{t_0}^t (R + S)[b](s)\, ds \} \leqq \int_{t_0}^t (R + S)[a](s) \exp \{ -\int_{t_0}^s (R + S)[b](r)\, dr \}\, ds,$$

and,

$$v(t) \leqq \int_{t_0}^t (R + S)[a](s) \exp \int_s^t (R + S)[b](r)\, dr\, ds.$$

Since $x(t) \leqq a(t) + b(t)\, v(t)$, the proof is complete. Th. 1 contains the results from the papers [1], [4], [5], [6], [10], [16]–[20], [22], [24]–[28] as special cases.

Theorem 2. *Let $v_k(t, t_1, \ldots, t_k)$, $V_k(t, t_1, \ldots, t_k) \in C(D^{k+1}, R^+)$ and let V_k be nondecreasing in t,*

$$V_k(t, t_1, \ldots, t_k) \geqq \max_{t_k \leqq s \leqq t} v_k(s, t_1, \ldots, t_k), \qquad k = 1, \ldots, n.$$

Let $\hat{R}[u](t)$ be defined in the same way as $R[u](t)$ in Th. 1 replacing there v_k by V_k.

Then the inequality (2) *implies that*

(4) $$x(t) \leqq a(t) + b(t) \int_{t_0}^{t} \hat{R}[a](s) \exp \int_{s}^{t} \hat{R}[b](r)\,dr\,ds, \qquad t \in [t_0, T).$$

Proof. It is seen from (2) that for each $t \in [t_0, h]$, $h < T$ the following inequality is valid

$$x(t) \leqq a(t) + b(t)\Big[\int_{t_0}^{t} V_1(h, t_1)\, x(t_1)\,dt_1 + \ldots$$
$$+ \int_{t_0}^{t}\int_{t_0}^{t_1} \cdots \int_{t_0}^{t_n} V_n(h, t_1, \ldots, t_n)\, x(t_n)\,dt_n \ldots dt_1\Big].$$

Since $\dfrac{\partial V_k}{\partial t} \equiv 0$ for $t \in [t_0, h]$ and $k = 1, \ldots, n$, Th. 1 forces the validity of (4) for $t \in [t_0, h]$, especially for $t = h$. Thus the theorem is proved since h may be chosen arbitrarily in $[t_0, T)$.

Note. Let $A(t)$, $B(t)$ be continuous, positive and nondecreasing functions, $A(t) \geqq \max_{t_0 \leqq s \leqq t} a(s)$, $B(t) \geqq \max_{t_0 \leqq s \leqq t} b(s)$. Then the inequality (4) can be simplified to the following one

$$x(t) \leqq A(t) \exp\{B(t) \int_{t_0}^{t} \hat{R}[1](s)\,ds\}.$$

In fact, in view of (4) we have

$$x(t) \leqq A(t) + B(t) \int_{t_0}^{t} \hat{R}[A](s) \exp \int_{s}^{t} \hat{R}[B](r)\,dr\,ds \leqq$$
$$\leqq A(t) + B(t) \int_{t_0}^{t} A(t)\hat{R}[1](s) \exp \int_{s}^{t} B(t)\hat{R}[1](r)\,dr\,ds =$$
$$= A(t)\big[1 - \exp\{B(t) \int_{s}^{t} \hat{R}[1]\,ds\}\big|_{s=t_0}^{s=t}\big] =$$
$$= A(t) \exp\{B(t) \int_{t_0}^{t} \hat{R}[1](s)\,ds\}.$$

For $n = 1$ see [21].

Theorem 3. *Let* $v_k(t, t_1, \ldots, t_k) \in C(D^{k+1}, R^+)$, $\dfrac{\partial v_k}{\partial t}(t, t_1, \ldots, t_k) \in C(D^{k+1}, R^-)$. *Suppose*

$$\int_{t_0}^{\infty}\int_{t_1}^{\infty} \cdots \int_{t_{k-1}}^{\infty} v_k(t, t_1, \ldots, t_k)\,dt_k \ldots dt_1 < \infty,$$
$$\int_{t_0}^{\infty}\int_{t_1}^{\infty} \cdots \int_{t_{k-1}}^{\infty} \frac{\partial v_k}{\partial t}(t, t_1, \ldots, t_k)\,dt_k \ldots dt_1 > -\infty$$

and define for admissible $u(t) \in C(J, R^+)$

$$P_2[u](t) = \int_t^\infty v_2(t, t, t_2)\, u(t_2)\, \mathrm{d}t_2,$$

$$P_k[u](t) = \int_t^\infty \int_{t_2}^\infty \dots \int_{t_{k-1}}^\infty v_k(t, t, t_2, \dots, t_k)\, u(t_k)\, \mathrm{d}t_k \dots \mathrm{d}t_2, \qquad k = 3, \dots, n,$$

