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ISOMORPHIC ALGEBRAIC PRE-CLOSURES AND EQUIVALENT SET-SYSTEMS

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Theorem 1.1 in [3] chap. II says that each closure system $\mathcal{S}$ on a given set S defines a closure operation on this set and conversely each closure operation on S defines a closure system on this set, thus there is given a one-to-one correspondence between all closure operations on S and all closure systems on this set. In the mentioned theorem this correspondence is expressed explicitely. From here it follows that the system of all closure operations on a given set S can be mapped injectively into the system exp exp S such that two closures are isomorphic if and only if the corresponding set-systems are equivalent (in the sense of paper [6]) i.e. $\mathcal{S}_1, \mathcal{S}_2 \in \exp \exp S$ are equivalent if there exists a permutation f of the set S such that $\mathcal{S}_2 = \{f(X) : X \in \mathcal{S}_1\}$ or $\mathcal{S}_1 = \{f(X) : X \in \mathcal{S}_2\}$. This equivalence is denoted by $\sim$. A natural question is whether the above described monorelational embedding is extendable onto a certain system of more general structures so called pre-closure operations. In paper [5] this problem is solved for topological closures and Čech's topologies. The aim of this paper is to show that there exists a system (closed with respect to isomorphisms) of the cardinality $2^{\operatorname{card} S}$ (for an infinite carrier set S) of algebraic pre-closure operations to which it is possible to extend the just mentioned embedding into (exp exp S, $\sim$). Terms and notations concerning algebraic closure operations are taken from papers [2], [3].

Let S be a set, C be a map of exp S into itself and n be a positive integer. By C^n will be denoted the n-fold composition of C with itself. A mapping C: exp $S \to$ $\to$ exp S is a pre-closure if for any $P \subset S$ and $Q \subset S$ these conditions are satisfied:

$$P \subset C(P) \text{ and } P \subset Q \text{ implies } C(P) \subset C(Q).$$

A pre-closure C which satisfies $C^2(P) \subset C(P)$ for all P contained in S is a closure. A pre-closure (or closure) which satisfies the compactness condition,

for any $P \subset S$ and for any $x \in C(P)$ there is a finite set $Q \subset P$ such that $x \in C(Q)$, will be called algebraic. This condition is equivalent to the condition: $C(P) =$ $= \bigcup \{C(Q) : Q \subset P, \operatorname{card} Q < \aleph_0\}$ for each $P \subset S$.

By a pre-closure (closure) space we mean an ordered pair (S, C), where C is a pre-closure (closure) on the set S. A set P is said to be closed in the space (S, C) or

C-closed if $C(P) = P$. The system of all closures on the set S will be denoted by $\mathscr{C}(S)$. Let pre-closure spaces (S_1, C_1), (S_2, C_2) and a mapping $f: S_1 \to S_2$ be given. The mapping f is said to be an isomorphism of the space (S_1, C_1) onto the space (S_2, C_2) if f is bijective and $f(C_1(P)) = C_2(f(P))$ for each set $P \subset S_1$. If $S_1 = S_2 = S$, pre-closure spaces (S, C_1), (S, C_2) are isomorphic, we say that pre-closures C_1, C_2 are isomorphic and we write $C_1 \cong C_2$. A system $\mathscr{X}$ of pre-closures on a set S is said to be closed with respect to closure isomorphisms if $C \in \mathscr{X}$, $C_1 \in \mathscr{C}(S)$, $C \cong C_1$ implies that $C_1 \in \mathscr{X}$. A pre closure C will be called n-iterable if n is the least positive integer such that C^n is a closure.

Let T be a non-empty subset of S. A decomposition of the set T determines a decomposition in the set S (in the sense of [1] chap. I). This decomposition will be denoted by $\bar{T}$ in accordance with [1]. A kernel of a set $P \subset S$ (denoted by [P]) in the decomposition $\bar{T}$ (where $T \subset S$, $T \neq \emptyset$) is a union of all the blocks of $\bar{T}$ which are subsets of the set P.

