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On evolution inclusions associated with time dependent convex subdifferentials*

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Abstract. In this paper we study evolution inclusions driven by a time varying convex subdifferential and a multivalued perturbation. We have two existence theorems; one for nonconvex valued perturbations and the other for convex valued perturbations. Then we compare those two solution sets ("relaxation theorem"). Our results extend earlier ones by Attouch—Damlamian, Aubin—Cellina, Moreau, Watanabe and Yotsutani.

Keywords: Convex subdifferential, evolution, maximal monotone operator, demiclosed operator, measurable selector, graph measurability, strong solution, relaxation theorem, weak norm

Classification: 34G20, 47H20

1. Introduction.

In this paper we study nonlinear evolution inclusions of the form

$$(*)\qquad \left\{\begin{array}{l} -\dot{x}(t) \in \partial\phi(t,x(t)) + F(t,x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array}\right\}$$

where $\phi(\cdot,\cdot)$ is a normal integrand convex in $x, \partial\phi(t,x)$ denotes the convex subdifferential of $\phi(t,\cdot)$ at x and $F(t,x)$ is a set valued perturbation.

Problems of this form have been studied by several authors, because they appear in various applications. Moreau [13] studied the case where $\phi(t,x) = \delta_{K(t)}(x)$ and no perturbation is present. Inclusions of that form appear in problems of theoretical mechanics. The case of a $\phi(\cdot)$ independent of t and of a single valued Lipschitz in x, perturbation $f(t,x)$ is treated in the monograph of Brezis[4]. Watanabe[20] considered the same problem, but with a time varying $\phi(\cdot,\cdot)$. The problem with multivalued perturbation was first considered by Attouch—Damlamian [1]. In their work, the integrand ϕ is independent of t and the multivalued perturbation has convex values. Their result was extended by Vrabie [18], who instead of the convex subdifferential $\partial\phi(x)$ considers a general m-accretive operator $A(x)$, while the multivalued perturbation is still convex valued.

In this paper, using the results of Kenmochi [11] and Yotsutani [21], we extend all the above mentioned works. On the one hand we allow ϕ to be time dependent and on the other hand we consider nonconvex valued perturbations. We also establish the existence of solutions for the case when the perturbation is convex valued and finally we compare the solution sets of the two problems ("relaxation theorem"). An example from control theory is also presented.

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2. Preliminaries.

Let (Ω, Σ) be a measurable space and X a separable Banach. A multifunction $F : \Omega \to 2^X \setminus \{0\}$ is said to be "graph measurable", if $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X. Let $\mu(\cdot)$ be a finite measure on (Ω, Σ). By S_F^1 we will denote the set of all Bochner integrable selectors of $F(\cdot)$, i.e. $S_F^1 = \{f \in L^1(X) : f(\omega) \in F(\omega)\mu - \text{a.e.}\}$. This set may be empty. It is nonempty if and only if $F(\cdot)$ is graph measurable and $\omega \to \inf\{\|x\| : x \in F(\omega)\}$ belongs in L^1_+.

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c.)(resp. lower semicontinuous (l.s.c.)) if and only if for every $U \subseteq Z$ open, $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ is open in Y (resp. $G^-(U) = \{y \in Y : G(y) \cap U \neq \emptyset\}$ is open in Y). If Y, Z are first countable, then the above definition of lower semicontinuity is equivalent to saying that if $y_n \to y$, then $G(y) \subseteq \underline{\lim} G(y_n) = \{z \in Z : z = \lim z_n, z_n \in A_n, n \geq 1\}$. When Z is a Banach space with the strong topology, we write $G(y) \subseteq s - \underline{\lim} G(y_n)$. Also if Z is regular and $G(\cdot)$ is closed valued, then upper semicontinuity implies that for all $y_n \to y$ in $Y, \overline{\lim} G(y_n) = \{z \in Z : z = \lim z_{n_k}, n_1 < n_2 < \cdots < n_k < \ldots\} \subseteq G(y)$. When Z is a Banach space with the weak topology, then we write that $w - \overline{\lim} G(y_n) \subseteq G(y)$. For details we refer to Delahaye—Denel [6].

Let $\phi : X \to \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$. We say that $\phi(\cdot)$ is proper if it is not identically $+\infty$. Assume that $\phi(\cdot)$ is proper, convex and l.s.c. (usually denoted by $\phi \in \Gamma_0(X)$). By $\operatorname{dom} \phi$ we will denote the effective domain of $\phi(\cdot)$, i.e. $\operatorname{dom} \phi = \{x \in X : \phi(x) < \infty\}$. Also $\partial\phi(x) = \{x^* \in X^* : (x^*, y - x) \leq \phi(y) - \phi(x), y \in \operatorname{dom} \phi\}$ is the subdifferential of $\phi(\cdot)$ at x. We say that $\phi(\cdot)$ is of compact type if for every $\lambda \in \mathbf{R}$, the level set $\{x \in X : \|x\|^2 + \phi(x) \leq \lambda\}$ is compact. For more information concerning those convex analytic concepts we refer to Laurent [12] and Rockafellar [16].

3. Nonconvex perturbation.

Let $T = [0, b]$ and H a separable Hilbert space. By a "strong solution" of $(*)$, we understand a function $x(\cdot) \in C(T, H)$ s.t. $x(\cdot)$ is strongly absolutely continuous on $(0, b), x(t) \in \operatorname{dom} \phi(t, \cdot)$ a.e. and satisfies $-\dot{x}(t) \in \partial\phi(t, x(t)) + f(t)$ a.e. , $x(0) = x_0$ with $f(\cdot) \in S^1_{F(\cdot, x(\cdot))}$. Recall that an absolutely continuous function from T into H is a.e. strongly differentiable (see for example Diestel—Uhl [7]).