$$P[u](t) = v_1(t, t)\, u(t) + \sum_{k=2}^n P_k[u](t),$$

$$Q_1[u](t) = \int_t^\infty \frac{\partial v_1}{\partial t}(t, t_1)\, u(t_1)\, \mathrm{d}t_1,$$

$$Q_k[u](t) = \int_t^\infty \int_{t_1}^\infty \dots \int_{t_{k-1}}^\infty \frac{\partial v_k}{\partial t}(t, t_1, \dots, t_k)\, u(t_k)\, \mathrm{d}t_k \dots \mathrm{d}t_1, \qquad k = 2, \dots, n,$$

$$Q[u](t) = \sum_{k=1}^n Q_k[u](t).$$

Let $a(t), b(t), x(t) \in C(J, R^+)$,

$$\int_t^\infty (P - Q)[a](s)\, \mathrm{d}s < \infty, \qquad \int_t^\infty (P - Q)[b](s)\, \mathrm{d}s < \infty$$

and

$$x(t) \leqq a(t) + b(t)\Big[\int_t^\infty v_1(t, t_1)\, x(t_1)\, \mathrm{d}t_1 + \dots + \int_t^\infty \int_{t_1}^\infty \dots \int_{t_{n-1}}^\infty v_n(t, t_1, \dots, t_n)\, x(t_n)\, \mathrm{d}t_n \dots \mathrm{d}t_1\Big].$$

Then

$$x(t) \leqq a(t) + b(t) \int_t^\infty (P - Q)[a](s) \exp \int_s^t (Q - P)[b](r)\, \mathrm{d}r\, \mathrm{d}s.$$

Proof. We shall proceed in the same manner as in the proof of Th. 1. Let

$$v(t) = \int_t^\infty v_1(t, t_1)\, x(t_1)\, \mathrm{d}t_1 + \dots + \int_t^\infty \int_{t_1}^\infty \dots \int_{t_{n-1}}^\infty v_n(t, t_1, \dots, t_n)\, x(t_n)\, \mathrm{d}t_n \dots \mathrm{d}t_1.$$

Then

$$v'(t) = -P[x](t) + Q[x](t) \geqq -P[a + bv](t) + Q[a + bv](t) = (Q - P)[a](t) + (Q - P)[bv](t).$$

Since $v(t)$ is nonnegative, nonincreasing and $(Q - P)[u_1](t) \geqq (Q - P)[u_2](t$ $u_1(t) \leqq u_2(t)$, it is $(Q - P)[bv](t) \geqq (Q - P)[b](t)\, v(t)$. Hence

$$v'(t) \geqq (Q - P)[b](t)\, v(t) + (Q - P)[a](t)$$

and

$$[v(t)\exp\int_t^\infty (Q-P)[b](s)\,\mathrm{d}s]' \geqq (Q-P)[a](t)\exp\int_t^\infty (Q-P)[b](s)\,\mathrm{d}s.$$

Taking in account that $v(\infty) = 0$, we receive from this inequality

$$v(t) \leqq \int_t^\infty (P-Q)[a](s)\exp\int_s^t (Q-P)[b](r)\,\mathrm{d}r\,\mathrm{d}s$$

and the theorem is proved.

For $n = 1$ this theorem has been proved by Сарбарлы [23].

In the special case $a(t) \equiv c \in R^+$, $b(t) \equiv 1$ and $v_k(t, t_1, \dots, t_k) = f_1(t_1)\dots f_k(t_k)$, where $f_1, \dots, f_n \in C^0(J, R^+)$, Th. 1 gives for $x(t)$ satisfying the inequality (2) the estimate

$$x(t) \leqq c\exp\{\int_{t_0}^t f_1(t_1)\,\mathrm{d}t_1 + \dots + \int_{t_0}^t f_1(t_1)\int_{t_0}^{t_1} f_2(t_2)\dots\int_{t_0}^{t_{n-1}} f_n(t_n)\,\mathrm{d}t_n\dots\mathrm{d}t_1\}.$$

This result can be sharpened by using another method of the proof (see [7] for $n = 2$).

Theorem 4. *Let $f_1, \dots, f_n$ $C^0(J, R^+)$ and let*

$$x(t) \leqq c + \int_{t_0}^t f_1(t_1)\,x(t_1)\,\mathrm{d}t_1 + \dots + \int_{t_0}^t f_1(t_1)\int_{t_0}^{t_1} f_2(t_2)\dots\int_{t_0}^{t_{n-1}} f_n(t_n)\,x(t_n)\,\mathrm{d}t_n\dots\mathrm{d}t_1$$

on J.