In what follows we suppose that S is an infinite set. Consider a system of triads of the form $\{T, \bar{T}, A\}$, where T is a non-void subset of S satisfying conditions card $T \geqq 2$, card $(S - T) \geqq \aleph_0$, $\bar{T}$ is a decomposition of T such that $X \in \bar{T}$ implies card $X < \aleph_0$ and A is a finite subset of S, lineary ordered, disjoint with T. We assign to every such a triad a mapping C: exp $S \to$ exp S which is defined as follows:

Let $A = \{a_1, a_2, \dots, a_n\}$ and the ordering of A be given by the n-tuple of indices $\{1, 2, \dots, n\}$. For $X \subset S$ such that $X \cap A \subset \{a_n\}$ and $[X] = \emptyset$ (i.e. X does not contain as a subset any element of $\bar{T}$), we put $C(X) = X$. If $X \cap A = \{a_{i_1}, \dots, a_{i_k}\}$, $i_k \leqq n$, then $C(X) = X \cup \{a_{i_1+1}, \dots, a_{i_k+1}\} \cup X_0$, where we put $a_{n+1} = a_n$ and $X_0 = \{a_0\}$ if $[X] \neq \emptyset$ and $X_0 = \emptyset$ otherwise. A triad $\{T, \bar{T}, A\}$ corresponding to the mapping C will be denoted by $\{T_C, \bar{T}_C, A_C\}$ or $\{T_C, \bar{T}_C, (A_C, <)\}$ if ti is necessary to express the ordering of the set A_C. Finally, denote by $\mathscr{A}_k(S)$ the system of mappings C of exp S into itself such that $A_C = \{a_0, a_1, \dots, a_k\}$ and put $\mathscr{A}(S) = \bigcup_{k \geqq 1} \mathscr{A}_k(S)$.

Lemma 1. *$\mathscr{A}(S)$ is a system of algebraic pre-closures on S, closed with respect to closure-isomorphisms, such that $\mathscr{A}(S) \cap \mathscr{C}(S) = \emptyset$ and to each positive integer k there exists in $\mathscr{A}(S)$ a k-iterable pre-closure.*