The following hypothesis on $\phi(t, x)$ will be in effect throughout this work (see Yotsutani [21], hypothesis (A)).

$\underline{H(\phi)}$: $\phi : T \times H \to \bar{\mathbf{R}} = \mathbf{R} U \{+\infty\}$ is a function s.t.

(1) for every $t \in T, \phi(t, \cdot)$ is proper, convex, l.s.c. and of compact type,

(2) for any positive integer r, there exists a constant $K_r > 0$, an absolutely continuous function $g_r : T \to \mathbf{R}$ with $\dot{g}_r \in L^\beta(T)$ and a function of bounded variation $h_r : T \to \mathbf{R}$ s.t. if $t \in T, x \in \operatorname{dom} \phi(t, \cdot)$ with $\|x\| \leq r$ and $s \in [t, b]$, then there exists $\hat{x} \in \operatorname{dom} \phi(s, \cdot)$ satisfying

$$\|\hat{x} - x\| \leq |g_r(s) - g_r(t)|(\phi(t, x) + K_r)^\alpha$$

and $\phi(s, \hat{x}) \leq \phi(t, x) + |h_r(s)|(\phi(t, x) + K_r)$
where $\alpha \in [0, 1]$ and $\beta = 2$ if $\alpha \in [0, 1/2]$ or $\beta = 1/1 - \alpha$ if $\alpha \in [1/2, 1]$.

This hypothesis on $\phi(t,x)$ is more general than the corresponding ones in Watanabe [20] and Kenmochi [11], and was first used by Yotsutani [21].

In this section we prove an existence result for $(*)$, for the case where $F(t,x)$ is not convex valued. We will need the following hypothesis on the multifunction $F(\cdot,\cdot)$.

$\underline{H(F)_1}$: $F : T \times H \to P_f(H)$ is a multifunction s.t.

(1) $(t,x) \to F(t,x)$ is graph measurable,
(2) $x \to F(t,x)$ is l.s.c.,
(3) $|F(t,x| = \sup\{\|h\| : h \in F(t,x)\} \le \psi(t)$ a.e. for all $x \in X$, with $\psi(\cdot) \in L^2_+$.

Theorem 3.1. *If hypotheses $H(\phi); H(F)_1$ hold and $x_0 \in \operatorname{dom}\phi(0,\cdot)$, then $(*)$ admits a strong solution.*

PROOF : Let $h(\cdot) \in L^2(H)$ s.t. $\|h(t)\| \le \psi(t)$ a.e. and consider the evolution

$$(*)_h \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t,x(t)) + h(t) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}.$$

From the existence theorem of Yotsutani [21] (p. 626), we know that $(*)_h$ has a unique strong solution $p(h)(\cdot) \in C(T,H)$. Let $B(\psi) = \{h \in L^2(H) : \|h(t)\| \le \psi(t) \text{ a.e. }\}$ and set $W = p(B(\psi)) \subseteq C(T,H)$. We claim that W is compact in $C(T,H)$. To this end first we will show that W is equicontinuous. So for every $t,t' \in T, t < t'$ and every $x(\cdot) \in W$ we have:

$$\|x(t') - x(t)\| \le \int_t^{t'} \|\dot{x}(s)\|\, ds = \int_0^b \|\chi_{[t,t']}(s)\dot{x}(s)\|\, ds \le$$
$$\le \Big(\int_0^b \chi_{[t,t']}(s)^2\, ds\Big)^{1/2} \cdot \Big(\int_0^b \|\dot{x}(s)\|^2\, ds\Big)^{1/2} \le (t'-t)^{1/2} M$$

with M independent of $x \in W$, since $B(\psi)$ is a bounded subset of $L^2(H)$(see Yotsutani [21], Lemmata 5.3 and 6.1). Hence, W is indeed an equicontinuous subset of $C(T,H)$. Also if $t = 0$, we get $\|x(t')\| \le \bar{M}$ for all $x \in W$ and $t' \in T$.

Next we will show that for each $t \in T, W(t) = \{x(t) : x(\cdot) \in W\}$ is relatively compact in H. From inequality 7.9 of Yotsutani[21] and again since $B(\psi)$ is bounded in $L^2(H)$, we have for all $x(\cdot) \in W$:

$$\|x(t)\|^2 + \phi(t,x(t)) \le \bar{M}^2 + M_1(b + \|h\|_{TV} + \|\dot{g}\|_\beta + \|\psi\|_2) + \phi(0,x_0) = M_2,$$

where M_1 depends only on $\|\psi\|_2, \|x_0\|$ and $\phi(0,x_0)$ (see Yotsutani [21]) and $\|\cdot\|_{TV}$ denotes the total variation norm. So

$$W(t) \subseteq L(M_2) = \{x \in H : \|x\|^2 + \phi(t,x) \le M_2\}$$

and the larger set $L(M_2)$ is compact, since by hypothesis $H(\phi)(1), \phi(t,\cdot)$ is of compact type for every $t \in T$. Hence for all $t \in T, W(t)$ is relatively compact in H.

