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ON PARTITIONS WITH LIMITED SUMMANDS

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INTRODUCTION

Let us denote by N the set of all positive integers, $N_0 = N \cup \{0\}$. *Partition* of a number $n \in N$ into k summands is every k-tuple of positive integers $a_1, \ldots, a_k \in N$ such that

$$a_1 + a_2 + \ldots + a_k = n$$

and *it does not depend on the order of summands*. With respect to this fact it can be supposed that every partition is normally written in the form of non-increasing sequence

$$a_1 \geqq a_2 \geqq \cdots \geqq a_k.$$

Let $k, n \in N$ be arbitrary. Let us denote by $P(n, k)$ the number of partitions of the integer n into k summands. It is evident that for example

$$P(n, 1) = P(n, n) = 1 \qquad \text{for every } n \in N$$

and

$$P(n, k) = 0 \qquad \text{for } n < k.$$

It is well known (see e.g. [1], Chap. 4) that values $P(n, k)$ can be calculated from the recurrent formula

$$P(n, k) = \sum_{i=1}^{k} P(n - k, i). \tag{1}$$

Let us denote by $p(n)$ the number of all partitions of the integer n, i.e.

$$p(n) = \sum_{k=1}^{n} P(n, k). \tag{2}$$

Finally let us denote by $Q(n, k)$ the number of all partitions of the integer n into at most k summands, i.e.

$$Q(n, k) = \sum_{i=1}^{k} P(n, i). \tag{3}$$

Thus the formula (1) can be rewritten in the form

(4) $$P(n, k) = Q(n - k, k)$$

i.e.

(5) $$Q(n, k) = P(n + k, k).$$

With respect to (5) we define for every $k \in N$

(6) $$Q(0, k) = 1.$$

From the equations (2), (3) and (5) it follows

(7) $$p(n) = Q(n, n) = P(2n, n).$$

The following geometric interpretation of partitions is useful in many considerations. A partition $a_1 \geqq a_2 \geqq \dots \geqq a_k$ of integer $n \in N$ into k summands can be denoted by a point diagram in the plane so that on i-th row there are a_i points and each row starting from the same line parallel to the long edge of the page. Thus e.g. the partition

$$5 \geqq 3 \geqq 2 \geqq 2 \geqq 1 \geqq 1$$

of the number 14 is denoted by the following diagram

```
o o o o o
o o o
o o
o o
o
o
```

If we now rotate the diagram of a partition $a_1 \geqq a_2 \geqq \dots \geqq a_k$ so that the rows become columns we get the diagram of the partition $b_1 \geqq b_2 \geqq \dots \geqq b_j$ which is called the partition *conjugated* to the partition $a_1 \geqq a_2 \geqq \dots \geqq b_k$.

Thus e.g. the partition $6 \geqq 4 \geqq 2 \geqq 1 \geqq 1$ is conjugated to the partition $5 \geqq 3 \geqq 2 \geqq 2 \geqq 1 \geqq 1$.

§ 1. PARTITIONS WITH LIMITED SUMMANDS

Definition. Let $k, n, s \in N$ be arbitrary. Let us denote by $P_s(n, k)$ the number of all partitions of the integer n into k summands not exceeding the integer s.

Further we put

(8) $$Q_s(n, k) = \sum_{i=1}^{k} P_s(n, i),$$

(9) $$p_s(n) = Q_s(n, n).$$

Finally for arbitrary $k \in N$, $n, s \in N_0$ we define

$$Q_s(n, k) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0, s = 0. \end{cases} \tag{10}$$

Remark. Thus the integer $P_s(n, k)$ gives the number of solutions of the equation

$$n = x_1 + x_2 + \dots + x_k$$

such that

$$s \geqq x_1 \geqq x_2 \geqq \dots \geqq x_k \geqq 1, \qquad x_1, x_2, \dots, x_k \in N,$$

the integer $Q_s(n, k)$ gives the number of solutions of this equation such that

$$s \geqq x_1 \geqq x_2 \geqq \dots \geqq x_k \geqq 0, \qquad x_1, x_2, \dots, x_k \in N_0.$$

It is evident that there holds

Theorem 1. *Let $k, n, s \in N$ be arbitrary. Then*

$$P_s(n, k) = \begin{cases} 0 & \text{for } s < \frac{n}{k}, \\ P(n, k) & \text{for } s > n - k, \end{cases} \tag{11}$$

$$Q_s(n, k) = Q(n, k) \qquad \text{for } s \geqq n. \tag{12}$$

Theorem 2. *Let $k, n, s \in N$ be arbitrary. Then it holds*

$$Q_s(n, k) = Q_k(n, s). \tag{13}$$

Proof: Let A be the set of all the partitions of the integer n into at most k summands not exceeding s, B be the set of all partitions of the integer n into at most s summands not exceeding the integer k. Let us define the mapping $f : A \to B$ as follows: For $x \in A$, $f(x)$ is the partition conjugated to the partition x. Then evidently f is a bijection A onto B, thus the sets A, B have the same number of elements.

