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A NOTE ON PERIODIC SOLUTION OF SECOND ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

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We consider the second order non-linear differential equation

$$x'' + f(t, x, x') = 0, \tag{1}$$

where f is a continuous real-valued function with domain $[-T, T] \times R^2$, $T > 0$.

Further, we shall assume that all solutions of initial value problems for (1) extend to $[-T, T]$.

Under the above assumptions we establish the following theorem

Theorem 1. *Assume*

(i) $$f(-t, -x, x') = -f(t, x\ x')$$

(ii) $$f(t, x, x')$$

is locally Lipschitzian with respect to (x, x'), *i.e. for each compact subset* Ω *of* R^2, *there exists positive constants* K *and* L *(depending on* Ω*) such that*

$$|f(t, x, x') - f(t, y, y')| \leqq K|x - y| + L|x' - y'|, \quad -T \leqq t \leqq T. \tag{2}$$

Then, there exists ω_0, $0 < \omega_0 \leqq T$, *such that for every*, $0 < \omega \leqq \omega_0$ *Equation* (1) *has a unique solution* $x(t)$ *satisfying the periodic boundary conditions*

$$x\left(-\frac{\omega}{2}\right) = x\left(\frac{\omega}{2}\right), \quad x'\left(-\frac{\omega}{2}\right) = x'\left(\frac{\omega}{2}\right). \tag{3}$$

Proof. First we consider Equation (1) with the following boundary condition

$$x(0) = x\left(\frac{\omega}{2}\right) = 0. \tag{4}$$

Let $M > 0$, and $N > 0$ be given. Let $Q = \text{Max}\ \{|f(t, x, x')|: -T \leqq t \leqq T, |x| \leqq M, |x'| \leqq N\}$ and let $G(t, s)$ be the Green's function

$$G(t, s) = \frac{2}{\omega}\begin{cases} t\left(\frac{\omega}{2} - s\right), & 0 \leqq t \leqq s \leqq \frac{\omega}{2}, \\ s\left(\frac{\omega}{2} - t\right), & 0 \leqq s \leqq t \leqq \frac{\omega}{2}, \end{cases} \tag{5}$$

and

$$G_t(t, s) = \frac{2}{\omega}\begin{cases} \frac{\omega}{2} - s, & 0 \leqq t \leqq s \leqq \frac{\omega}{2}, \\ -s, & 0 \leqq s \leqq t \leqq \frac{\omega}{2}. \end{cases}$$

Let $B = \{\Phi \in C'\left[0, \frac{\omega}{2}\right]: |\Phi(t)| \leqq M,\ |\Phi'(t)| \leqq N\}$, and define the operator S on B by

$$(6)\quad \begin{aligned} (S\Phi)(t) &= \int_0^{\frac{\omega}{2}} G(t, s)\, f(s, \Phi(s), \Phi'(s))\, ds, \\ (S\Phi')(t) &= \int_0^{\frac{\omega}{2}} G_t(t, s)\, f(s, \Phi(s), \Phi'(s))\, ds. \end{aligned}$$

Then

$$|(S\Phi)(t)| \leqq \frac{\omega^2}{32} Q \leqq M,\ |(S\Phi')(t)| \leqq \frac{\omega}{8} Q \leqq N. \tag{7}$$

Hence S maps B continuously into itself provided

$$\omega < M\ .\ \inf \left\{\sqrt{\frac{32}{Q} M}, 8\frac{N}{Q}\right\}. \tag{8}$$

Let K and L be the Lipschitz constants for f corresponding to the compact set $\Omega \subset R^2$

$$\Omega = \{(x, x') : |x| \leqq M,\ |x'| \leqq N\}.$$

If for $\Phi \in B$, we let $\|\Phi\| = \text{Max}\left\{|\Phi(t)|,\ |\Phi'(t)| : t \in \left[0, \frac{\omega}{2}\right]\right\}$, one may easily show that S is a contraction with respect to $\|\cdot\|$ on B provided that ω is chosen so that