Proof. Let n be an arbitrary positive integer, $C \in \mathscr{A}_n(S)$. From the above construction it follows immediately that C is a pre-closure on S. Let $P \subset S$ be a non-void set. If $P \cap A_C \subset \{a_n\}$ and $[P] = \emptyset$ (in the decomposition $\bar{T}_C$) then the same holds for each finite $Q \subset P$, hence $C(P) = \bigcup\{C(Q) : Q \subset P, \text{card } Q < \aleph_0\}$. Let $P \cap A = \{a_{i_1}, \dots, a_{i_k}\}$, $i_k \leqq n$ and $[P]_C \neq \emptyset$. Then by the construction of (S, C) we have $C(P) = P \cup \{a_0, a_{i_1+1}, \dots, a_{i_k+1}\}$. Let $x \in C(P)$ be an arbitrary point. If $x = a_0$, then $x \in C(Y)$, where $Y \in \bar{T}_C$ is a finite subset of P. If $x = a_j$, $j \in \{i_1 + 1, \dots, i_k + 1\}$, then $x \in C\{a_{j-1}\}$, where $a_{j-1} \in P$. If moreover $x \in P$, then clearly $x \in C\{x\}$. In all the possible cases considered with respect to the set P we get in the similar way that $x \in C(P)$ is followed by $x \in C(Q)$ for a suitable finite subset $Q \subset P$. Thus (S, C) is an algebraic pre-closure space. Since for arbitrary $X \in \bar{T}_C$ there holds $C(X) = X \cup \{a_0\}$ and $C^2(X) = X \cup \{a_0, a_1\}$ where $a_1 \neq a_0$, $a_1 \notin X$, the pre-closure C is not a closure and we get $\mathscr{A}(S) \cap \mathscr{C}(S) = \emptyset$. Let D be a pre-closure isomorphic to C, $f : (S, C) \to (S, D)$ the corresponding closure-isomorphism. Put $T' = f(T_C)$, $\bar{T}' = \{f(X) : X \in \bar{T}_C\}$, $b_i = f(a_i)$, $i = 0, 1, \ldots, n$ and $B = \{b_i : i = 0, 1, \ldots, n\}$. Since f is a permutation of the set S, we have that there exists a closure, say C_1, which corresponds to the triad $\{T', \bar{T}', B\}$ in the above sense. Let $P \subset S$ be an arbitrary set. If $P \cap B \subset \{b_n\}$ and P does not contain any element of $\bar{T}$ i.e. $[P]' = \emptyset$, then $C_1(P) = P$. Then $f^{-1}(P) \cap A \subset \{a_n\}$ and $[f^{-1}(P)] = \emptyset$ thus $C(f^{-1}(P)) = f^{-1}(P)$. Hence $D(P) = D(f(f^{-1}(P))) = f(C(f^{-1}(P))) = P = C_1(P)$. Now, let $P \cap B = \{b_{i_1}, \ldots, b_{i_k}\}$ be valid with $i_k \leqq n$ and $[P] = \bigcup_{j=1}^{r} X_j$, where $X_1, X_2, \ldots, X_r$ are elements of $\bar{T}'$. Then $f^{-1}(P) \cap A = \{a_{i_1}, \ldots, a_{i_k}\}$, where $f(a_{i_1}) = b_{i_1}, \ldots, f(a_{i_k}) = b_{i_k}$ and $f^{-1}(X_j)$, $j = 1, 2, \ldots, r$ are all the elements of $\bar{T}_C$ contained in $f^{-1}(P)$. We have $D(P) = D(f(f^{-1}(P))) = f(C(f^{-1}(P))) = f(f^{-1}(P) \cup \{a_0, a_{i_1+1}, \ldots, a_{i_k+1}\}) = P \cup \{b_0, b_{i_1+1}, \ldots, b_{i_k+1}\} = C_1(P)$. We get in this way that for each subset P of S there holds $D(P) = C_1(P)$, thus $D = C_1$ and we have that $\mathscr{A}(S)$ is closed with respect to closure isomorphisms. Let k be a positive integer. Consider arbitrary $C \in \mathscr{A}_{k-1}(S)$. Then $C^{k+1}(T) = T \cup \{a_0, a_1, \ldots, a_{k-1}\} = C^{k+1}(T)$ and $C^m(T) = T \cup \{a_0, a_1, \ldots, a_{m-1}\} \subsetneqq T \cup \{a_0, a_1, \ldots, a_m\} = C^{m+1}(T)$ for $m < k$. Since $C^k(P) = C^{k+1}(P)$ for each $P \subset S$, i.e. C^k is a closure and C^m is not if $m < k$, we have that C is a k-iterable pre-closure, q.e.d.

Now, put $\mathscr{T}(S) = \mathscr{C}(S) \cup \mathscr{A}(S)$ and define a mapping F of $\mathscr{T}(S)$ into exp exp S by $F(C) = \{X : X \subset S,\ C(X) = X\} \cup \bigcup_{k \geqq 1} \{X : X \subset S,\ C^k(X) \neq S,\ C^{k+1}(X) = S\}$ for every $C \in \mathscr{T}(S)$.

Lemma 2. *Let $C_1, C_2 \in \mathscr{A}(S)$, $C_1 \neq C_2$. Then it holds $F(C_1) \neq F(C_2)$.*

Proof. Let $\{T_1, \bar{T}_1, (A_1, <_1)\}$, $\{T_2, \bar{T}_2, (A_2, <_2)$ be triads corresponding to C_1, C_2 respectively, where C_1, C_2 are arbitrary different pre-closures from $\mathscr{A}(S)$. Consider all possible cases:

(1) $T_1 = T_2$, $\bar{T}_1 = \bar{T}_2$, $(A_1, <_1) \neq (A_2, <_2)$,
(2) $T_1 = T_2$, $\bar{T}_1 \neq \bar{T}_2$,
(3) $T_1 \neq T_2$.