Finally we will show that W is closed in $C(T,H)$. So let $\{x_n\}_{n\geq 1} \subseteq W$ and assume that $x_n \to x$ in $C(T,H)$. By definition we have:

$$-\dot{x}_n(t) \in \partial\phi(t,x_n(t)) + h_n(t) \text{ a.e.}, x_n(0) = x_0, h_n \in B(\psi) n \geq 1.$$

From inequality of 7.5 of Yotsutani [21], we know that $\|\dot{x}_n\|_2 \leq M_3$ for all $n \geq 1$. Since in the Hilbert space $L^2(H)$ bounded sets are sequentially w-compact (Alaoglu and Eberlein—Smulian theorems), by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} y$ and $h_n \xrightarrow{w} h$ in $L^2(H)$. It is clear that $y = \dot{x}$. Note that from Lemmata 3.2 and 3.4 and inequality 7.8 of Yotsutani [21], we deduce that $\phi(\cdot, x_n(\cdot)) \in L^1$ for all $n \geq 1$. So from Lemma 4.4. of that paper we get that for all $n \geq 1 - \dot{x}_n(\cdot) - h_n(\cdot) \in \partial I_\phi(x_n(\cdot))$, where $I_\phi(z) = \int_0^b \phi(t, z(t))\, dt$ if $\phi(\cdot, z(\cdot)) \in L^1, +\infty$ otherwise. Recall that the convex subdifferential being maximal monotone, is demiclosed (see for example Barbu [3], Proposition 3.5, p. 75). Since $(x_n, -\dot{x}_n - h_n) \xrightarrow{sxw} (x, -\dot{x} - h)$ in $L^2(H) \times L^2(H)$, in the limit as $n \to \infty$ we get $-\dot{x}(\cdot) - h(\cdot) \in \partial I_\phi(x(\cdot))$. A new application of Lemma 4.4 of Yotsutani [21] (see also proposition 1.1 of Kenmochi [11]), tells us that $-\dot{x}(t) \in \partial\phi(t, x(t)) + h(t)$ a.e., $x(0) = x_0, h \in B(\psi)$. So W is closed in $C(T,H)$. Invoking the Arzela—Ascoli theorem, we conclude that W is compact in $C(T,H)$. Hence by Mazur's theorem (see Diestel—Uhl [7], Theorem 12, p. 51), we have that $\hat{W} = \overline{\text{conv}} W \subseteq C(T,H)$ is compact, too.

Next let $G : \hat{W} \to 2^{L^1(H)}$ be defined by $G(x) = S^1_{F(\cdot,x(\cdot))}$. Clearly, since $F(\cdot, x(\cdot))$ is closed valued, so is $G(\cdot)$. Also we claim that for all $x \in \hat{W}, G(x) \neq \emptyset$. To this end, consider the map $k : T \to T \times H$ defined by $k(t) = (t, x(t))$. Clearly this is measurable. Then let $\theta : T \times H \to T \times H$ be defined by $\theta(t,y) = (k(t), y)$, i.e. $\theta(\cdot) = (k(\cdot), id(\cdot))$. So $\theta(\cdot)$ is measurable. Now observe that:

$$\begin{aligned} & GrF(\cdot, x(\cdot)) = \{(t,y) \in T \times H : (k(t), y) \in GrF\} \\ \Longrightarrow & GrF(\cdot, x(\cdot)) = \{(t,y) \in T \times H : \theta(t,y) \in GrF\} \\ \Longrightarrow & GrF(\cdot, x(\cdot)) = \theta^{-1}(GrF). \end{aligned}$$

But by hypothesis $H(F)_1(1), GrF \in B(T) \times B(H)$, while we just saw that $\theta(\cdot,\cdot)$ is measurable. So $\theta^{-1}(GrF) \in B(T) \times B(Y)$. Hence by Aumann's selection theorem (see Wagner [19]), we have that $F(\cdot, x(\cdot))$ admits a measurable selector, which because of hypothesis $H(F)_1(3)$ belongs in $L^2(H)$. Thus $G(x) \neq \emptyset$ for all $x \in \hat{W}$.

Next we will show that $G(\cdot)$ is l.s.c.. So let $x_n \to x$ in $\hat{W}$. Because by hypothesis $H(F)_2(2) F(t,\cdot)$ is l.s.c. on H, from Theorem 4.1 of [14], we have $G(x) \subseteq s-\underline{\lim} G(x_n) \Longrightarrow G(\cdot)$ is l.s.c.. Apply Fryszkowski's continuous selection theorem [10] to get $g : \hat{W} \to L^1(H)$ continuous s.t. $g(x) \in G(x)$ for all $x \in \hat{W}$. For $x(\cdot) \in \hat{W}$ consider the following Cauchy problems:

$$\left\{ \begin{array}{l} -\dot{y}(t) \in \partial\phi(t, y(t)) + g(x)(t) \text{ a.e.} \\ y(0) = x_0 \end{array} \right\}.$$

From Yotsutani [21], we know that this problem has a unique solution $q(x)(\cdot) \in \hat{W}$. Then $\hat{W} \to \hat{W}$ and we claim that it is continuous. So let $x_n \to x$ in $\hat{W}$. Then because of the continuity of the selector $g(\cdot)$, we have $g(x_n)(\cdot) \xrightarrow{s} g(x)(\cdot)$ in $L^2(H)$. Also $-\dot{y}_n(\cdot) - g(x_n)(\cdot) \in \partial I_\phi(y_n)$. Since $\{y_n\}_{n\geq 1} \subseteq \hat{W}$ and the latter is compact in $C(T,H)$, by passing to a subsequence if necessary, we may assume that $\dot{y}_n \xrightarrow{w} \dot{y}$ in $L^2(H)$. So in the limit as $n \to \infty$, we get $(y, -\dot{y} - g(x)) \in Gr\partial I_\phi \Longrightarrow -\dot{y}(t) \in \partial\phi(t, x(t)) + g(x)(t)$ a.e., $y(0) = x_0 \Longrightarrow y = q(x)$. So $q(\cdot)$ is indeed continuous on $\hat{W}$. Apply Schauder's fixed point theorem to get $x \in \hat{W}$ s.t. $\hat{x} = q(\hat{x})$. Clearly this $\hat{x}(\cdot) \in C(T,H)$ is the desired solution of $(*)$. ∎

4. Convex perturbation.

In this section we prove an existence theorem for the case where the multivalued perturbation is convex valued. The hypothesis on $F(\cdot,\cdot)$ is now the following:

$\underline{H(F)_2}$: $F : T \times H \to P_{fc}(H)$ is a multifunction s.t.

