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ON TOPOLOGIES CONVEXLY COMPATIBLE WITH THE ORDERING

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Sets with both ordering and topology have been investigated by several authors (e.g. [1]–[3], [5], [8], [10]–[12]). In some papers the topology is derived from an ordering, in other ones the topology is in a certain sense compatible with an ordering.

In this note two types of compatibility of a topology with an ordering are introduced (convex compatibility and convex weak compatibility). Under a topology we understand here a topology in the sense of Čech. Our conditions of compatibility are analogical to those delt with in papers [1], [2], [11] for topologies in Bourbaki's sense.

Let $(A, \leqq)$ be a fixed partially ordered set. The system of all topologies on A will be denoted by $\mathscr{T}(A)$, the symbols $\alpha(A, \leqq)$ and $\beta(A, \leqq)$ will be used for the system of all topologies on A convexly compatible and convexly weakly compatible with the ordering $\leqq$, respectively.

In the first section a formula for the number of topologies on a finite set with the trivial ordering is given. Conditions, under which any of the equalities $\alpha(A, \leqq) = \beta(A, \leqq)$, $\alpha(A, \leqq) = \mathscr{T}(A)$, $\beta(A, \leqq) = \mathscr{T}(A)$ holds, are found in the second section. In the section 3 there are described all orderings $\preceq$ on A such that $\alpha(A, \leqq) = \alpha(A, \preceq)$ and $\beta(A, \leqq) = \beta(A, \preceq)$.

The system of all subsets of a set F is denoted by 2^P, for the cardinality of P we use the symbol card P.

Let P be a given set. A mapping $u : 2^P \to 2^P$ is said to be a *topology* on P, if the following three axioms are satisfied:

(1) $u\emptyset = \emptyset$,

(2) $M \subset P \Rightarrow M \subset uM$,

(3) $M_1 \subset M_2 \subset P \Rightarrow uM_1 \subset uM_2$.

If u is a topology on P, the pair (P, u) is called a *topological space*. The system of all topologies on P is denoted by $\mathscr{T}(P)$.

A set $O \subset P$ is said to be a *neighborhood* of a point $x \in P$ in the space (P, u), if $x \notin u(P - O)$. The notation $\mathscr{D}_u(x)$ is used for the system of all neighborhoods of x in (P, u).

We shall often use the following statement (A), which enables us to introduce a topology into a set P (cf. [7], 4.1.).

(A) 1. *Let (P, u) be a topological space, $x \in P$. The system $\mathscr{D}_u(x)$ has the following properties:*

(i) $\mathscr{D}_u(x) \neq \emptyset$,

(ii) $O \in \mathscr{D}_u(x) \Rightarrow x \in O$,

(iii) $O \subset O_1, O \in \mathscr{D}_u(x) \Rightarrow O_1 \in \mathscr{D}_u(x)$.

2. *Let P be an arbitrary set and let $\mathscr{D}(x)$ be a nonvoid family of subsets of P, assigned to each point $x \in P$, satisfying:*

(1) $O \in \mathscr{D}(x) \Rightarrow x \in O$,

(2) $O \subset O_1, O \in \mathscr{D}(x) \Rightarrow O_1 \in \mathscr{D}(x)$.

If we define a mapping $u : 2^P \to 2^P$ in such a manner that $x \in uM$ ($M \subset P$) iff $P - M \notin \mathscr{D}(x)$, then u is a topology on P and for each $x \in P$ it is $\mathscr{D}_u(x) = \mathscr{D}(x)$.

1.

Theorem. *Let n be a positive integer and let P be a set with* card $P = n$. *The number of all topologies on P is s^n, where s is the number of antichains of the Boolean algebra of all subsets of a set of the cardinality $n - 1$.*

Proof. By (A) each topology on P is uniquely determined by the set $\{\mathscr{D}(x) : x \in P\}$, where $\mathscr{D}(x)$ is a nonempty system of subsets of P fulfilling conditions (1), (2) from (A). Let x be a fixed element from P and let $S = S(x)$ be the number of nonempty systems of subsets of P fulfilling (1), (2). Evidently S does not depend on the choice of $x \in P$, thus the number of all topologies on P is S^n. We shall show that $S = s$. The partially ordered set of all subsets of $P = \{x = x_0, x_1, \ldots, x_{n-1}\}$, that contain x, is obviously isomorphic to the Boolean algebra of all subsets of the set $\{x_1, \ldots, x_{n-1}\}$. The system $\mathscr{D}(x)$ is determined by the set of its minimal elements. This set corresponds to an antichain of the Boolean algebra of all subsets of the set $\{x_1, \ldots, x_{n-1}\}$. Therefore $S = s$.

