Jean Mawhin

Contractive mappings and periodically perturbed conservative systems

Archivum Mathematicum, Vol. 12 (1976), No. 2, 67--73

Persistent URL: http://dml.cz/dmlcz/106930

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XII: 67—74, 1976

CONTRACTIVE MAPPINGS AND PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

JEAN MAWHIN, Louvain-la-Neuve

(Received November 3, 1975)

1. INTRODUCTION

This paper is devoted to the use of Banach fixed point theorem [2, 17] in proving a mapping theorem for nonlinear operators of the form $L - N$ in a Hilbert space H, with L linear and N (possibly) nonlinear. The abstract result is then applied to give an elementary and direct proof of a result of Lazer and Sanchez [13] concerning the existence of periodic solutions for some periodically perturbed conservative systems.

The abstract mapping theorem is in the spirit of the work by Hammerstein [9], Golomb [7], Dolph [4], Kolodner [12], Ehrmann [5, 6] and others on nonlinear Hammerstein equations. In contrast with those papers, our result is formulated for mappings of the form L − N instead of Hammerstein mappings I − KN (K linear), emphasizing the fact that L needs not to be invertible. Moreover no compactness is required on the mappings which should make possible the use of this theorem in problems such that ker L is not finite-dimensional. Lastly the spectrum of L is not required to be discrete and, together with the classical theorem of fixed point for contractions, we use in the proof only basic facts of the theory of self-adjoint linear mappings.

The considered application to some periodic differential equations has its origin in Loud's work [16] on Duffing's equation

$$x'' + g(x) = E \cos t$$

with the derivative g' satisfying conditions of the form

$$(n + \delta)^2 \leqq g'(x) \leqq (n + 1 - \delta)^2$$

for some $\delta > 0$ and some positive integer n. Loud proved for this equation existence and uniqueness results by elementary but rather long arguments based upon the use of the operator of translation along solutions and the properties of the variational equations. Those results were partly extended by Leach [15] using the method of continuation and, like in Loud's papers, polar coordinates in the plane, which makes difficult the extension of their methods to systems. This extension was performed by

Lazer and Sanchez [13] using the alternative method [8] and a version of Brouwer fixed point theorem to solve the corresponding bifurcation equations. The uniqueness of the solution was only obtained later by Lazer [14] under more general conditions for which existence also holds, as shown recently by Ahmad [1] using the method of continuation and Lazer's paper [14]. Lastly, in the frame of the assumptions of Lazer and Sanchez [13], Kannan [10] has recently proved simultaneously the existence and uniqueness of the solution by the use of Cesari's alternative method [3] and an invariance of domain theorem. The result of Lazer and Sanchez, together with uniqueness, is given in this paper as a very simple application of our abstract mapping theorem and we obtain in this way a functional analytic proof based upon very simple arguments as well as the usual Picard iteration for getting approximate solutions. Of course, other boundary value problems for n-dimensional systems of the form

$$x'' + \operatorname{grad} G(x) = e(t),$$

including Neumann boundary conditions, could be treated similarly.

2. A MAPPING THEOREM IN HILBERT SPACE

Let H be a (real) Hilbert space with inner product $(,)$ and norm $|.|$, L: $\operatorname{dom} L \subset H \to H$ a linear, self-adjoint operator and $N : H \to H$ a mapping having on H a bounded linear Gâteaux derivative N' such that, for each x in H, $N'(x)$ is a symmetric operator. We shall denote respectively by $\varrho(A)$, $\sigma(A)$, $r_\sigma(A)$ the resolvant set, the spectrum and the spectral radius [11] of any linear operator A in H, and we shall write $A \geqq 0$ if and only if $(Ax, x) \geqq 0$ for every $x \in H$ and $A \geqq B$ if and only if $A - B \geqq 0$.