Let the case (1) occur. Suppose $A_1 \neq A_2$ and put $P = T_1 \cup A_1 = T_2 \cup A_1$, $Q = T_1 \cup A_1 \cup A_2 = T_2 \cup A_1 \cup A_2$. Since $T_1 \cap A_1 = T_1 \cap A_2 = \emptyset$, it holds $P \neq Q$. Further $C_1(P) = P$ thus $P \in F(C_1)$ and $C_2(P) = Q$, $C_2^2(P) = C_2(Q) =$

$= Q \neq S$ for $\operatorname{card}(S - T_i) \geqq \aleph_0$, $i = 1, 2$. Thus $P \notin F(C_2)$. Let $A_1 = A_2$, $<_1 \neq <_2$. Put $\{a_0, a_1, \ldots, a_n\} = A_1$, $\{b_0, b_1, \ldots, b_n\} = A_2$. There exists a pair of indices i, j such that $a_i = b_j$, $i \neq j$. Let $i \in \{0, 1, 2, \ldots, n\}$ be the greatest non-negative integer with the property $a_i = b_j$, where $j < i$. Put $P = \{a_i, a_{i+1}, \ldots, a_n\}$. There is $C_1(P) = P$ but $C_2(P) \neq P$ and $C_2^k(P) \neq S$ for every k. Hence $P \in F(C_1)$, $P \notin F(C_2)$ and we have $F(C_1) \neq F(C_2)$ in the case (1).

Let the case (2) occur. There exists a block $X \in \bar{T}_1$ which does not belong to $\bar{T}_2$. If $[X]_2 = \emptyset$, then the set X is C_2-closed thus $X \in F(C_2)$ whereas $X \notin F(C_1)$. If $[X]_2 \neq \emptyset$, there exists a block $Y \in \bar{T}_2$ contained as a subset in X. Then $[Y]_1 = \emptyset$ and thus Y is a C_1-closed set i.e. $Y \in F(C_1)$ but $Y \notin F(C_2)$, thus $F(C_1) \neq F(C_2)$ again.

Suppose now that (3) holds, i.e. $T_1 \neq T_2$. Admit first that $(A_1, <_1) = (A_2, <_2)$. Without loss of generality it can be supposed $T_1 - T_2 \neq \emptyset$. Let $x \in T_1 - T_2$, $X \in \bar{T}_1$ be such a block that $x \in X$. Let $[X]_2 = \emptyset$. Then $X \notin F(C_1)$, $X \in F(C_2)$ since $C_2(X) = X$. If $[X]_2 \neq \emptyset$, then there exists a set $Y \in \bar{T}_2$ with $Y \subset X$. Then $[Y]_1 = \emptyset$ thus $C_1(Y) = Y$ and we have $Y \in F(C_1)$, $Y \notin F(C_2)$. Now, consider the case $(A_1, <_1) \neq (A_2, <_2)$. If $A_1 = A_2$, $<_1 \neq <_2$, then we get in the same way as in the case (1) that there exists a set $P \subset S$ with $P \in F(C_1)$, $P \notin F(C_2)$. Let $A_1 \neq A_2$, $A_1 = \{a_0, a_1, \ldots, a_n\}$, $A_2 = \{b_0, b_1, \ldots, b_m\}$. If there exists $a_i \in A_1 - A_2$ with $i < n$ then $C_2\{a_i\} = \{a_i\}$ for $\operatorname{card} X \geqq 2$ whenever $X \in \bar{T}_2$ and thus $\{a_i\} \in F(C_2)$ but $\{a_i\} \notin F(C_1)$. Let $a_i \in A_1 - A_2$ implies $i = n$. If there exists $b_j \in A_2 - A_1$ with $j < m$, then similarly as above $\{b_j\} \in F(C_1)$ but $\{b_j\} \notin F(C_2)$. If $b_j \in A_2 - A_1$ implies $j = m$, and $a_i \in A_1 - A_2$ is followed by $i = n$, then we have $\{a_{n-1}, a_n\} \in F(C_1)$ and $\{a_{n-1}, a_n\} \notin F(C_2)$. Now, consider the case $A_1 - A_2 = \{a_n\}$. Then there exist positive integers i, j with $i < n$, $j < n$, $i \neq j$ such that $b_{m-1} = a_i$, $b_m = a_j$ and thus $\{b_{m-1}, b_m\}$ is a C_2-closed set, i.e. $\{b_{m-1}, b_m\} \in F(C_2)$ but $\{b_{m-1}, b_m\} \notin F(C_1)$. Therefore $F(C_1) \neq F(C_2)$ in the case (3) and we have that the mapping F restricted onto the system $\mathscr{A}(S)$ is injective.