Remark. The problem of the determination of the number of antichains in the Boolean algebra of all subsets of a finite set was investigated by several authors (of., e.g., [6], [9]). In the paper [9] there is derived a formula for the number of all topologies on a finite set, but more complicated than the above one.

2.

2.1. Definition. Let $(A, \leqq)$ be a partially ordered set. A topology u on A will be said to be *convexly compatible with the ordering* $\leqq$, if it has the following property:

(α) If $a, b \in A$ and if U is a neighborhood of a with $b \notin U$, then there exists a convex neighborhood V of a such that $b \notin V$.

2.2. Definition. Let $(A, \leqq)$ be a partially ordered set. A topology u on A will be called *convexly weakly compatible with the ordering* $\leqq$, if it has the following property:

(β) If a and b are comparable elements of A and if U is a neighborhood of a with $b \notin U$, then there exists a convex neighborhood V of a such that $b \notin V$.

For an arbitrary fixed partially ordered set $(A, \leqq)$ let us denote $\alpha(A, \leqq)$ and $\beta(A, \leqq)$ the set of all topologies on A, which are convexly compatible and convexly weakly compatible with the ordering $\leqq$, respectively. Clearly, $\alpha(A, \leqq) \subset \beta(A, \subset)$.

The converse inclusion does not hold in general, as shown by the following theorem.

If X, Y are partially ordered sets, we denote by $X \oplus Y$ their ordinal sum (cf. [4]).

2.3. Theorem. *Let $(A, \leqq)$ be a partially ordered set. Then $\alpha(A, \leqq) = \beta(A, \leqq)$ if and only if one of the following conditions holds:*

(1) *Every element of A is maximal or minimal.*

(2) *It is $A = A_1 \oplus A_2 \oplus A_3$, where A_1, A_3 are antichains, A_2 is a nonempty chain (A_1, A_3 can be empty).*

Proof. Suppose that $(A, \leqq)$ satisfies (1) or (2). Take $u \in \beta(A, \leqq)$ and noncomparable elements $a, b \in A$ such that there exists a neighborhood $U \in \mathscr{D}_u(a)$ not containing b. Then b is maximal or minimal and hence it cannot belong to the convex hull $[U]$ of U, which is evidently a neighborhood of a. Therefore $u \in \alpha(A, \leqq)$.

Conversely, suppose that $\alpha(A, \leqq) = \beta(A, \leqq)$ and $(A, \leqq)$ is not a chain. Let a, b be noncomparable elements of A. We shall show that each of a, b is maximal or minimal. Define $\mathscr{D}(a) = \{A - \{b\}, A\}$, $\mathscr{D}(z) = \{A\}$ for every $z \in A$, $z \neq a$. The topology u such that $\mathscr{D}_u(y) = \mathscr{D}(y)$ for every $y \in A$ obviously belongs to $\beta(A, \leqq)$ and hence by assumption $u \in \alpha(A, \leqq)$. This implies that $A - \{b\}$ is a convex set, i.e. b is maximal or minimal. Analogously a is maximal or minimal. Denote A_1 and A_3 the set of all minimal and maximal elements of A, respectively. If $A_1 \cup A_3 = A$, we have (1). Assume $A_1 \cup A_3 \neq A$. Denote $A_2 = A - (A_1 \cup A_3)$ and pick any $c \in A_2$. Since c is neither maximal nor minimal, it is comparable with each element of A. Thus $c > x$ and $c < y$ for every $x \in A_1$ and $y \in A_3$. Further arbitrary two elements of A_2 are comparable. We conclude $A = A_1 \oplus A_2 \oplus A_3$.

The following theorem gives a necessary and sufficient condition under which each topology on a partially ordered set $(A, \leqq)$ is convexly compatible and convexly weakly compatible with the ordering $\leqq$, respectively.

2.4. Theorem. *Let* $(A, \leqq)$ *be a partially ordered set. The following conditions are equivalent:*

(i) $\alpha(A, \leqq) = \mathscr{T}(A)$.

(ii) $\beta(A, \leqq) = \mathscr{T}(A)$.

(iii) *Every element of A is maximal or minimal.*

Proof. Since $\alpha(A, \leqq) \subset \beta(A, \leqq)$, the condition (i) implies (ii). To prove that (ii) implies (iii), suppose that there exists an element $b \in A$ that is neither maximal nor minimal. Then there exist $a, x \in A$ such that $a < b < x$. Put $\mathscr{D}(a) = \{A - \{b\}, A\}$, $\mathscr{D}(z) = \{A\}$ for every $z \in A$, $z \neq a$. The topology u such that $\mathscr{D}_u(y) = \mathscr{D}(y)$ for each $y \in A$ obviously does not belong to $\beta(A, \leqq)$. Finally we shall prove that (iii) implies (i). Take a topology $u \in \mathscr{T}(A)$ and arbitrary elements $a, b \in A$ such that there exists $U \in \mathscr{D}_u(a)$ not containing b. By (iii), b does not belong to the convex hull $[U]$ of U. Hence $u \in \alpha(A, \leqq)$.