(1) $t \to F(t,x)$ is graph measurable,
(2) $x \to F(t,x)$ is u.s.c. from H into H_w (here H_w denotes the space H with the weak topology),
(3) $|F(t,x)| \leq \psi(t)$ a.e. for all $x \in H$ and with $\psi(\cdot) \in L^1_+$.

Theorem 4.1. *If hypotheses $H(\phi), H(F)_2$ hold and $x_0 \in \operatorname{dom}\phi(0,\cdot)$, then $(*)$ admits a strong solution.*

PROOF : Again let $B(\psi) = \{h \in L^2(H) : \|h(t)\| \leq \psi(t) \text{ a.e.}\}$ and for $h \in B(\psi)$ consider the following evolution equation:

$$(*)_h \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t, x(t)) + h(t) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}$$

We know that this has a unique solution $p(h)(\cdot) \in \hat{W} \subseteq C(T,H)$ where $\hat{W}$ is as in the proof of Theorem 3.1. Consider the multifunction $L : B(\psi) \to 2^{L^1(H)}$ defined by $L(h) = S^1_{F(\cdot,p(h))(\cdot))}$. Let $s_n(\cdot)$ be simple function s.t. $s_n(t) \xrightarrow{s} p(h)(t)$ a.e.. Since by hypothesis $H(F)_2(2), F(t,\cdot)$ is u.s.c. from H into H_w, we have that $w - \overline{\lim} F(t, s_n(t)) \subseteq F(t, p(h)(t))$ a.e.. Hence Theorem 4.2 of [14] tells us that $w - \overline{\lim} S^1_{F(\cdot,s_n(\cdot))} \subseteq S^1_{F(\cdot,p(h)(\cdot))}$. Since for each $n \geq 1, s_n(\cdot)$ is a simple function and by hypothesis $H(F)_2(1) F(\cdot, x)$ is graph measurable, $t \to F(t, s_n(t))$ is graph measurable and so by Aumann's selection theorem it has measurable selectors which by $H(F)_2(3)$ belong in $L^2(H)$. Also since for every $n \geq 1, S^1_{F(\cdot,s_n(\cdot))} \subseteq B(\psi)$ and the latter is w-compact in $L^2(H)$, we deduce that $w - \overline{\lim} S^1_{F(\cdot,s_n(\cdot))} \neq \emptyset \Longrightarrow S^1_{F(\cdot,p(h)(\cdot))} \neq \emptyset$. Hence $L(h) \neq \emptyset$ for all $h \in B(\psi)$ and in fact it is easy to check that $L(h)$ is closed and convex since $F(\cdot,\cdot)$ is $P_{fc}(H)$-valued. Let $B(\psi)_w$ denote $B(\psi)$ with the relative $L^2(H)$-topology. We claim that $L(\cdot)$ is u.s.c. from $B(\psi)_w$ into itself. Given that $B(\psi)_w$ is sequentially w-compact in $L^2(H)$, it suffices to show that GrL is sequentially closed in $B(\psi)_w \times B(\psi)_w$ (see Aubin—Cellina [2] and Delahaye—Denel [6]). So let $\{(h_n, f_n)\}_{n\geq 1} \subseteq GrL, (h_n, f_n) \xrightarrow{w \times w} (h, f)$ in

$L^2(H) \times L^2(H)$. Since for each $n \geq 1, p(h_n) \in \hat{W}$ and the latter is compact in $C(T,H)$, by passing to a subsequence if necessary, we may assume that $p(h_n) \to q$ in $C(T,H)$ and as before exploiting the demiclosedness of $\partial I_\phi(\cdot)$ and the uniqueness of the solution of $(*)_h$, we can easily see that $q = p(h)$. Using Theorem 3.1 of [14], we have $f(t) \in \overline{\text{conv}}w - \overline{\lim}F(t,p(h_n)(t)) \subseteq F(t,p(h)(t))$ a.e. $\Longrightarrow f \in S^1_{F(\cdot,p(h)(\cdot))} \Longrightarrow$ $(h,f) \in GrL \Longrightarrow L(\cdot)$ is u.s.c.. Apply the Kakutani—KyFan fixed point theorem to get $\hat{h} \in B(\psi)$ s.t. $L(\hat{h}) = \hat{h}$. Clearly then $p(\hat{h})(\cdot) \in C(T,H)$ is the desired strong solution of $(*)$. ■

5. Relaxation.

To the multivalued Cauchy problem $(*)$, we associate the one with convexified dynamics, i.e.

$$(*)_c \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in \partial\phi(t,x(t)) + \overline{\text{conv}}F(t,x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}.$$

Let $S(x_0) \subseteq C(T,H)$ be the solution of $(*)$ and $S_c(x_0) \subseteq C(T,H)$ the solution set of $(*)_c$. It is natural to ask how much we increase $S(x_0)$ by convexifying the orientor field. We have a nice answer to this question for a large class of evolution inclusions that appear often in applications (in particular in control theory).