Theorem 1. *Suppose there exist real numbers $\lambda < \mu$ such that*

$$]\lambda, \mu[\subset \varrho(L), \qquad \lambda, \mu \in \sigma(L)$$

and real numbers p, q with

$$\lambda < q \leqq p < \mu$$

such that, for each $x \in H$,

$$qI \leqq N'(x) \leqq pI.$$

Then, $L - N$ is one-to-one,

$$(L - N)(\operatorname{dom} L) = H$$

and $(L - N)^{-1}$ is globally Lipschitzian.

Proof. If $\nu \in]\lambda, \mu[$ and $y \in H$, the equation

$$Lx - Nx = y \tag{1}$$

is clearly equivalent to

$$(L - \nu I)x - (N - \nu I)x = y,$$

i.e. to

$$Ax - Bx = y$$

if A: dom $L \subset H \to H$ is the linear self-adjoint operator defined by

$$A = L - \nu I$$

and $B : H \to H$ is the (Gâteaux) differentiable mapping

$$B = N - \nu I$$

with linear, bounded and symmetric Gâteaux derivative at $x \in H$

$$B'(x) = N'(x) - \nu I.$$

The proof consists now in the following four steps.

1. A^{-1} *exists, is bounded and* $|A^{-1}| = \max\{(\mu - \nu)^{-1}, (\nu - \lambda)^{-1}\}$.

By our assumptions, $\nu \in \varrho(L)$ and hence $A^{-1} = (L - \nu I)^{-1}$ exists and is bounded. Moreover,

$$A - \alpha I = L - (\nu + \alpha) I$$

and hence

$$]\lambda - \nu, \mu - \nu[\subset \varrho(A)$$

which implies, A being self-adjoint,

$$\sigma(A) \subset]-\infty, \lambda - \nu] \cup [\mu - \nu, +\infty[$$

But A is necessary closed and hence $\alpha \neq 0$ belongs to $\sigma(A)$ is and only if α^{-1} belongs to $\sigma(A^{-1})$ [11], which implies that

$$\sigma(A^{-1}) \subset [(\lambda - \nu)^{-1}, (\mu - \nu)^{-1}]$$

the boundary points of the interval belonging to $\sigma(A^{-1})$. Now A^{-1} is also self-adjoint and hence

$$|A^{-1}| = r_\sigma(A^{-1}) = \max\{(\nu - \lambda)^{-1}, (\mu - \nu)^{-1}\}.$$

2. *B is globally Lipschitzian with Lipschitz constant* $\gamma = \max(|p - \nu|, |q - \nu|)$.

Using the mean-value theorem [18], we have, if $x, x' \in H$,

$$|Bx - Bx'| \leqq |B'(x + \tau(x' - x))| \, |x - x'| \leqq \sup_{z \in H} |N'(z) - \nu I| \, |x - x'|$$

where $\tau \in]0, 1[$. But,

$$(q - \nu) I \leqq N'(z) - \nu I = B'(z) \leqq (p - \nu) I$$

and hence for each $x \in H$,

$$(q - \nu) | x |^2 \leqq (B'(z) x, x) \leqq (p - \nu) | x |^2$$

which implies, using the self-adjointness of $B'(z)$,

$$| B'(z) | = \sup_{|x|=1} | (B'(z) x, x) | \leqq \max (| q - \nu |, \ | p - \nu |) = \gamma.$$

3. *Equation* (1) *is uniquely solvable for each* $y \in H$.

Clearly equation (1) is equivalent to the fixed point problem

$$x = A^{-1}(Bx + y) \tag{2}$$

and we shall show that for a convenient choice of ν, the right-hand member of (2) is a contraction in H. If $x, x' \in H$,

$$\begin{aligned} | A^{-1}(Bx + y) - A^{-1}(Bx' + y) | &\leqq | A^{-1} | | Bx - Bx' | \leqq \\ &\leqq \max \{(\nu - \lambda)^{-1}, (\mu - \nu)^{-1}\} . \max (| q - \nu |, | p - \nu |) . | x - x' | = \\ &= \max (| q - \nu |, | p - \nu |) [\min (\nu - \lambda, \mu - \nu)]^{-1} | x - x' | \end{aligned}$$

Hence $A^{-1}(B(.) + y)$ will be a contraction if and only if

$$\max (| q - \nu |, | p - \nu |) < \min (\nu - \lambda, \mu - \nu)$$

which is easily shown equivalent to

$$(p + \lambda)/2 < \nu < (q + \mu)/2.$$

In particular, one can take $\nu = (p + q)/2$ or $\nu = (\lambda + \mu)/2$. For such a value of ν it follows directly from Banach fixed point theorem that equation (2), and hence equation (1) has an unique solution.