Lemma 3. *Let $C_1, C_2 \in \mathscr{T}(S)$. Then $C_1 \cong C_2$ if and only if $F(C_1) \sim F(C_2)$.*

Proof. If C_1, C_2 are closure operations i.e. $C_1, C_2 \in \mathscr{C}(S)$, then $F(C_1)$, $F(C_2)$ are systems of all C_1-closed, C_2-closed sets respectively, thus $C_1 \cong C_2$ if and only if $F(C_1) \sim F(C_2)$. This is the well-known assertion. Suppose that $C_1 \in \mathscr{A}(S) - \mathscr{C}(S)$, $C_2 \in \mathscr{A}(S) - \mathscr{C}(S)$ are isomorphic pre-closures. Denote by $\{T_1, \bar{T}_1, (A_1, <_1)\}$, $\{T_2, \bar{T}_2, (A_2, <_2)\}$ triads corresponding to C_1, C_2 respectively. It was shown in the proof of lemma 1 that if $C_1 \cong C_2$, then there exists a permutation f of the set S such that $T_2 = f(T_1)$, $\bar{T}_2 = \{f(X) : X \in \bar{T}_1\}$ and $(A_2, <_2)$ is order-isomorphic to $(A_1, <_1)$ with the isomorphism f. Let $A_1 = \{a_0, a_1, \ldots, a_n\}$, $A_2 = \{b_0, b_1, \ldots, b_n\}$. Let P be a C_2-closed set and put $Q = f^{-1}(P)$. The following three cases are possible:

(1) $P \cap A_2 \subset \{b_n\}$ and $[P \cap T_2]_2 = \emptyset$,

(2) $b_i \in P \cap A_2$, $i < n$ implies $b_{i+1} \in P \cap A_2$ and $[P \cap T_2]_2 = \emptyset$,

(3) $[P \cap T_2]_2 \neq \emptyset$ and $A_2 \subset P$.

Considering the properties of the mapping f we get that $Q \subset \{a_n\}$ and $[Q \cap T_1]_1 = \emptyset$ in the case (1). Similarly, $a_i \in Q \cap A_1$, $i < n$ implies $a_{i+1} \in Q \cap A_1$ and $[Q \cap T_1]_1 = \emptyset$ if the case (2) occurs and $[Q \cap T_1]_1 \neq \emptyset$, $A_1 \subset Q$ in the case (3). Thus $Q = f^{-1}(P)$ is a C_1-closed set. Now, let $P \in F(C_2)$ be such a set that $C_2^k(P) \neq S$, $C_2^{k+1}(P) = S$, where $k \leqq n - 1$. Then $P = S - \{b_0, b_1, \dots, b_k\}$ hence $f^{-1}(P) = S - \{a_0, a_1, \dots, a_k\}$ and thus $f^{-1}(P) \in F(C_1)$. Therefore $F(C_2) \subset \{f(X) : X \in F(C_1)\}$. If P is a set of the form $P = f(Q)$, where Q is a suitable set belonging to $F(C_1)$, we get, similarly as above, considering all possible cases with respect to the set Q that the inclusion $F(C_2) \supset \{f(X) : X \in F(C_1)\}$ holds. Thus we have $F(C_1) \sim F(C_2)$.