For this we will need the following hypotheses. Here Y is a separable Banach space

$\underline{H(g)_1}$: $g : T \times H \times Y \to H$ is a map s.t.

(1) $(t,u) \to g(t,x,u)$ is measurable,

(2) for every $u \in U =$ bounded subset of $Y, \|g(t,x',u) - g(t,x,u)\| \leq k_U(t)\|x' - x\|$ a.e. with $k_U(\cdot) \in L^1_+$ and $x', x \in X$,

(3) $\|g(t,x,u)\| \leq a(t) + b(t)\|u\|$ for all $x \in X$, with $a(\cdot), b(\cdot) \in L^2_+$.

$\underline{H(U)_1}$: $U : T \to 2^Y \setminus \{\emptyset\}$ is a graph measurable function s.t. $U(t) \subseteq V \in P_{wkc}(Y)$ a.e..

Set $F(t,x) = \cup\{g(t,x,u) : u \in U(t)\}$. This will be the orientor field in our evolution inclusion $(*)$. Note that because of Aumann's selection theorem and hypothesis $H(U), S^1_U \neq \emptyset$. Let $u \in S^1_U$ and define $\hat{g}(t,x) = g(t,x,u(t))$. Because of hypothesis $H(g), g(\cdot,\cdot)$ is measurable in t, Lipschitz in X and $\|\hat{g}(t,x)\| \leq a(t) + b(t)|V| = \psi(t)$ a.e., $\psi(t) \in L^2_+$. Then by Theorem 3.1 of the evolution equation $\dot{x}(t) \in \partial\phi(t,x(t)) + \hat{g}(t,x(t))$ a.e., $x(0) = x_0$ has a solution and so $S(x_0) \neq \emptyset$. Furthermore since clearly $S(x_0) \subseteq S_c(x_0)$, we also have $S_c(x_0) \neq \emptyset$.

Theorem 5.1. *$\underline{\text{If}}$ hypotheses $H(\phi), H(g)_1, H(U)_1$ hold and $x_0 \in \text{dom}\,\phi(0,\cdot)$, $\underline{\text{then}}$ $\emptyset \neq \overline{S(x_0)} = \overline{S_c(x_0)}$ both closures taken in $C(T,H)$.*

PROOF : Let $x(\cdot) \in S_c(x_0)$ and consider $\hat{F}(t) = F(t,x(t))$. We have:

$$\begin{aligned} Gr\hat{F} &= \{(t,z) \in T \times H : z \in \hat{F}(t) = F(t,x(t))\} \\ &= \{(t,z) \in T \times H : z = g(t,x(t),u) \text{ for some } u \in U(t)\} \\ &= \text{proj}_{T\times H}[\{(t,z,u) \in T \times H \times V : z = g(t,x,u)\} \cap r(GrU \times H)] \end{aligned}$$

where $r : T \times Y \times H \to T \times H \times Y$ is defined by $r(t,u,z) = (t,z,u)$. Since $U(\cdot)$ is graph measurable, $r(GrU \times H) \in B(T) \times B(H) \times B(V)$, while because of hypothesis $H(g)_1$, we have that $\{(t,z,u) \in T \times H \times V : z = g(t,x(t),u)\} \in B(T) \times B(H) \times B(V)$. So $\{(t,z,u) \in T \times H \times V : z = g(t,x,u)\} \cap r(GrU \times H) \in B(T) \times B(H) \times B(V)$. Note that since Y is separable, Theorem 3. p. 434 of Dunford—Schwartz [**8**] tells us that V with the relative weak topology, denoted by V_w is compact, metrizable. Also since Y is separable, it admits a Kadec norm and so by Corollary 2.4 of Edgar [**9**] we have $B(Y_w) = B(Y) \Longrightarrow B(V_w) = B(Y_w) \cap V = B(Y) \cap V = B(V)$. So finally using the Arsenin—Novikov theorem (see Saint—Beuve [**17**]), we get that $\text{proj}_{T\times H}[\{(t,z,u) \in T \times H \times V : z = g(t,x,u)\} \cap r(GrU \times H)] \in B(T) \times B(H) \Longrightarrow Gr\hat{F} \in B(T) \times B(H)$. So we can apply Theorem 2 of Chuong [**5**] to get that $S^1_{F(\cdot,x(\cdot))}$ is dense in $S^1_{\overline{\text{conv}}F(\cdot,x(\cdot))}$ for the weak norm $|\cdot|_w$ on $L^1(H)$ defined by $|f|_w = \sup\{\|\int_t^{t'} f(s)\,ds\| : 0 \le t \le t' \le b\}$. So if $x(\cdot) = p(h)(\cdot)$ with $h \in S^1_{\overline{\text{conv}}F(\cdot,x(\cdot))}$, we can find $h_n \in S^1_{F(\cdot,x(\cdot))}$ s.t. $h_n \xrightarrow{|\cdot|_w} h$ as $n \to \infty$. Let $L_n(t) = \{u \in U(t) : h_n(t) = g(t,x(t),u)\}$. Using hypotheses $H(g)$ and $H(U)$, it is easy to check that $GrL_n \in B(T) \times B(Y)$. Apply Aumann's selection theorem to find $u_n : T \to Y$ measurable s.t. $u_n(t) \in L_n(t)$ for all $t \in T$. Then $h_n(t) = g(t,x(t),u_n(t))$ a.e. $n \ge 1$.