From Theorem 2 there follows a well known theorem (see e.g. [2], p. 268):

Theorem 3. *The number of partitions of the integer $k + n$ into k summands is equal to the number of partitions of the integer n into at most k summands and to the number of all the partitions of the integer n into summands not exceeding the integer k.*

Proof: It is necessary to prove that

$$P(n + k, k) = Q(n, k) = p_k(n).$$

The first equality has been mentioned in (5) yet, the second follows from (9), (12) and (13):

$$p_k(n) = Q_k(n, n) = Q_n(n, k) = Q(n, k).$$

The calculation of the values $P_s(n, k)$ enables the following reccurent formula.

Theorem 4. *Let $k, n, s \in N$ be arbitrary. Then*

$$P_{s+1}(n, k) = \sum_{i=1}^{k} P_s(n - k, i), \tag{14}$$

when

$$P_1(n, k) = \begin{cases} 1 & \text{for } n = k, \\ 0 & \text{for } n \neq k. \end{cases} \tag{15}$$

Proof: The equality (15) is evident. Thus let us prove the formula (14) which can be written in the form

$$P_{s+1}(n, k) = Q_s(n - k, k). \tag{16}$$

Let A be the set of all partitions of the integer $n - k$ into at most k summands not exceeding the number s, let B be the set of all partitions of the integer n into k summands not exceeding the number $s + 1$. Let us define the mapping $f : A \to B$ as follows: Let $x \in A$ be the following partition

$$a_1 \geqq a_2 \geqq \dots \geqq a_p, \qquad p \leqq k.$$

Then $f(x) \in B$ is the partition

$$b_1 \geqq b_2 \geqq \dots \geqq b_k$$

defined as follows

$$b_i = \begin{cases} a_i + 1 & \text{for } i = 1, \dots, p, \\ 1 & \text{for } i = p + 1, \dots, k. \end{cases}$$

Then evidently f is a bijection of A onto B so that (16) and consequently also (14) are valid.

From the equations (10) and (14) there follows immediately

Corollary. *For arbitrary $k \in N$ and arbitrary $n, s \in N_0$ it holds*

$$Q_s(n, k) = P_{s+1}(n + k, k). \tag{17}$$

The following relation is more suitable for practical calculation than the formula (14).

Theorem 5. *Let $k, n, s \in N$ be arbitrary. Then it holds*

$$P_{s+1}(n, k) = P_{s+1}(n-1, k-1) + P_s(n-k, k). \tag{18}$$

Proof: By (14) it holds

$$P_{s+1}(n, k) = \sum_{i=1}^{k} P_s(n-k, i) = \sum_{i=1}^{k-1} P_s(n-k, i) + P_s(n-k, k).$$

$P_3(n, k)$

k \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	2	1	1	0	0	0	0	0	0	0	0	0
3	0	0	1	1	2	2	2	1	1	0	0	0	0	0	0
4	0	0	0	1	1	2	2	3	2	2	1	1	0	0	0
5	0	0	0	0	1	1	2	2	3	3	3	2	2	1	1
6	0	0	0	0	0	1	1	2	2	3	3	4	3	3	2
7	0	0	0	0	0	0	1	1	2	2	3	3	4	4	4
8	0	0	0	0	0	0	0	1	1	2	2	3	3	4	4
9	0	0	0	0	0	0	0	0	1	1	2	2	3	3	4
10	0	0	0	0	0	0	0	0	0	1	1	2	2	3	3
$p_3(n)$	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27

$P_4(n, k)$

k \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	2	2	2	1	1	0	0	0	0	0	0	0
3	0	0	1	1	2	3	3	3	3	2	1	1	0	0	0
4	0	0	0	1	1	2	3	4	4	5	4	4	3	2	1
5	0	0	0	0	1	1	2	3	4	5	6	6	6	6	5
6	0	0	0	0	0	1	1	2	3	4	5	7	7	8	8
7	0	0	0	0	0	0	1	1	2	3	4	5	7	8	9
8	0	0	0	0	0	0	0	1	1	2	3	4	5	7	8
9	0	0	0	0	0	0	0	0	1	1	2	3	4	5	7
10	0	0	0	0	0	0	0	0	0	1	1	2	3	4	5
$p_4(n)$	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54