Now, suppose $F(C_1) \sim F(C_2)$ for $C_1, C_2 \in \mathscr{A}(S)$. There exists a permutation f of the set S such that $F(C_2) = \{f(X) : X \in F(C_1)\}$. Put $T = f(T_1)$, $\mathbf{T} = \{f(X) : X \in \mathbf{T}_1\}$. Let $(A, <)$ be a chain such that $f: (A_1, <_1) \to (A, <)$ is an order-isomorphism of A_1 onto A. Denote by C a pre-closure on S determined by $\{T, \mathbf{T}, (A, <)\}$. As it was shown in the proof of lemma 1, the pre-closure C is isomorphic to the pre-closure C_1. Further, it is easy to see that $F(C) = \{f(X) : X \in F(C_1)\}$, thus we have $F(C) = \{f(X) : X \in F(C_1)\} = F(C_2)$ and by lemma 2 it holds $C = C_2$, hence pre-closures C_1, C_2 are isomorphic. Finally, let $C_1 \in \mathscr{A}(S)$, $C_2 \in \mathscr{C}(S)$, i.e. C_1 non $\cong C_2$. We are going to show that $F(C_1)$ non $\sim F(C_2)$. Admit on the contrary that there exists a permutation f of the set S with $F(C_2) = \{f(X) : X \in F(C_1)\}$. Let $A_1 = \{a_0, a_1, \dots, a_n\}$, Since $C_1(S - \{a_0, a_1\}) = S - \{a_1\}$, $C_1^2(S - \{a_0, a_1\}) = C_1(S - \{a_1\}) = S$ thus $S - \{a_0, a_1\} \in F(C_1)$, we have $S - \{f(a_0), f(a_1)\} \in F(C_2)$. Further $C_1(T_1 \cup A_1) = T_1 \cup A_1$, hence the set $f(T_1) \cup f(A_1)$ is C_2-closed. Then also $f(T_1) \cup \{f(a_2), \dots, f(a_n)\} = [S - \{f(a_0), f(a_1)\}] \cap (f(T_1) \cup f(A_1))$ is a C_2-closed set. From here $T_1 \cup \{a_2, \dots, a_n\} = f^{-1}f(T_1 \cup \{a_2, \dots, a_n\}) \in F(C_1)$, which contradicts the definition of $F(C_1)$. Hence $F(C_1)$ non $\sim F(C_2)$. The proof is complete.

Lemma 4. *Let $C_1, C_2 \in \mathscr{T}(S)$, $C_1 \neq C_2$. Then $F(C_1) \neq F(C_2)$.*

Proof. If $C_1, C_2 \in \mathscr{C}(S)$, then $F(C_1), F(C_2)$ are systems of all closed sets in corresponding closure spaces, thus $C_1 \neq C_2$ implies $F(C_1) \neq F(C_2)$. If $C_1, C_2 \in \mathscr{A}(S)$, $C_1 \neq C_2$, then $F(C_1) \neq F(C_2)$ by lemma 2. If $C_1 \in \mathscr{A}(S)$, $C_2 \in \mathscr{C}(S)$, then with respect to lemma 3 $F(C_1) = F(C_2)$ is followed by $C_1 \cong C_2$ which is a contradiction, thus $F(C_1) \neq F(C_2)$.