Next for every $n \ge 1$ consider the following evolution equation:

$$\left\{ \begin{array}{l} -\dot{x}_n(t) \in \partial\phi(t,x_n(t)) + g(t,x_n(t),u_n(t)) \text{ a.e.} \\ x_n(0) = x_0 \end{array} \right\}$$

We know (see for example Theorem 3.1), that this has at least one solution $x_n(\cdot) \in C(T,H)$. We claim that this solution is unique. To see this suppose $v_n(\cdot) \in C(T,H)$ is another solution. Making use of the monotonicity of the subdifferential operator, we have

$$\begin{aligned} &0 \le (-\dot{x}_n(t) - g(t,x_n(t),u_n(t)) + \dot{v}_n(t) + g(t,v_n(t),u_n(t)), x_n(t) - v_n(t)) \text{ a.e.} \\ &\Longrightarrow \frac{1}{2}\frac{d}{dt}\|x_n(t) - v_n(t)\|^2 \le (g(t,v_n(t),u_n(t)) - g(t,x_n(t),u_n(t)), x_n(t) - v_n(t)) \text{ a.e.} \\ &\Longrightarrow \|x_n(t) - v_n(t)\|^2 \le 2\int_0^t (g(s,x_n(s),u_n(s)) - g(s,v_n(s),u_n(s)), x_n(s) - v_n(s))\,ds \\ &\qquad \le 2\int_0^t k_V(s)\|x_n(s) - v_n(s)\|^2\,ds. \end{aligned}$$

Invoking Gronwall's inequality, we deduce that $x_n = v_n$ for all $n \ge 1$. So indeed the solution is unique. Note that $\{x_n\}_{n\ge1} \subseteq \hat{W}$ (see the proof of Theorem 3.1). So by passing to a subsequence if necessary, we may assume that $x_n \to y$ in $C(T,H)$. Let $z_n(\cdot) \in \hat{W} \subseteq C(T,H)$ be the unique solution of

$$\left\{ \begin{array}{l} -\dot{z}_n(t) \in \partial\phi(t,z_n(t)) + h_n(t) \text{ a.e.} \\ z_n(0) = x_0 \end{array} \right\}.$$

So $z_n = p(h_n)$. We claim that since $h_n \xrightarrow{|\cdot|_w} h$, then $h_n \xrightarrow{w} h$ in $L^2(H)$. To see this let $s \in L^2(H)$ be a step function. Then we have

$$|(s, h_n - h)| \leq \sum_{k=1}^{n} |\int_{t_{k-1}}^{t_k} (h_n(s) - h(s))\, ds| \cdot \|s_k\| \leq |h_n - h|_w \cdot \sum_{k=1}^{n} \|s_k\| \to 0.$$

Since step functions are dense in $L^2(H)$, we get our claim. Then since $h_n \xrightarrow{w} h$ in $L^2(H)$, as before exploiting the demiclosedness of $\partial I_\phi(\cdot)$, we have $z_n = p(h_n) \to p(h) = x$ in $C(T, H)$. Then we have:

$$\begin{aligned}
&0 \leq (-\dot{x}_n(t) - g(t, x_n(t), u_n(t)) + \dot{z}_n(t) + h_n(t), x_n(t) - z_n(t)) \text{ a.e.} \\
\Longrightarrow &\frac{1}{2}\frac{d}{dt}\|x_n(t) - z_n(t)\|^2 \leq (g(t, x(t), u_n(t)) - g(t, x_n(t), u_n(t)), x_n(t) - z_n(t)) \\
\Longrightarrow &\|x_n(t) - z_n(t)\|^2 \leq \int_0^t k_V(s)\|x(s) - x_n(s)\| \cdot \|x_n(s) - z_n(s)\|\, ds \\
\Longrightarrow &\|y(t) - x(t)\|^2 \leq \int_0^t k_V(s)\|y(s) - x(s)\|^2\, ds \\
\Longrightarrow &x = y \\
\Longrightarrow &x_n \to x \text{ in } C(T, H) \text{ and } x_n \in S(x_0) n \geq 1.
\end{aligned}$$

So we conclude that $\overline{S(x_0)} = \overline{S_c(x_0)}$ the closures in $C(T, H)$. ■

Strengthening our hypotheses on the data, we can guarantee that $S_c(x_0)$ is closed, hence compact in $C(T, H)$.

$\underline{H(g)_2}$: $g : T \times H \times Y$ is a map st.

(1) $t \to g(t, x, u)$ is measurable,
(2) for all $u \in U$ = a bounded set of Y, $\|g(t, x', u) - g(t, x, u)\| \leq k_U(t)\|x' - x\|$ a.e. with $k_U(\cdot) \in L^1_+$ and $x, x' \in H$,
(3) $(x, u) \to g(t, x, u)$ is sequentially continuous from $H \times Y_w$ into H_w,
(4) $\|g(t, x, u)\| \leq a(t) + b(t)\|u\|$ for all $x \in H$, with $a(\cdot), b(\cdot) \in L^2_+$.

$\underline{H(U)_2}$: $U : T \to P_{wk}(Y)$ is a graph measurable function s.t. $U(t) \subseteq V \in P_{wkc}(Y)$ a.e..