$P_{10}(n, k)$

k	n=1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
2	0	1	1	2	2	3	3	4	4	5	5	5	4	4	3
3	0	0	1	1	2	3	4	5	7	8	10	12	13	14	15
4	0	0	0	1	1	2	3	5	6	9	11	15	18	22	25
5	0	0	0	0	1	1	2	3	5	7	10	13	18	23	29
6	0	0	0	0	0	1	1	2	3	5	7	11	14	20	26
7	0	0	0	0	0	0	1	1	2	3	5	7	11	15	21
8	0	0	0	0	0	0	0	1	1	2	3	5	7	11	15
9	0	0	0	0	0	0	0	0	1	1	2	3	5	7	11
10	0	0	0	0	0	0	0	0	0	1	1	2	3	5	7
$p_{10}(n)$	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164

But $\sum_{i=1}^{k-1} P_s(n - k, i) = P_{s+1}(n - 1, k - 1)$ again by (14).

Now without any difficulties we can fill in tables of the values $P_s(n, k)$. If $s \in N$ is an arbitrary fixed integer, then the integer $P_s(i, j)$ is in the doint of intersection of i-th column and j-th row of the table $P_s(n, k)$.

Let us give an example of tables $P_3(n, k)$, $P_4(n, k)$ and $P_{10}(n, k)$.

It is evident that in any table $P_s(n, k)$ there is only finitely many non-zero integers in every row and in every column. It holds

Theorem 6. *Let $s \in N$ be arbitrary. Then in i-th column of the table $P_s(n, k)$ there is $i - \left[\frac{i}{s}\right] + 1$ of non-zero values and in i-th row of this table $i\,(s - 1) + 1$ of non-zero values. (At the same time for real x the symbol $[x]$ is the greatest integer smaller or equal to x.)*

Proof: I. Let $n, s \in N$ be arbitrary fixed numbers. It is evident that $P_s(n, k) \neq 0$ for $\left[\frac{n}{s}\right] \leqq k \leqq n$, i.e. for $n - \left[\frac{n}{s}\right] + 1$ values of the argument k.

II. Let $k, s \in N$ be arbitrary fixed numbers. Evidently $P_s(n, k) \neq 0$ for $n = k, k + 1, \ldots, k \,.\, s$, i.e. for $ks - (k - 1) = k(s - 1) + 1$ values of the argument n.

From here there follows

Corollary. *Let $k, s \in N$ be arbitrary. Then $P_s(n, k) \neq 0$ iff $n = k + i$, $i = 0, 1, \ldots, \ldots, k(s - 1)$.*

Since in every table $P_s(n, k)$ there exist in every row and in every column only finitely many non-zero values thus sums of all values of arbitrary column and arbitrary row i.e.

$$\sum_{k=1}^{\infty} P_s(n, k) \qquad \text{and} \qquad \sum_{n=1}^{\infty} P_s(n, k)$$

are finite. The sum of the column, however, is by definition the number $p_s(n)$ and by Theorem 3 $p_s(n) = P(n + s, s)$, i.e. it holds

$$\sum_{k=1}^{\infty} P_s(n, k) = p_s(n) = P(n + s, s). \tag{19}$$

In § 2 we shall calculate the sum of all values of one row and we shall also show that the sequence of non zero values in an arbitrary row is "symmetric" (it is the same if we read it from the front or from behind).

§ 2. PARTITIONS AND CARDINAL POWERS OF FINITE CHAINS

By a *poset* is meant a set with reflexive antisymmetric transitive relation called an *ordering* and usually denoted by symbol $\leqq$.

If A is a poset, $l(A)$ denotes its *length* and $h(x)$ is the *height* of an element $x \in A$. $\breve{A}$ denotes the poset *dually ordered* to A, i.e. $x \leqq y$ in $\breve{A}$ iff $y \leqq x$ in A. If posets A, B are *isomorphic*, we write $A \cong B$.

Finally a cardinal number of a set A is denoted by symbol card A. Definitions of all here nondefined notions see e.g. in [3].

Definition. Let G, H be posets. The *cardinal power* G^H is the poset of all isotone mappings of the set H into the set G with relation $\leqq$ defined as follows:

For $f, g \in G^H$ there holds $f \leqq g$ if and only if $f(x) \leqq g(x)$ in G for every $x \in H$.

Definition. Let $n \in N_0$ be arbitrary. The poset $\{0 < 1 < 2 < \ldots < n - 1\}$ is denoted by symbol $\boldsymbol{n}$.