(9) $$\frac{\omega^2}{32}(K + L) < 1, \quad \frac{\omega}{8}(K + L) < 1.$$

Hence, if ω satisfies both (8) and (9), then (1), (4) has a unique solution $x(t)$ with

$$|x(t)| \leqq M, \quad |x'(t)| \leqq N.$$

Now, since $-f(-t, -x, x') = f(t, x, x')$, so by (1) if $z(t) = -x(-t)$ then $z(t)$ is also a solution of (2), and since by (4) $z(0) = -x(0) = x(0)$, $z'(t) = x'(-t)$, and $z'(0) = z'(0)$, it follows from the uniqueness that $x(t) = -x(-t)$ for $-\frac{\omega}{2} \leqq t \leqq \leqq \frac{\omega}{2}$. Therefore

$$x\left(-\frac{\omega}{2}\right) = -x\left(\frac{\omega}{2}\right) = 0 = x\left(\frac{\omega}{2}\right)$$

and

$$x'\left(-\frac{\omega}{2}\right) = x'\left(\frac{\omega}{2}\right)$$

which proves Theorem 1.

Corollary 1. *With the assumptions of Theorem* 1 *assume $f(t, x, x')$ to be periodic of period ω, i.e.*

$$f(t + \omega, x, x') = f(t, x, x')$$

Then Equation (1) *possesses a unique periodic solution of period ω.*

Proof. Define $x(t)$ as before on the interval $\left(-\frac{\omega}{2}, 0\right)$ by the equality $x(-t) = = -x(t)$ and continuous over the whole interval $(-\infty, +\infty)$ as a periodic function with period ω. Then by (i) and (4) it is easy to show that $x(t)$ is a periodic solution of Equation (1); see for example M. A. Krasnosel'skij (cf. [4], pp. 313–314).

Let us now consider a few applications of Theorem 1.

(A_1) We consider the equation

(10) $$x'' + g(x) = p(t).$$

Let $p(t)$ be continuous and $g(x)$ locally Lipschitzian in x. Further, assume

$$-g(-x) = g(x), \quad -p(-t) = p(t)$$

for all x and t. Then there exists an $\omega_0 > 0$, such that if $p(t)$ is periodic of period ω, $0 < \omega \leqq \omega_0$, Equation (10) has a unique periodic solution of period ω.

Example 1. We consider

$$x' + x^3 = \sin 2t \tag{11}$$

Let $f(t, x) = x^3 - \sin^2 t$, and $M > 0$ be given. Then $Q = \mathrm{Max}\,\{|f(t, x)| : 0 < t \leqq \pi, |x| \leqq M\} = M^3 + 1$ and the Lipschitz constant L corresponding to the compact set $\Omega = \{x : |x| \leqq M\}$ is equal to $3M^2$. Therefore from inequalities (8) and (9) we obtain

$$\frac{\pi^2}{32}(M^3 + 1) \leqq M \quad \text{and} \quad 3\frac{\pi^2}{32} M^2 < 1.$$

Now, the above inequalities are satisfied for many values of M, for example $M = \frac{1}{2}$. Therefore Equation (11) possesses a unique periodic solution $x(t)$ of period π such that $|x(t)| \leqq M$.

(A_2) We consider Equation

$$x'' + f(x)\,x'^n + ax = p(t),\ a \in R,\ n \geqq 0. \tag{12}$$

Let $p(t)$ be continuous and $f(x)$ locally Lipschitzian in x. Furthermore, assume

$$-f(-x) = f(x), \qquad -p(-t) = p(t)$$

for all x and t. Then there exists an $\omega_0 > 0$, such that if $p(t)$ is ω-periodic, $0 < \omega \leqq \leqq \omega_0$, Equation (12) has a unique periodic solution of period ω.

(A_3) We consider the forced Liénard's equation

$$x'' + f(x, x')\,x' + g(x, x') = p(t). \tag{13}$$

Let $p(t)$ be continuous and f, g locally Lipschitzian in x and x'. Furthermore assume

$$-f(-x, x') = f(x, x'),\ -g(-x, x') = g(x, x'),\ -p(-t) = p(t)$$

for all x and t. Then there exists an $\omega_0 > 0$, such that if $p(t)$ is periodic of period ω, $0 < \omega \leqq \omega_0$, Equation (13) has a unique periodic solution of period ω.

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