Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 3, 511--515

Persistent URL: http://dml.cz/dmlcz/106886

Remarks on some nonlinear Dirichlet problems with unbounded nonlinearties*

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Abstract. We present necessary and/or sufficient conditions for the existence of solutions to the Dirichlet problem $u''+u+g(u)=h, \quad u(0)=0=u(\pi)$, where g is a nondecreasing function.

Keywords: Nonlinear boundary value problem, Dirichlet problem, duality principle, alternative method

Classification: 34B15

We consider here the nonlinear Dirichlet problem

$$u''+u+g(u)=h, \quad u(0)=u(\pi)=0 \tag{1}$$

where g is continuous and $h \in C[0,\pi]$.

We shall assume that

$$g \text{ is nondecreasing,} \tag{2}$$

$$\text{there exist constants } \gamma \in [0,3), C \in R \text{ such that } |g(u)| \leq \gamma \cdot |u| + C. \tag{3}$$

In [**14**], problem (1) was studied and it was proved that (1) has a solution provided that g is odd, $\lim_{u\to\infty} g(u) = \infty$, and $\gamma < 0.24347$. In [**5**] that estimate was improved to $\gamma < 0.443$, and the assumption of oddness was removed.

By integrating (1) we see that a necessary condition for (1) to have a solution is that

$$\omega = \omega(h) = \frac{1}{2}\int_0^\pi h(x)\sin x\,dx \in [g(-\infty), g(\infty)].$$

If g is bounded, then

$$g(-\infty) < \omega < g(+\infty) \tag{4}$$

is a sufficient condition for (1) to have a solution. If g is not bounded, then a restriction on γ is needed since for $g(u)=3u$ and $h(x)=\sin 2x$, problem (1) has no solution. As a consequence of the result of [**2**] we have that (2), (3) and (4) imply that (1) has at least one solution.

In the present paper, we prove the following result.

The author is thankful to the referee for helpful comments and suggestions

*Research partially supported by DGICYT, project PS88-0054

Theorem. *Suppose that* (2) *and* (3) *holds. If* h *admits a decomposition of the form* $h = h_1 + h_2$ *and there exists* $\delta > 0$ *with*

$$\int_0^\pi h_1(x) \sin x \, dx = 0 \tag{5}$$

and

$$g(-\infty) + \delta \leq h_2(x) \leq g(\infty) - \delta \text{ for all } x \in [0, \pi] \tag{6}$$

then there exists at least one solution of (1).

As an immediate consequence, we get

Corollary 1. *If* (2) *and* (3) *holds, and* g *satisfies*

$$-g(-\infty) = g(\infty) = \infty \tag{7}$$

then (1) *has a solution.*

Corollary 2. *Under assumptions* (2) *and* (3), *we have:*

a) $\omega \in \mathrm{Int}(\mathrm{Range}\, g)$ *is a sufficient condition for* (1) *to have a solution,*
b) $\omega \in \mathrm{Range}\, g$ *is a necessary condition for* (1) *to have a solution,*
c) *if* $\omega \in \mathrm{Bdry}(\mathrm{Range}\, g)$, *then* (1) *has a solution if and only if* $g(0) = \omega$.

Thus, we complete previous results in the following ways.

1. The condition $\limsup_{|u| \to \infty}(g(u)/u) = \gamma > 0$ given in [**5**] implies (7), and consequently Corollary 1 is a generalization of Th. 1 in the paper of Cesari and Kannan.
2. The estimate $\gamma < 3$ is the best estimate possible as shown by the previous example.
3. Corollary 2 includes some results for nonlinearities bounded only from one side. For instance, the cases $g(u) = \alpha u^-, \alpha < 0$, [**1**], [**7**], and $g(u) = e^u$ [**9**].
4. For g monotone (either nondecreasing or nonincreasing) hypothesis (3) of [**9**] implies that $\omega \in \mathrm{Range}\, g$ and we can apply Corollary 2.
5. Corollary 2 gives necessary and/or sufficient conditions for the existence of solutions to (1) and cover all the possible situations. Previous results [**1**], [**5–9**], [**14**] were only partial existence results.
6. To prove our main result we give a new proof by using a duality principle [**3**]. Although the proof of the Theorem is an adaptation of the method used by Brezis, we give it for the convenience of the reader and the sake of completeness.

Moreover, the method is applicable to partial differential equations. For instance, one can use such a duality principle to study the following Dirichlet boundary value problem for elliptic equations:

$$\Delta u + \lambda_1 u + g(u) = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where Ω is a smooth domain in $\mathbf{R}^n$ and λ_1 is the smallest eigenvalue of the linear problem $\Delta u + \lambda u = 0$ in $\Omega, u = 0$ on $\partial\Omega$.

PROOF of the Theorem : We shall use a duality principle that can be found in [**3**]. Let $E = L^2(I), I = [0,\pi]$, and $(\cdot,\cdot)$ the usual inner product in E. Set $D(A) = H^2(I) \cap H_0^1(I)$, and define $Au = u'' + u$, and $Bu = g(u)$. Thus, (1) is equivalent to the operator equation

$$Au + Bu = h. \tag{8}$$

Note that $B = \nabla J$, where J is a C^1 convex function on E. We have that $N(A) = \langle\xi\rangle$ where $\xi(x) = \sin x$, and $R(A) = \{u \in E : (u,\xi) = 0\}$. Thus, $A : D(A) \cap R(A) \to R(A)$ is one-to-one and onto and $K = A^{-1}$ is defined from $R(A)$ to $D(A) \cap R(A)$.