4. $(L - N)^{-1}$ *is globally Lipschitzian with constant* $(| A^{-1} |^{-1} - \gamma)^{-1}$.

If $y, y', x, x' \in H$ are such that

$$Lx - Nx = y, \qquad Lx' - Nx' = y',$$

then,

$$| x - x' | = | A^{-1}(Bx - Bx' + y - y') | \leqq | A^{-1} | (\gamma | x - x' | + | y - y' |)$$

and hence

$$\begin{aligned} |(L - N)^{-1} y - (L - N)^{-1} y' | = | x - x' | &\leqq | A^{-1} | (1 - \gamma | A^{-1} |)^{-1} | y - y' | = \\ &= (| A^{-1} |^{-1} - \gamma)^{-1} | y - y' |. \end{aligned}$$

3. PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

Let us consider the vector differential equation

$$x'' + \operatorname{grad} G(x) = y \tag{3}$$

where $G : R^n \to R$ is of class C^2, $y : R \to R^n$ is continuous and 2π-periodic, and where $H : R^n \to \mathscr{L}(R^n, R^n)$ is defined by

$$H(x) = \left(\frac{\partial^2 G}{\partial x_i \partial x_j}(x)\right) \qquad (i, j = 1, \ldots, n).$$

Theorem 2. *If there exist* $m \in N$ *and real numbers*

$$m^2 < r \leqq s < (m + 1)^2 \tag{4}$$

such that, for each $x \in R^n$,

$$rI \leqq H(x) \leqq sI \tag{5}$$

then equation (3) *has an unique* 2π*-periodic solution.*

Proof. Let H be the (Hilbert) space of (equivalence classes of) mappings x from $J = [0, 2\pi]$ into R^n such that

$$\| x(.) \|^2 = \sum_{i=1}^{n} x_i^2(.)$$

is (Lebesgue) integrable over J, with the inner product

$$(x, y) = (2\pi)^{-1} \int_0^{2\pi} [\sum_{i=1}^{n} x_i(t)\, y_i(t)]\, dt$$

and let

dom $L = \{x \in H : x$ is absolutely continuous and 2π-periodic as well as x' and $x'' \in H\}$,

$L : \operatorname{dom} L \subset H \to H, x \to x.''$

It is a known result that L is self-adjoint and that

$$\sigma(L) = \{-m^2 : m \in N\}.$$

Moreover, if $x, x' \in R^n$,

$$\| \operatorname{grad} G(x) - \operatorname{grad} G(x') \| \leqq \sup_{z \in R^n} \| H(z) \| \| x - x' \| \leqq k \| x - x' \|$$

with $\|.\|$ the Euclidian norm in R^n and some $k > 0$, and hence

$$\| \operatorname{grad} G(x) \| \leqq k \| x \| + k', \qquad x \in R^n,$$

which implies that the mapping N defined by

$$Nx = -\operatorname{grad} G(x(.))$$

maps H into itself. Hence, if we find a solution $x \in \operatorname{dom} L$ of equation

$$Lx - Nx = y, \tag{6}$$

it follows from the continuity of x, grad G and y that x'' will be continuous and 2π-periodic, and thus will be a classical 2π-periodic solution of (3). Conversely every classical 2π-periodic solution of (3) satisfies (6) and our problem is thus reduced to the unique solvability of (6). It is shown in [13] that N has a Gâteaux derivative given by