Then

$$x(t) \leqq c[1 + \int_{t_0}^t f_1(t_1)\,\mathrm{d}t_1 + \int_{t_0}^t f_1(t_1)\int_{t_0}^{t_1} f_2(t_2)\exp\int_{t_0}^{t_2} f_2(t_3)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 + \tag{5}$$

$$+ \dots + \int_{t_0}^t f_1(t_1)\dots\int_{t_0}^{t_{n-2}} f_{n-1}(t_{n-1})\exp\int_{t_0}^{t_{n-1}} [f_{n-1}(t_n) + f_n(t_n)]\,\mathrm{d}t_n\dots\mathrm{d}t_1.$$

Proof. Let us define for any $u(t) \in C(J, R^+)$

$$L_n u(t) = u(t) + \int_{t_0}^t f_n(s)\,u(s)\,\mathrm{d}s,$$

$$L_k u(t) = u(t) + \int_{t_0}^t f_k(s)\,L_{k+1}u(s)\,\mathrm{d}s, \qquad k = n-1, \dots, 2,$$

$$L_1 u(t) = c + \int_{t_0}^t f_1(s)\,L_2 u(s)\,\mathrm{d}s$$

and let us denote

$$u_1(t) = L_1 x(t), \qquad u_k(t) = L_k u_{k-1}(t), \qquad k = 2, \dots, n$$

so that

$$u_k(t) \leqq u_{k+1}(t).$$

The functions $u_i(t)$ satisfy the following system of differential inequalities

$$\begin{aligned}
&u_1' \leqq f_1 u_2, \\
(6) \qquad &u_i' \leqq f_1 u_2 + \dots + f_i u_{i+1}, \qquad i = 2, \dots, n-1, \\
&u_n' \leqq f_1 u_2 + \dots + f_{n-2} u_{n-1} + (f_{n-1} + f_n) u_n.
\end{aligned}$$

In fact, since $x(t) \leqq u_1(t)$, it holds $u_1' = f_1 L_2 x \leqq f_1 L_2 u_1 = f_1 u_2$. Assuming that the first $k-1$ ($k \leqq n-1$) inequalities are valid, we have

$$u_k' = u_{k-1}' + f_k L_{k+1} u_{k-1} \leqq f_1 u_2 + \dots + f_{k-1} u_k + f_k L_{k+1} u_k = \\ = f_1 u_2 + \dots + f_k u_{k+1}.$$

For $k = n$ it is

$$u_n' = u_{n-1}' + f_n u_{n-1} \leqq f_1 u_2 + \dots + f_{n-1} u_n + f_n u_n$$

and the statement is proved. From the latter inequality it follows

$$u_n' \leqq \sum_{i=1}^{n} f_i u_n,$$

so that

$$(7) \qquad u_n(t) \leqq c \exp \int_{t_0}^{t} \sum_{i=1}^{n} f_i(s)\, ds.$$

If we know the estimate for u_{i+1}, we receive from the inequality (6) written in the form

$$u_i' \leqq \sum_{k=1}^{i-1} f_k u_i + f_i u_{i+1},$$

that

$$(8) \quad u_i(t) \leqq \exp \{ \sum_{k=1}^{i-1} \int_{t_0}^{t} f_k(s)\, ds \} [c + \int_{t_0}^{t} \exp \{ - \sum_{k=1}^{i-1} \int_{t_0}^{s} f_k(r)\, dr\, f_i(s)\, u_{i+1}(s)\, ds],$$

for $i = n-1, \dots, 2$, and,

$$u(t) \leqq c + \int_{t_0}^{t} f_1(s)\, u_2(s)\, ds.$$

Hence starting from (7) and using (8) for $i = n-1, \dots, k$ we receive

$$u_{n-k}(t) \leqq c \exp \{ \sum_{i=1}^{n-k-1} \int_{t_0}^{t} f_i(s)\, ds \} [1 + \int_{t_0}^{t} f_{n-k}(t_{n-k}) \times \\ \times \exp \int_{t_0}^{t_{n-k}} f_{n-k+1}(\tau)\, d\tau \langle 1 + \dots + \{1 + \int_{t_0}^{t_{n-2}} f_{n-1}(t_{n-1}) \times \\ \times \exp \{ \int_{t_0}^{t_{n-1}} [f_{n-1}(\tau) + f_n(\tau)]\, d\tau\, dt_{n-1} \} \dots \rangle\, dt_{n-k}]$$

and

$$u(t) \leqq c[1 + \int_{t_0}^{t} f_1(t_1) \langle 1 + \int_{t_0}^{t_1} f_2(t_2) \exp \int_{t_0}^{t_2} f_2(\tau)\, d\tau \{1 + \dots [1 + \int_{t_0}^{t_{n-2}} f_{n-1}(t_{n-1}) \times$$
$$\times \exp \int_{t_0}^{t_{n-1}} (f_{n-1}(\tau) + f_n(\tau))\, d\tau]\, dt_{n-1} \dots \}\, dt_2 \rangle\, dt_1]$$

which is equivalent to (5). The proof is complete.

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