It evidently holds

Theorem 1. *Let G, H be finite chains,* card $G = n$, card $H = k$. *Then it is*

$$G^H \cong \mathbf{n}^{\mathbf{k}}.$$

Further it is evident (see e.g. [3], Theorem 2, p. 57) that there holds

Theorem 2. *Let G, H be posets. Then* $\breve{G^H} \cong \breve{G}^{\breve{H}}$.

Corollary. *Let G, H be finite chains. Then we have* $\breve{G^H} \cong G^H$.

Proof: The assertion follows from Theorem 2 and from the evident fact that for every finite chain A there holds $\breve{A} \cong A$.

Definition. Let $k, n \in N$ be arbitrary. The poset $\mathscr{A}(n, k)$ is defined as follows: The elements of the poset $\mathscr{A}(n, k)$ are all k tuples $(a_1, a_2, \ldots, a_k)$ such that $a_i \in \mathbf{n}$ for $i = 1, 2, \ldots, k$ and it holds $a_1 \geqq a_2 \geqq \ldots \geqq a_k$. The ordering $\leqq$ on $\mathscr{A}(n, k)$ is defined in the following way:
$(a_1, \ldots, a_k) \leqq (b_1, \ldots, b_k)$ in $\mathscr{A}(n, k)$ iff $a_i \leqq b_i$ in $\mathbf{n}$ for every $i = 1, 2, \ldots, k$.

Theorem 3. *Let $k, n \in N$ be arbitrary. Then it holds*

$$\mathbf{n}^{\mathbf{k}} \cong \mathscr{A}(n, k).$$

Proof: Let us define a mapping $F : \mathbf{n}^{\mathbf{k}} \to \mathscr{A}(n, k)$ as follows: For $f \in \mathbf{n}^{\mathbf{k}}$ we put $F(f) = (f(k-1), f(k-2), \ldots, f(0))$. Then F is evidently an isomorphism of posets $\mathbf{n}^{\mathbf{k}}$ and $\mathscr{A}(n, k)$.

Remark. In what follows we shall identify posets $\mathbf{n}^{\mathbf{k}}$ and $\mathscr{A}(n, k)$ and we shall not differentiate the element $f \in \mathbf{n}^{\mathbf{k}}$ from k tuple $(f(k-1), \ldots, f(0)) \in \mathscr{A}(n, k)$.

Example. On Fig. 1 we have the Hasse diagram of the cardinal power $\mathbf{4}^{\mathbf{3}}$. The element (a_1, a_2, a_3) is briefly written in the form $a_1a_2a_3$.

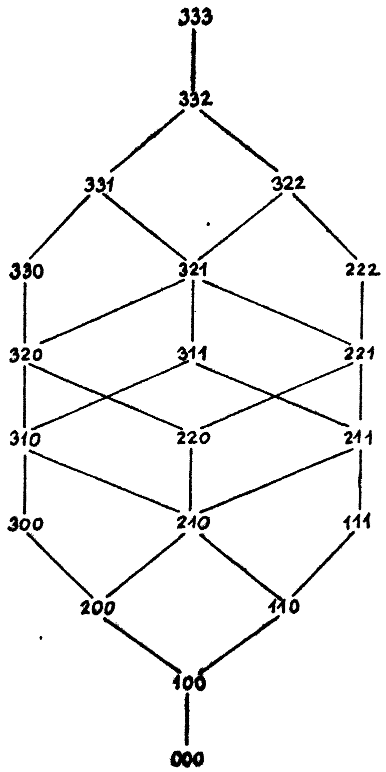

Fig. 1

If $k, n \in N$ are arbitrary, then $(0, 0, \ldots, 0)$ is evidently the least element and $(n-1, n-1, \ldots, n-1)$ the greatest element of the cardinal power $\mathbf{n}^k$ and evidently one and only one element $(1, 0, \ldots, 0)$ is of the height 1.

Evidently it holds in general

Theorem 4. *Let $k, n \in N$ be arbitrary, let $f = (a_1, \ldots, a_k) \in \mathbf{n}^k$. Then*

$$h(f) = \sum_{i=1}^{k} a_i. \tag{20}$$

Theorem 5. *Let $k, n \in N$ be arbitrary. Then it holds*

$$l(\mathbf{n}^k) = k(n-1), \tag{21}$$

$$\operatorname{card} \mathbf{n}^k = \binom{n+k-1}{n}. \tag{22}$$

Proof: I. The length of the power $\mathbf{n}^k$ is evidently equal to the height of the greatest element $(n-1, \ldots, n-1)$. The relation (21) now follows from (20).