Theorem 5.2. *If hypotheses $H(\phi), H(g)_2, H(U)_2$ hold and $x_0 \in \operatorname{dom}\phi(0,\cdot)$ then $S_c(x_0)$ is compact in $C(T, H)$ and $\emptyset \neq \overline{S(x_0)} = S_c(x_0)$ the closure taken in $C(T, H)$.*

PROOF : Since $S_c(x_0) \subseteq \hat{W} \in P_{kc}(C(T, H))$ and having Theorem 5.1, all we have to show is that $S_c(x_0)$ is closed in $C(T, H)$. To this end, let $\{x_n\}_{n\geq 1} \subseteq S_c(x_0)$ and assume that $x_n \to x$ in $C(T, H)$. Then we have

$$\left\{ \begin{array}{l} -\dot{x}_n(t) \in \partial\phi(t, x_n(t)) + h_n(t) \text{ a.e.} \\ x_n(0) = x_0 \end{array} \right\}$$

with $S^1_{\overline{\text{conv}}F(\cdot,x_n(\cdot))}$. Observe that for all $n \geq 1, h_n \in B(\psi), \psi(\cdot) = a(\cdot) + b(\cdot)|V| \in L^2_+$. So by passing to a subsequence if necessary, we may assume that $h_n \xrightarrow{w} h$ in $L^2(H)$. From Theorem 3.1. of [**14**], we have $h(t) \in \overline{\text{conv}}w - \overline{\lim}F(t, x_n(t))$ a.e.. We claim that $F(t, \cdot)$ is u.s.c. from H into H_w. Observe that $F(t,x) \subseteq B(\phi)(t) = \{z \in H : \|z\| \leq \psi(t)\}$ a.e. for all $x \in X$, and the latter is w-compact in H. So in order to prove our claim, it is enough to show that $GrF(t, \cdot)$ is sequentially closed in $H \times H_w$. Hence let $(x_n, z_n) \in GrF(t, \cdot) n \geq 1, (x_n, z_n) \xrightarrow{sxw} (x, z)$ in $H \times H$. Then $z_n = g(t, x_n, u_n), u_n \in U(t)$. By passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u \in U(t)$. Then from hypothesis $H(g)_2(3)$ we get that $z_n = g(t, x_n, u_n) \xrightarrow{w} g(t, x, u) = z \Longrightarrow (x, z) \in GrF(t, \cdot) \Longrightarrow GrF(t, \cdot)$ is sequentially closed in $H \times H_w \Longrightarrow F(t, \cdot)$ is indeed u.s.c. from H into H_w. Thus we get $h(t) \in \overline{\text{conv}}F(t, x(t))$ a.e. $\Longrightarrow x = p(h) \in S_c(x_0) \Longrightarrow S_c(x_0)$ is closed in $C(T, H)$. ∎

Remarks. Suppose that $K : T \to P_{fc}(H)$ is an absolutely continuous multifunction with modulus $m(\cdot) \in L^1_+$, i.e. $|d(x, K(t)) - d(y, K(\tau))| \leq \|x - y\| + \int_\tau^t m(s)\,ds$ (where for any $A \in 2^H \setminus \{\emptyset\}, d(x, A) = \inf_{\alpha \in A} \|x - a\|$). Set $\phi(t, x) = \delta_{K(t)}(x)$ where $\delta_{K(t)}(x) = 0$ if $x \in K(t), +\infty$ otherwise. Thanks to the absolute continuity hypothesis on $K(\cdot)$, it is easy to check that $\phi(\cdot, \cdot)$ satisfies $H(\phi)$ (in this case $\dot{g}_r(\cdot) = m(s), \beta = 1, h_r = 0$). Also from convex analysis (see Laurent [**12**] and Rockafellar[**16**]), we know that $\partial\phi(t, x) = \partial\delta_{K(t)}(x) = N_{K(t)}(x) =$ the normal cone to $K(t)$ at x. So the evolution inclusion $(*)$ takes the following form:

$$(*)' \qquad \left\{ \begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}$$

Hence $(*)'$ is a special case of $(*)$. Inclusions of the form $(*)'$ were studied by Moreau [**13**] (with $F = 0$, i.e. no perturbation), Aubin—Cellina [**2**] (with $H = \mathbf{R}^n, K(t) = K$ for all $t \in T$ and $F(\cdot, \cdot)$ convex valued; they are called "differential variational inequalities") and by Papageorgiou [**15**] (with $H = \mathbf{R}^n$). All these works are extended by the present paper, as well as those of Watanabe [**20**], Yotsutani [**21**] and the ones mentioned in the introduction. So the results of this paper unify and extend a series of recent works on evolution inclusions.

6. Example.

Let Z be a bounded domain in $\mathbf{R}^n$ with smooth boundary ∂Z. Consider the following parabolic control problem:

$$(**) \quad \left\{ \begin{array}{l} \dfrac{\partial x(t,z)}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial z_j}\Big(a_{ij}(t,z)\frac{\partial x(t,z)}{\partial z_i}\Big) + \beta(x(t,z)) \ni f(t,z,x(t,z),u(t,z)) \\ x(0,z) = x_0(z), x(t,z)_\Gamma = 0 \text{ and } u(t,z), u\text{-measurable} \end{array} \right\}$$

Assume that $a_{ij} \in L^\infty(T \times Z), a_{ij} = a_{ji}, \sum_{i,j=1}^n a_{ij}(t,z)\eta_i\eta_j \geq c\|\eta\|^2$ for every $(t,z) \in T \times Z$ and every $\eta \in \mathbf{R}^n$ and $|a_{ij}(t,z) - a_{ij}(s,z)| \leq k|t - s|$ a.e. on $Z, k \geq 0$. Also $\beta = \partial j$ where $j : \mathbf{R} \to \bar{\mathbf{R}}_+$ is proper, l.s.c. and convex.

