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AN APPLICATION OF INACCESSIBLE ALEPHS

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1. Introduction. Objects which are homogeneous in an appropriate sense play an essential role in contemporary mathematics as, e.g. algebraic structures, homogeneous spaces and linear differential operators with constant coefficients. This note is devoted to the strictly opposite objects: Let us call a topological space *totally inhomogeneous* if any sufficiently small neighbourhoods of any pair of distinct points are not mutually homeomorph. There is a little use of such a space and even its existence seems to be nontrivial. We shall construct, however, much more curious space:

2. Theorem. *There exists a non empty Hausdorff topological space M such that*

(i) *every point $x \in M$ has a base of neighbourhoods consisting of open-closet sets,*

(ii) *if N, N' are homeomorph open subsets of the space M, then $N = N'$ and the only homeomorphism between N, N' is the identity mapping.*

3. Notations. Let us take the sequence of all cardinal numbers

$$0, 1, 2, \ldots, \aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots,$$

where the infinite cardinals (alephs) are indexed by ordinal numbers

$$0, 1, 2, \ldots, \omega, \omega + 1, \ldots.$$

Both classes of ordinal and cardinal numbers are well-ordered by the known relation $\leqq$. Every cardinal number will be considered as a symbol of a class of mutually equipollent sets, e.g. elements in a given set are numbered by cardinals. In future we shall use only infinite cardinals.

Denote by $\omega(\alpha)$ the least ordinal of the property so that the number of ordinals preceding $\omega(\alpha)$ is exactly equal to $\aleph_\alpha$. Evidently $\omega(\alpha) \geqq \alpha$ and from the inequality $\omega(\alpha) > \alpha$ there follows $\aleph_{\omega(\alpha)} > \aleph_\alpha$ and hence $\omega(\omega(\alpha)) > \omega(\alpha)$. So we have the sequence of ordinals

$$0 < \omega(0)\, \omega < \omega(\omega(0)) < \omega(\omega(\omega(0))) < \ldots.$$

Suppose that their limit exists and denote them by ι.

Obviously $\omega(\iota) = \iota$ and the ordinal ι is the least ordinal with this property. The aleph $\aleph_\iota$ is called *inaccessible*.

4. Definition of the space M. Let M be the set of all ordinal numbers ξ such that $\xi < \iota$. We are to define a topology on the set M.

Take a point $\alpha \in M$. Choose an ordinal ϑ from the interval $\omega(\alpha) \leqq \vartheta < \omega(\alpha + 1)$ and let $N(\alpha, \vartheta)$ be the following set

$$N(\alpha, \vartheta) = Q_0 \cup Q_1 \cup (\bigcup_{i=2}^{\infty} Q_i),$$

where Q_0 is the one-point set $\{\alpha\}$, Q_1 is the set of all ordinals β lying in the interval $\vartheta \leqq \beta < \omega(\alpha + 1)$ and Q_{i+1} $(i \geqq 1)$ is the set of all ordinals β which satisfy some of the inequalities $\omega(\gamma) \leqq \beta < \omega(\gamma + 1)$, where the ordinal γ ranges over the set Q_i.

It can be easily verified that the intersection of any two sets of the form $N(\alpha_i, \vartheta_i)$ $(i = 1, 2)$ is either empty or is again a set of this kind (see also the following Lemma). Therefore the set M can be topologized in an obvious manner: *Open sets* of the space M are defined to be arbitrary sums $\bigcup_{\alpha \in I, \xi \in \mathscr{I}} N(\alpha, \vartheta_\xi)$ $(I \subset M, \mathscr{I}$ is an index set).

Evidently, all sets $N(\alpha, \vartheta)$ (α fixed, $\omega(\alpha) \leqq \vartheta \leqq \omega(\alpha + 1)$) form a base of open neighbourhoods at the point α.

5. Lemma. *If the set* $N(\alpha, \vartheta) \cap N(\alpha', \vartheta')$ *is non-empty, then either* $N(\alpha, \vartheta) \subset$ $\subset N(\alpha', \vartheta')$ *or* $N(\alpha, \vartheta) \supset N(\alpha', \vartheta')$.