Lemma 5. *It holds:* card $\mathscr{A}(S) = 2^{\text{card}\, S}$, card $[\mathscr{A}(S)/\cong] =$ card S *and* $\mathscr{X} \in \mathscr{A}(S)/\cong$ *implies* card $\mathscr{X} \geqq$ card S.

Proof. Let $T \subset S$ be a set of an infinite cardinality $\mathfrak{m}$. The system of all such decompositions of the set T blocks of which have finitely many elements has the cardinality $\mathfrak{m}$. Denote by $\mathscr{F}$ the system of triads $\{T, \mathbf{T}, A\}$ satisfying the above conditions, namely $\aleph_0 \leqq \text{card } T$, $\text{card }(S - T) \geqq \aleph_0$, $X \in \mathbf{T}$ implies $2 \leqq \text{card } X < \aleph_0$, $2 \leqq \text{card } A < \aleph_0$ and $A \cap T = \emptyset$. Clearly, $\text{card } \mathscr{F} \geqq 2^{\text{card } S}$ because there is at least $2^{\text{card } S}$ different sets T satisfying the just mentioned conditions. On the other hand $\text{card } \mathscr{F} \leqq 2^{\text{card } S} . \text{card } S . \aleph_0 = 2^{\text{card } S}$. From the equality $\text{card } \mathscr{F} = \text{card } \mathscr{A}(S)$ it follows the first assertion. Consider the decomposition $\mathscr{A}(S)/\cong$. Denoting by $\mathscr{F}$ a decomposition of $\mathscr{F}$ such that $\{T_1, \mathbf{T}_1, A_1\} \in \mathscr{F}$ and $\{T_2, \mathbf{T}_2, A_2\} \in \mathscr{F}$ belong to the same block of $\mathscr{F}$ if there exists a permutation f of the set S with $T_2 = f(T_1)$, $\mathbf{T}_2 = \{f(X) : X \in \mathbf{T}_1\}$, $A_2 = f(A_1)$, we have $\text{card } \mathscr{A}(S)/\cong = \text{card } \mathscr{F} = \text{card } S . \text{card } S . \aleph_0 = \text{card } S$. Let $\mathscr{X} \in \mathscr{A}(S)/\cong$. Denote by $\mathscr{Y}$ the corresponding block of $\mathscr{F}$. Let $\{T, \mathbf{T}, A\} \in \mathscr{Y}$. Consider these two possible cases: (1) $\text{card }(S - T) = \text{card } S$, (2) $\aleph_0 \leqq \text{card }(S - T) < \text{card } S$. In case (1) we chose an arbitrary element $a \in T$ and assign to every element $x \in S - (T \cup A)$ a triad $\{T_x, \mathbf{T}_x, A_x\}$, where $T_x = f_x(T)$, $\mathbf{T}_x = \{f_x(X) : X \in \mathbf{T}\}$, $A_x = A$ and f_x is a permutation of the set S defined by: $f_x(s) = s$ for $s \in S, x \neq S \neq a$ and $f_x(a) = x$, $f_x(x) = a$. Evidently $\text{card } S \leqq \text{card } \mathscr{Y}$. Let case (2) occur. We construct other triads from $\{T, \mathbf{T}, A\}$ in the following way. Let $X, Y \in \mathbf{T}$, $X \neq Y$, $a \in X$, $b \in Y$. Put $T_1 = T$, $A_1 = A$, $X_1 = (X - \{a\}) \cup \{b\}$, $Y_1 = (Y - \{b\}) \cup \{a\}$ and finally $\mathbf{T}_1 = (\mathbf{T} - \{X, Y\}) \cup \{X_1, Y_1\}$. If C, C_1 are corresponding pre-closures, then it holds $C(X_1) = X_1 \neq X_1 \cup \{a_0\} = C_1(X_1)$ and $C \cong C_1$. Since $\text{card } T = \text{card } \mathbf{T} = \text{card } S$, we get again $\text{card } S \leqq \text{card } \mathscr{Y}$. Hence $\text{card } S \leqq \text{card } \mathscr{X}$, q.e.d.