Let $H = L^2(Z)$ and define $\phi : T \times H \to \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ by

$$\phi(t,x)=\begin{cases} \sum_{i,j=1}^{n} \int_Z a_{ij}(t,z)\frac{\partial x}{\partial z_i}\frac{\partial x}{\partial z_j}\,dz + \int_Z j(x(z))\,dz \text{ if } x\in H_0^1(Z), j(x(\cdot)) \in L^1(Z) \\ +\infty \text{ otherwise.} \end{cases}$$

It is easy to check that $\phi(t,\cdot)$ is proper, convex and l.s.c.. Also note that since $j \geq 0$ for each $\lambda > 0$, the level set $L_\lambda = \{x \in L^2(Z) : \|x\|_2^2 + \phi(t,x) \leq \lambda\}$ is bounded in $L^2(Z)$ while from our strong ellipticity hypothesis $\{|\nabla x| : x \in L_\lambda\}$ is bounded in $L^2(Z)$. Therefore L_λ is compact in $L^2(Z)$ and so we have that $\phi(t,\cdot)$ is of compact type. Let $\hat{x}_0 = x_0(\cdot) \in H_0^1(Z)$.

Next using Poincaré's inequality we can check that $\phi(t,x) \geq \hat{c}\|x\|^2_{H_0^1(Z)}\hat{c} > 0$ and that $|\phi(t,x) - \phi(s,x)| \leq \sum_{i,j=1}^{n} \int_Z |a_{ij}(t,z) - a_{ij}(s,z)||\frac{\partial x}{\partial z_i}||\frac{\partial x}{\partial z_j}|\,dz \leq \hat{k}|t-s| \cdot \|x\|^2_{H_0^1(Z)} \leq \hat{k}|t-s|\phi(t,x)$. So $\phi(t,x)$ satisfies hypothesis $H(\phi)$. Furthermore as in Barbu [3], we can check that $\partial\phi(t,x) = \{\sum_{i,j=1}^{n} \frac{\partial}{\partial z_j}(a_{ij}(t,z)\frac{\partial x}{\partial z_i}) + g(z) : g \in L^2(Z), g(z) \in \beta(x(z)) \text{ a.e. }\}$. Assume $f(t,z,x,u) = g(t,z,x) + \int_Z r(t,z,z',u(z'))\,dz'$ with $g : T \times Z \times \mathbf{R} \to \mathbf{R}$. Caratheodory function s.t. $|g(t,z,x)| \leq a(t,z)$ a.e. $a(\cdot,\cdot) \in L^2(T \times Z)$, and $r : T \times Z \times Z \times \mathbf{R} \to \mathbf{R}$. Caratheodory s.t. $r(t,z,z',u)| \leq R(t,z,z')(a + b|u|)$ a.e. with $R(\cdot,\cdot,\cdot) \in L^2(T \times Z \times Z)$. Let $\hat{g} : T \times L^2(Z) \to L^2(Z)$ and $\hat{h} : T \times L^2(Z)$ be the Nemitsky operators corresponding to $g(t,z,x)$ and $h(t,z,u) = \int_Z r(t,z,z',u(z'))\,dz'$ respectively. We know $g(t,\cdot)$ is continuous while $\hat{h}(t,\cdot)$ is completely continuous (Krasnoselski—Ladyzenski *theorem*). *Thus* $\hat{f}(t,x,u)$ satisfies hypothesis $H(g)_2$. Finally let $U(t,z) = [m_1(t,z), m_2(t,z)]$ *with* $m_1, m_2 \in L^\infty(T\times Z), m_1 < m_2$. Let $\hat{U}(t) = \{u \in L^2(Z) : m_1(t,z) \leq u(z) \leq m_2(z)\}$. Then $\hat{U}(\cdot)$ is measurable with $P_{wkc}(L^2(Z))$-values and $\hat{U}(t) \subseteq V = \{u \in L^2(Z) : \|u\|_2 \leq \max(\|m_1\|_\infty, \|m_2\|_\infty)b^{1/2}\}$.

We can now write $(**)$ in the following abstract form:

$$(\hat{**}) \qquad \left\{\begin{array}{l} -\dot{x}(t) \in \partial\phi(t,x(t)) + \hat{f}(t,x(t),u(t)) \text{ a.e.} \\ x(0) = \hat{x}_0, u(t) \in \hat{U}(t) \text{ a.e. } u(\cdot) \in L^2(T, L^2(Z)) \end{array}\right\}$$

To those control system we associate its "relaxed" version with convexified dynamics; namely the system

$$(\hat{**})_r \qquad \left\{\begin{array}{l} -\dot{x}(t) \in \partial\phi(t,x(t)) + \overline{\mathrm{conv}}\hat{f}(t,x(t),U(t)) \text{ a.e.} \\ x(0) = \hat{x}_0 \end{array}\right\}$$

Now suppose we are given a cost functional $\theta : L^2(T,H) = L^2(T \times Z) \to \mathbf{R}$ defined by $\theta(x) = \int_Z L(z, x(b,z))\,dz$ (terminal cost), with $L : Z \times \mathbf{R} \to \bar{\mathbf{R}}$ proper, l.s.c. and $L(z,x) \geq \phi(z) - M|x|$ a.e. $\phi(\cdot) \in L^1(Z), M > 0$. Then $\theta(\cdot)$ is l.s.c. (Fatou's lemma). If we minimize $\theta(\cdot)$ over the set $S(\hat{x}_0)$ of trajectories of $(\hat{**})$, we need not have a solution, since $S(x_0)$ need not be compact. However minimizing it over the set of trajectories of $(\hat{**})_r$ we will have an optimal solution since $S_r(\hat{x}_0)$ is compact in $C(T,H)$, hence in $L^2(H)$ too (see Theorem 5.1). Furthermore the values of the two problems are equal since $\overline{S(\hat{x}_0)} = S_r(x_0)$ in $C(T,H)$ (Theorem 5.2). So by convexifying the dynamics we captured the asymptotic behavior of the minimizing sequences of the original optimal control problem.

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