II. It holds $\operatorname{card} \mathbf{n}^k = \operatorname{card} \mathscr{A}(n, k)$. But $\operatorname{card} \mathscr{A}(n, k)$ is equal to the number of all combinations of k-th class with repetition formed of n elements which is, as well-known, equal to the combination number $\binom{n+k-1}{k}$.

Definition. Let A be a poset, let $i \in N_0$ be arbitrary. The set

$$H_i(A) = \{x; x \in A, h(x) = i\}$$

is called the *i-th layer* of the poset A.

Theorem 6. *Let $k, n \in N$ be arbitrary. Then for every $i \in N_0$ it holds*

$$\operatorname{card} H_i(\mathbf{n}^k) = Q_{n-1}(i, k). \tag{23}$$

Proof: Let $f \in \mathbf{n}^k$, $f = (a_1, \ldots, a_k)$ be arbitrary. By (20) we have $h(f) = \sum_{i=1}^{k} a_i$ so that all elements $(a_1, \ldots, a_k) \in \mathbf{n}^k$ such that

$$a_1 + \ldots + a_k = i, \qquad n - 1 \geqq a_1 \geqq a_2 \geqq \ldots \geqq a_k \geqq 0$$

have the height $i \in N_0$. By the remark ahead of Theorem 1, § 1, the number of these elements is equal to the integer $Q_{n-1}(i, k)$.

Corollary. *For arbitrary $k, n \in N$, $i \in N_0$ it holds*

$$\operatorname{card} H_i(\mathbf{n}^k) = P_n(k + i, k). \tag{24}$$

Proof: The assertion follows from the relations (17) and (23).

By (24) the sequence of non zero values in k-th row of the table $P_s(n, k)$ gives the number of elements in layers of the cardinal power $\mathbf{s}^k$.

Example. From Fig. 1 it is evident that the number of elements in layers of the cardinal power $\mathbf{4}^3$ is succesively

$$1, 1, 2, 3, 3, 3, 3, 2, 1, 1$$

which are only and only non zero values in the 3 rd row of the table $P_4(n, k)$ given in § 1.

Now we shall prove that non-zero values in every row of the arbitrary table $P_s(n, k)$ form a symmetric sequence. Let us realize at the same time that by Corollary of Theorem 6, § 1, it holds $P_s(n, k) \neq 0$ iff $n = k + i$, $i = 0, 1, \ldots, k(s - 1)$.

Theorem 7. *Let $k, s \in N$ be arbitrary. Then for every $i \in N_0$, $i \leqq k(s - 1)$ it holds*

$$P_s(k + i, k) = P_s(ks - i, k). \tag{25}$$

Proof: By (24) we have $P_s(k + i, k) = \text{card } H_i(\mathbf{s}^k)$. However by corollary of Theorem 2 it holds $\mathbf{s}^k \cong \breve{\mathbf{s}}^k$ so that

$$\text{card } H_i(\mathbf{s}^k) = \text{card } H_{k(s-1)-i}(\mathbf{s}^k).$$

By (24), however, we have

$$\text{card } H_{k(s-1)-i}(\mathbf{s}^k) = P_s(k + ks - k - i, k) = P_s(ks - i, k).$$

This proves the equality (25).

Corollary. *Let $k, s \in N$ be arbitrary. Then for every $n \in N_0$, $n \leqq ks$ holds*

$$Q_s(n, k) = Q_s(sk - n, k). \tag{26}$$

Proof: By (17) and (25) it holds

$$Q_s(n, k) = P_{s+1}(n + k, k) = P_{s+1}(ks + k - n, k) = Q_s(sk - n, k).$$

Theorem 8. *The sum of all values in k-th row of the table $P_s(n, k)$ is equal to $\binom{s + k - 1}{k}$ i.e.*

$$\sum_{n=1}^{\infty} P_s(n, k) = \binom{s + k - 1}{k}. \tag{27}$$

Proof: By Theorem 6, § 1, we have $\sum_{n=1}^{\infty} P_s(n, k) = \sum_{i=0}^{k(s-1)} P_s(k + i, k)$. But by (24), (21) and (22) it holds

$$\sum_{i=0}^{k(s-1)} P_s(k + i, k) = \sum_{i=0}^{k(s-1)} \text{card } H_i(\mathbf{s}^k) = \text{card} \bigcup_{i=0}^{k(s-1)} H_i(\mathbf{s}^k) = \text{card } s^k = \binom{s + k - 1}{k}.$$

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