Now, for $u \in E$, define $Pu = c\xi$ where $c = c(u) = (u,\xi)/(\xi,\xi)$. We shall denote by $\|\cdot\|_p$ the usual norm in $L^p(I), 1 \le p \le \infty$.

The solutions of (8) correspond to the critical points of the functional

$$\psi(v) = \frac{1}{2}(Kv,v) + J^\star(v+h)$$

subject to the constrain $v \in R(A)$, provided that B is one-to-one and onto. Here, $J^\star$ is the conjugate convex function of J.

For $\varepsilon > 0$, consider the perturbed equation

$$Au + g_\varepsilon(u) = h \tag{9}$$

where $g_\varepsilon(u) = g(u) + \varepsilon u$. Let f_ε be the inverse function of g_ε, and define the function $G_\varepsilon(u) = \int_0^x f_\varepsilon(t)\,dt$. Thus, (9) has a solution if the functional

$$\psi_\varepsilon(v) = \frac{1}{2}\int_0^\pi [Kv](x)\cdot v(x)\,dx + \int_0^\pi G_\varepsilon(v(x)+h(x))\,dx$$

has a critical point subject to the constrain $v \in R(A)$.

Reasoning as in [**3**, Lemma 1], we see that ψ_ε has a critical point provided that $\varepsilon < 3 - \gamma$. Therefore, (9) has a solution.

Now, for any solution u_ε of (9) we have the estimates

$$\|u_\varepsilon\|_1 \le C_1, \quad \|Au_\varepsilon\|_2 \le C_1, \quad \|g(u_\varepsilon)\|_2 \le C_1 \tag{10}$$

where C_1 is a constant independent of ε.

We write $u_\varepsilon = u_{1\varepsilon} + u_{2\varepsilon}$ with $u_{1\varepsilon} \in R(A)$ and $u_{2\varepsilon} \in N(A)$. The operator K is bounded as an operator from $R(A) \cap L^1(I)$ to $R(A) \cap L^\infty(I)$. Hence, there exists a constant C_2 such that $\|u_1\|_\infty \le C_2$. On the other hand, $u_{2\varepsilon} = c(u_{2\varepsilon})\xi$, and (10) implies that $c(u_{2\varepsilon})$ is bounded, that is, there exists C_3 such that $\|u_{2\varepsilon}\|_\infty \le C_3$. In consequence, $\|u_\varepsilon\| \le C_2 + C_3$. Using these estimates, we see that there exists a sequence $\{\varepsilon_n\} \to 0$ and $u \in E$ such that $\{u_{\varepsilon_n}\} \to u$ (weakly in E), $\{A(u_{\varepsilon_n})\} \to Au$ (weakly in E), and $\{u_{1\varepsilon_n}\} \to u_1$ (in E). Now, using the theory of monotone operators we can conclude that u is a solution of (8), that is, (1) has a solution. This completes the proof of the theorem. ■

PROOF of Corollary 1: Obvious from the Theorem. ■

PROOF of Corollary 2: Part b) follows by integrating (1), and c) can be proved as in [**10**, Th. 2] (see also [**11**]). Hence, we shall only prove part a). We shall distinguish three cases:

I) g is bounded ($\gamma = 0$). We can proceed as in the proof of Lemma 2 of [**10**] by using the alternative method [**4**].

II) g is bounded only from one side, that is, either

i) $g(-\infty) = -\infty, \quad g(\infty) < \infty$, or

ii) $g(-\infty) > -\infty, \quad g(\infty) = \infty$.

If (i) holds, then we can choose $\varepsilon > 0$ such that $\mu = g(\infty) - \varepsilon > \omega$, and consider the problem

$$u'' + u + G(u) = H, \quad u(0) = u(\pi) = 0 \tag{11}$$

where $G(u) = g(u) - \mu$ and $H(x) = h(x) - \mu$. Note that (11) and (1) are equivalent.

We have that $\omega(H) = \omega(h) - \mu$. Hence $c(H) = 4\omega(H)/\pi < 0$. Now, we write $H = H_1 + H_2$ where $H_2 = PH = c(H)\xi$ and $H_1 = H - H_2$. Thus, we get that $c(H) \leq H_2(x) \leq 0$ and $G(-\infty) = -\infty < c(H) \leq H_2(x) \leq 0 < \varepsilon = G(\infty)$. Therefore, (5) and (6) are satisfied and by the previous theorem we can conclude that (11) has at least one solution. This proves a) of Corollary 2 in case II-i).

Case ii) is similar if we define $G(u) = g(u) - \mu$ and $H(x) = h(x) - \mu$ with $\mu = g(-\infty) + \varepsilon$ such that $0 < \varepsilon < \omega - g(-\infty)$.

III) Condition (7) holds. Then $\omega \in \mathrm{Int}(\mathrm{Range}\, g) = \mathbf{R}$, and we can apply Corollary 1. ■

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(*Received* January 19, 1990)