$$\big(N'(x)\,u\big)(t) = -H\big(x(t)\big)\,u(t) \qquad \text{a.e. in } [0,2\pi],$$

and hence $N'(x)$ is symmetric and bounded. Also, by (5),

$$-s\,(2\pi)^{-1}\int_0^{2\pi} \|\,u(t)\,\|^2\,\mathrm{d}t \leqq -(2\pi)^{-1}\int_0^{2\pi} u^{\mathrm{T}}(t)\,H\,(x(t))\;u(t)\,\mathrm{d}t \leqq$$

$$\leqq -r\,(2\pi)^{-1}\int_0^{2\pi} \|\,u(t)\,\|^2\,\mathrm{d}t,$$

which implies that

$$-(m+1)^2\,I < -sI \leqq N'(x) \leqq -rI < -m^2 I.$$

Thus the assumptions of Theorem 1 hold with $\lambda = -(m+1)^2$, $\mu = -m^2$, $q = -s$, $p = -r$ and the result follows from direct application of Theorem 1.

REFERENCES

[1] Sh. Ahmad: *An existence theorem for periodically perturbed conservative systems*, Michigan Math. J. 20 (1973) 385—392.

[2] S. Banach: *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundam. Math. 3 (1922) 133—181.

[3] L. Cesari: *Alternative methods in nonlinear analysis*, Proc. Int. Conf. Differential Equations, Los Angeles, Sept. 74, Academic Press, 1975, 95-148.

[4] C. L. Dolph: *Non-linear integral equations of the Hammerstein type*, Trans. Amer. Math. Soc. 66 (1949) 289—307.

[5] H. Ehrmann: *On implicit function theorems and the existence of solutions of non-linear equations*, L'Enseignement Math. 9 (1963) 129—176.

[6] H. Ehrmann and H. E. Lahmann: *Anwendungen des Schauderschen Fixpunktsatzes auf gewisse nichtlineare Integralgleichungen,* L'Enseignement Math. 11 (1965) 267—280.
[7] M. Golomb: *Zur Theorie der nichtlinearen Integralgleichungen, Integralgleichungssysteme und allgemeinen Funktionalgleichungen,* Math. Z. 39 (1935) 45—75.
[8] J. K. Hale: "*Applications of alternative problems*", Brown Univ. Lecture Notes 71—1, Providence, RI, 1971.
[9] A. Hammerstein: *Nichtlineare Integralgleichungen nebst Anwendungen,* Acta Math. 54 (1930) 117—176.
[10] R. Kannan: *Periodically perturbed conservative systems,* J. Differential Equations, 16 (1974) 506—514.
[11] T. Kato: "*Perturbation theory for linear operators*", Springer, Berlin, 1966.
[12] I. Kolodner: *Equations of Hammerstein type in Hilbert spaces,* J. Math. Mech. 13 (1964) 701—750.
[13] A. C. Lazer and D. A. Sanchez: *On periodically perturbed conservative systems,* Michigan Math. J. 16 (1969) 193—200.
[14] A. C. Lazer: *Application of a lemma on bilinear forms to a problem in nonlinear oscillations,* Proc. Amer. Math. Soc. 33 (1972) 89—94.
[15] D. E. Leach: *On Poincaré's perturbation theorem and a theorem of W. S. Loud,* J. Differential Equations 7 (1970) 34—53.
[16] W. S. Loud: *Periodic solutions of nonlinear differential equations of Duffing type,* Proc. U.S.-Japan Semin. on Diff. and Funct. Equations, Benjamin, New York, 1967, 199—224.
[17] N. Rouche et J. Mawhin: „*Equations différentielles ordinaires*", vol. 1, Masson, Paris, 1973.
[18] M. M. Vainberg: "*Variational method and method of monotone operators in the theory of nonlinear equations*", Wiley, New York, 1973.

J. Mawhin
Université de Louvain
Institut Mathématique
Chemin du Cyclotron, 2,
B-1348 Louvain-la-Neuve,
Belgium