3.

In this section conditions for the validity of the relations $\alpha(A, \leqq) = \alpha(A, \preceq)$, $\beta(A, \leqq) = \beta(A, \preceq)$ are investigated, where $\leqq, \preceq$ are two partial orderings on A.

If M is a subset of A, then the convex hull of M in the partially ordered set $(A, \leqq)$ and $(A, \preceq)$ will be denoted by $[M]_{\leqq}$ and $[M]_{\preceq}$, respectively. We shall say that an element $x \in A$ lies between elements $a, b \in A$ in the partially ordered set $(A, \leqq)$, if either $a < x < b$ or $a > x > b$ holds. The relation of betweenness in $(A, \preceq)$ is defined analogously.

3.1. Theorem. *Let* $\leqq, \preceq$ *be two partial orderings on the set A. Then the following conditions are equivalent:*

(i) $\alpha(A, \leqq) \subset \alpha(A, \preceq)$.

(ii) *If a subset M of A is convex in* $(A, \leqq)$, *then M is convex in* $(A, \preceq)$ *as well.*

(iii) *If an element* $x \in A$ *lies between elements* $a, b \in A$ *in the partially ordered set* $(A, \preceq)$, *then the same holds in* $(A, \leqq)$.

(iv) $\beta(A, \leqq) \subset \beta(A, \preceq)$.

Proof. First we prove that the conditions (i) and (ii) are equivalent. Let $\alpha(A, \leqq) \subset$ $\subset \alpha(A, \preceq)$ and let M be an arbitrary convex subset of $(A, \leqq)$. If $M = \emptyset$, then M is obviously convex in $(A, \preceq)$, too. Thus we can suppose that $M \neq \emptyset$. Pick an arbitrary fixed element $a \in M$. Consider the topology u on A such that $\mathscr{D}_u(a) =$ $= \{O \subset A : M \subset O\}, \mathscr{D}_u(z) = \{A\}$ for each $z \in A, z \neq a$. Then evidently $u \in \alpha(A, \leqq)$ and consequently $u \in \alpha(A, \preceq)$. For an arbitrary element $b \in A - M$ there exists a neighborhood of a not containing b, hence there exists a set $X_b \in \mathscr{D}_u(a)$ convex in $(A, \preceq)$ such that $b \notin X_b$. Since $M \subset X_b$, we have $[M]_{\preceq} \subset X_b$ which shows that

$b \notin [M]_{\preceq}$. It follows $[M]_{\preceq} \subset M$. Hence M is convex also in $(A, \preceq)$. It is easy to see that (ii) implies (i).

Evidently the condition (iii) implies (ii). To verify the converse implication, suppose that $a \prec x \prec b$. By (*ii*) the set $[\{a, b\}]_{\leqq}$ is convex in $(A, \preceq)$. This together with $a, b \in [\{a, b\}]_{\leqq}$ yields that $x \in [\{a, b\}]_{\leqq}$. Since $x \in [\{a, b\}]_{\leqq} - \{a, b\}$, the elements a, b must be comparable in $(A, \leqq)$. Hence either $a < x < b$ or $a > x > b$.