As in § 4 of [5] we use, for the sake of brevity, the following notions. If P, Q are sets and ϱ, σ binary relations on P, Q respectively, then the mapping $f : P \to Q$ is called an embedding of the monorelational system (P, ϱ) into the monorelational system (Q, σ) if f is injective and for every pair of elements $a \in P$, $b \in Q$ it holds $a\varrho b$ if and only if $f(a) \, \sigma f(b)$.

We summarize the obtained results in the following theorem. Notice that we have proved in fact a stronger assertion because the below described system of pre-closures was explicitely constructed.

Theorem. *Let S be an infinite set. There exist a system $\mathscr{T}(S)$ of pre-closures on S containing $\mathscr{C}(S)$, closed with respect to closure-isomorphisms, and a mapping F of $\mathscr{T}(S)$ into* exp exp S, *such that it holds*:

1° *Each element of $\mathscr{T}(S) - \mathscr{C}(S)$ is an algebraic pre-closure on S and to every positive integer n there exists an n-iterable pre-closure contained in $\mathscr{T}(S) - \mathscr{C}(S)$.*

2° $\text{card }[\mathscr{T}(S) - \mathscr{C}(S)] = 2^{\text{card } S}$, $\text{card }[(\mathscr{T}(S) - \mathscr{C}(S))/\cong] = \text{card } S$ *and* $\mathscr{X} \in [\mathscr{T}(S) - \mathscr{C}(S)]/\cong$ *implies* $\text{card } \mathscr{X} \geqq \text{card } S$.

3° $F : \mathscr{T}(S) \to$ exp exp S *is an embedding of the monorelational system* $(\mathscr{T}(S), \cong)$ *into the monorelational system* (exp exp S, $\sim$) *and for every closure* $C \in \mathscr{C}(S)$ *it holds* $F(C) = \{X \subset S : C(X) = X\}$.

Proof follows from lemmas 1,3,4 and 5.

The paper [5], mentioned in the introduction, contains the following incorrectness. The system $\mathscr{T}_A(P)$, defined in § 3 p. 108 – 109 is not a system of A-topologies and thus final system $\mathscr{T}(P)$ does not contain any A-topology. All lemmas and especially the main theorem of the paper [5] are valid, however for their proofs it is necessary to change the definition of $\mathscr{T}_A(P)$ as follows:

Denote by $\mathscr{A}_1(P)$ a system of all A-topologies on P satisfying the following condition. There exists a pair $X_1, X_2 \subset P$ of non-void sets with $X_1 \cup X_2 \neq P$, card $(X_1 \cap X_2) = 1$ such that if $X \subset P$ then $uX = X \cup Y$, where

(i) $Y = \emptyset$ if $X \cap X_1 = \emptyset = X \cap X_2$,

(ii) $Y = X_i$, $i \in \{1, 2\}$ if $X \cap X_i \neq \emptyset$ and $X \cap X_j = \emptyset$ for $j \in \{1, 2\}, j \neq i$,

(iii) $Y = X_1 \cup X_2$ if $X \cap X_1 \neq \emptyset \neq X \cap X_2$.

To every A-topology u from the system $\mathscr{A}_1(P)$ there is assigned a pair of sets X_1, X_2 with the above described properties. We shall denote these sets by $L_1(u)$, $L_2(u)$ respectively. Put $\mathscr{T}_A(P) = \{u \in \mathscr{A}_1(P) : \text{card } L_1(u) \geqq 2, \text{ card } L_2(u) \geqq 2$, card $(L_1(u) \cap L_2(u)) = 1$ and $L_1(u) \cup L_2(u) \neq P\}$. It was proved by Vladimír Tichý that all assertions concerning $\mathscr{T}_A(P)$ from paper [5] are true after the above change of the definition of the system $\mathscr{T}_A(P)$.

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