Proof: Let β be the least ordinal from the intersection $N(\alpha, \vartheta) \cap N(\alpha', \vartheta')$, where $N(\alpha, \vartheta) = \bigcup_i Q_i$, $N(\alpha', \vartheta') = \bigcup_i Q'_i$. Suppose for the sake of certainty that $\beta \in Q_j \cap Q'_k$. The lemma evidently holds if either $j = 0$ or $k = 0$. But the supposition $j > 0$, $k > 0$ leads to contradiction because in this case there holds $\omega(\gamma) \leqq \beta < \omega(\gamma + 1)$, where $\gamma \in Q_j \cap Q'_k$, therefore $\gamma \in N(\alpha, \vartheta) \cap N(\alpha', \gamma')$ and at the same time $\gamma < \beta$.

6. Note on the topology of the space M. Preceding Lemma implies that M is a Hausdorff space and also that the sets $N(\alpha, \vartheta)$ are all both open and closed. So it remains to prove (ii).

7. Definition. The *weight* of a point $\alpha \in M$ is the least cardinal number $\aleph$ such that there exists a base of neighbourhoods of the point α whose number of elements is $\aleph$.

8. End of the proof. From the most fundamental properties of cardinal and ordinal numbers it follows that the weight of a point $\alpha \in M$ is exactly $\aleph_{\alpha+1}$. Different points have therefore different weights, too. Because the weight is a local topological invariant, two homeomorph open subsets of the space M must consist of the same points and any homeomorphism between them is identity.

9. Improvement of results. The number of elements of the topological space M constructed in the preceding proof is too large but the result can be improved by using some finest local topological invariants. Evidently there does not exist a finite

Hausdorff totally inhomogeneous space so that the best result which can be proved in this direction is:

10. Theorem. *There exists a countable Hausdorff topological space M with properties* (i), (ii) *from Theorem* 2.

11. An outline of the proof. We shall use some properties of powers of ordinals. Let ε be the limit of the sequence

$$\omega, \omega^{\omega}, \omega^{(\omega^{\omega})}, \omega^{(\omega^{(\omega^{\omega})})}, \ldots.$$

Evidently $\varepsilon^{\varepsilon} = \varepsilon$ and ε is the least infinite ordinal with this property. Let M be the set of all ordinal numbers ξ such that $\xi < \varepsilon$. Take a point $\alpha \in M$. Choose an ordinal ϑ from the interval $\omega^{\alpha} \leqq \vartheta < \omega^{\alpha+1}$ and let us put $N(\alpha, \vartheta) = \bigcup_i Q_i$ $(i = 0, 1, \ldots)$, where $Q_0 = \{\alpha\}$, Q_1 is the set of ordinals β satisfying $\vartheta \leqq \beta < \omega^{\alpha+1}$, Q_{i+1} is the set of all ordinals β satisfying some of the inequalities $\omega^{\gamma} \leqq \beta < \omega^{\gamma+1}$ $(\gamma \in Q_i)$. Again, open sets are defined to be sums of sets of the form $N(\alpha, \vartheta)$ and there holds an analogue of the Lemma 5, hence M is a Hausdorff space and satisfies also (i). For the proof of (ii) we are to introduce the following local topological invariant:

12. Definition. Suppose that M is a topological space such that every point $\alpha \in M$ has a neighbourhood N whose open subsets containing α are well ordered by set-theoretic inclusion $\subset$. Denote by $\tau(N)$ the corresponding ordinal number. The *depth* of the point α is then the least ordinal number $\tau(N)$ which can be obtained in this manner by an appropriate choice of the neighbourhood N.

13. The end of the proof. The depths are defined only in a very special case but our topological space M is exactly of this kind (use an analogue of Lemma 5). Obviously, the depth of a point $\alpha \in M$ is $\omega^{\alpha+1}$ and the further steps are analogous as in the paragraph 8.

14. Remark. By the process of β-compactification it may be proved that there exists a *compact space M* satisfying conditions of Theorem 2 because every point from the set $\beta M - M$ is of the weight $\geqq \aleph_1$ and these points are therefore invariant by any local homeomorphism. Question about the existence of countable and compact Hausdorff space M with properties (i), (ii) seems to be open.

REFERENCES

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