Finally we prove the equivalence of the conditions (iii), (iv). Let the condition (iii) hold. Take an arbitrary topology $u \in \beta(A, \leqq)$ and elements $a, b \in A$ comparable in $(A, \preceq)$ such that there exists a neighborhood $U \in \mathscr{D}_u(a)$ not containing b. If b is maximal or minimal in $(A, \preceq)$, then $A - \{b\}$ is a neighborhood of a and $A - \{b\}$ is convex in $(A, \preceq)$. Hence we can suppose that b is neither maximal nor minimal in $(A, \preceq)$. Then there exist elements $c, d \in A$ such that $c \prec b \prec d$. If $a \prec b$, from $a \prec b \prec d$ by the condition (iii) we get either $a < b < d$ or $a > b > d$. Analogously, from $a \succ b$ we obtain that b lies between a, c in $(A, \leqq)$. Since $u \in \beta(A, \leqq)$, $U \in \mathscr{D}_u(a)$, $b \notin U$ and a, b are comparable in $(A, \leqq)$, there exists a neighborhood $V \in \mathscr{D}_u(a)$, convex in $(A, \leqq)$, not containing b. Evidently $[V]_{\preceq} \in \mathscr{D}_u(a)$, $[V]_{\preceq}$ is a convex set in $(A, \preceq)$. It remains to show that $b \notin [V]_{\preceq}$. Suppose that for some elements $x, y \in V$ $x \prec b \prec y$ holds. By the condition (iii) b lies between x, y in $(A, \leqq)$. Then $b \in [V]_{\leqq} =$ $=$ V, which is a contradiction. Conversely, let us suppose that (iv) holds. Pick elements $a, x, b \in A$ with $a \prec x \prec b$. Let u be a topology on A such that $\mathscr{D}_u(a) =$ $= \{O \subset A : [\{a, b\}]_{\leqq} \subset O\}$, $\mathscr{D}_u(z) = \{A\}$ for every $z \in A$, $z \neq a$. Then evidently $u \in \beta(A, \leqq)$ and hence $u \in \beta(A, \preceq)$. It is $x \in [\{a, b\}]_{\leqq}$. For, if this were false, then, since $a \prec x$, $[\{a, b\}]_{\leqq} \in \mathscr{D}_u(a)$ and $u \in \beta(A, \preceq)$, we should have $x \notin [[\{a, b\}]_{\leqq}]_{\preceq}$, contrary to $a \prec x \prec b$. According to $x \in [\{a, b\}]_{\leqq}$, the elements a, b are comparable in $(A, \leqq)$ and it is $a < x < b$ or $a > x > b$.

3.2. Corollary. *Let* $\leqq, \preceq$ *be two partial orderings on the set* A. *Then the following conditions are equivalent:*

(i*) $\alpha(A, \leqq) = \alpha(A, \preceq)$.

(ii*) *A subset* M *of* A *is convex in* $(A, \leqq)$ *if and only if it is convex in* $(A, \preceq)$.

(iii*) *An element* x *lies between elements* a, b *in* $(A, \leqq)$ *if and only if the same holds in* $(A, \preceq)$.

(iv*) $\beta(A, \leqq) = \beta(A, \preceq)$.

3.3. Theorem. *Let* $\leqq, \preceq$ *be two partial orderings on the set* A *with* card $A \geqq 3$, *where* $(A, \preceq)$ *is directed. Then each of the conditions* (i)–(iv) *of the theorem 3.1. is equivalent to the condition that the identical mapping* $\iota : (A, \preceq) \to (A, \leqq)$ *is isotone or antitone.*

Proof. If the identical mapping $\iota : (A, \preceq) \to (A, \leqq)$ is isotone or antitone, then obviously the condition (iii) is satisfied. Conversely, let us suppose that the equivalent conditions (i)–(iv) hold. First we shall prove that $a, b \in A$, $a \prec b$ implies $a < b$ or

$a > b$. Suppose that for some $a, b \in A$ with $a \prec b$ each element of $A - \{a, b\}$ is noncomparable in $(A, \preceq)$ with some of a, b. Pick $c \in A - \{a, b\}$. If c is noncomparable in $(A, \preceq)$ with a, then for arbitrary d_1 with $d_1 \prec a$, $d_1 \prec c$ we have $d_1 \prec a \prec b$, a contradiction. Analogously we get a contradiction assuming that c is noncomparable in $(A, \preceq)$ with b. Hence if $a \prec b$, then there exists an element $c \in A$ such that $c \prec a \prec b$ or $a \prec c \prec b$ or $a \prec b \prec c$. In each case we get by (iii) that a, b are comparable in $(A, \leqq)$.

Now suppose that for some $a, b, c, d \in A$ it is $a \prec b$, $a < b$, $c \prec d$, $c > d$. Let e and f be an arbitrary lower and upper bound of a, c and b, d in $(A, \preceq)$, respectively. Assume that $e = a$ and $f = b$, simultaneously. Then $a \preceq c \prec d \preceq b$ and since clearly either $a \neq c$ or $b \neq d$, we get by (iii) $a \leqq c < d \leqq b$ or $a \geqq c > d \geqq b$, a contradiction. Hence either $e \prec a$ or $b \prec f$. Using (iii) we obtain from $e \preceq a \prec \prec b \preceq f$ that $e < f$. On the other hand $e \preceq c \prec d \preceq f$ implies $e > f$. This contradiction shows that ι is either isotone or antitone.

3.4. Corollary. *Let $\leqq, \preceq$ be two partial orderings on the set A with* card $A \geqq 3$ *such that either $(A, \leqq)$ or $(A, \preceq)$ is a directed set. Then each of the conditions* (i*) – (iv*) *of 3.2. is equivalent to the condition that the orderings $\leqq, \preceq$ are identical or dual.*

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