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Abstract. We construct a delta-convex function on $\mathbf{R}^2$ which is strictly differentiable at 0, but this property possesses none of its control functions. Further, we prove that if a function H on an open convex subset A of a normed linear space X is controlled by a bounded continuous function h_E on each bounded closed set E contained in A, then A is delta-convex on A. We construct various counterexamples showing that this result is the best possible generalization of its finite-dimensional counterpart, which is due to P. Hartman.

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Introduction.

Let A be a convex subset of a normed linear space X. A function $H\colon A \to \mathbf{R}$ is termed *delta-convex* on A if H can be expressed as a difference of two continuous convex functions on A. Let H and h be functions on A. We say that h is a *control function* to H on A, or, that h *controls* H on A, if both the functions $h - H$ and $h + H$ are continuous and convex. Then, of course, h is also continuous and convex and H is continuous. We may define equivalently delta-convex functions on A as those functions, which have control functions. The family of all delta-convex functions on A is the linear span of the set of all continuous convex functions on A.

A difficulty of the concept of delta-convexity consists in the fact that in general we cannot found a "canonical" control function to H, which would be controlled by all control functions to H.

The notion of delta-convex function was introduced by A.D. Aleksandrov [**1**] in n-dimensional case. It was observed by L. Zajíček [4], that the infinite-dimensional version of this notion allows to characterize the sets of Gâteaux nondifferentiability of continuous convex functions on separable Banach spaces.

A survey of results in the theory of delta-convex functions and mappings (see Remark 17 below) can be found in an important article by L. Veselý and L. Zajíček [**3**].

In this paper we solve Problem 3 and Problem 5 from [**3**]. In Section 1 we show an example of a delta-convex function H on $\mathbf{R}^2$ such that the function H is strictly differentiable at 0, but none of its control functions is (Fréchet) differentiable at 0.

In Section 2 we compare local and global delta-convexity . A function H is said to be *locally delta-convex* on an open set A, if every point $z \in A$ has an open convex neighborhood V in A such that H is delta-convex on V. P. Hartman [**2**] proved that if $A \subset \mathbf{R}^n$ is an open convex set and H is a locally delta-convex function on A,

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then it is delta-convex. We show that analogous statement is not true in infinite-dimensional spaces even if A is bounded and H is locally delta-convex on the whole space. A further example shows that there is a function H on l^2 such that H is delta-convex on each bounded convex subset of l^2 but H is not delta-convex on l^2. The only positive result remains: If a function H on a normed linear space X is controlled by a *bounded* continuous convex function on *each bounded convex set*, then H is delta-convex on X. The same result holds for mappings (cf. Remark 17).

Differentiability.

Let X be a normed linear space. We denote $B(a,r) = \{x \in X : |x-a| < r\}$. Let F be a function defined on an open set $A \subset X$. A continuous linear functional L is said to be a *strict derivative* of F at a point a if for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $x, y \in A \cap B(a,\delta)$ we have

$$|F(y) - F(x) - L(y-x)| \le \varepsilon |y-x| .$$

If a convex function is differentiable (= Fréchet differentiable) at a point a, then it is strictly differentiable at a ([**3**, Prop. 3.8]). This cannot be said about delta-convex functions. An example of a delta-convex function on $\mathbf{R}^2$ which is differentiable at 0 but not in the strict sense in given in [**3**, Note 6.4]. Such a function cannot be controlled by a function differentiable at 0.

In this section we construct a delta-convex function H on $\mathbf{R}^2$ such that the function H is strictly differentiable at 0, but none of its control functions is differentiable at 0.

Example 1. Find a sequence $\{k_j\}$ of positive integers such that $\cos(2\pi/k_j) \ge 1 - 2^{-j-3}$ and denote

$$M = \left\{ \left(2^{-j}\cos(2\pi k/k_j),\, 2^{-j}\sin(2\pi k/k_j)\right) : j \in \mathbf{N},\, k \in \{1,\dots,k_j\} \right\} .$$

Set

$$F(x) = |x| + 4|x|^2 .$$

For each $z \in \mathbf{R}^2 \setminus \{0\}$ we define

$$G_z(x) = F(z) + \left\langle F'(z),\, x - z \right\rangle = (8|z|+1)\frac{\langle x, z\rangle}{|z|} - 4|z|^2 .$$

Since F is convex and G_z is a tangent function to F at z, we have $G_z \le F$ on $\mathbf{R}^2$. Set

$$G(x) = \sup\{G_z(x) : z \in M\}$$

and

$$H(x) = G(x) - |x|.$$

Obviously G is a convex functions on $\mathbf{R}^2$. It follows that the function H is delta-convex. In what follows we will derive further properties of the functions G and H.

Lemma 2. *Let M be as in Example 1 and $x \in \mathbf{R}^2$. Suppose $0 < |x| < 1$. Then there is $z \in M$ such that*

$$|z| \leq |x| \leq 2|z|$$

and

$$\frac{\langle x, z\rangle}{|x||z|} > 1 - \frac{1}{8}|z|.$$

PROOF : Find $j \in \mathbf{N}$ such that

$$2^{-j} \leq |x| < 2^{-j+1}.$$

Further find $z \in M$ such that $|z| = 2^{-j}$ and the angle between the radiusvectors of z and x is less than $2\pi/k_j$, i.e.

$$\frac{\langle x, z\rangle}{|x||z|} > \cos(2\pi/k_j)$$

Then we have

$$|z| \leq |x| \leq 2|z|$$

and

$$\frac{\langle x, z\rangle}{|x||z|} > 1 - 2^{-j-3} = 1 - \frac{1}{8}|z|. \qquad \blacksquare$$

Lemma 3. *The function G from Example 1 satisfies*

$$|x| + |x|^2 \leq G(x) \leq |x| + 4|x|^2 = F(x)$$

for all $|x| < 1$.

PROOF : Fix $x \in \mathbf{R}^2$ with $0 < |x| < 1$. The inequality $G(x) \leq F(x)$ is obvious. Find $z \in M$ as in Lemma 2. Then we have

$$\begin{aligned} G(x) \geq G_z(x) &= (8|z|+1)\frac{\langle x, z\rangle}{|z|} - 4|z|^2 \geq (8|z|+1)(1 - \frac{1}{8}|z|)|x| - 4|z|^2 \\ &= |x| + |z|(8|x| - \frac{1}{8}|x| - |x||z| - 4|z|) \geq |x| + 2|z||x| \geq |x| + |x|^2. \end{aligned} \qquad \blacksquare$$

Lemma 4. *Let G, G_z be the functions from Example 1. Let $r \in (0,1)$. If*

$$0 < |z| \leq \frac{r}{9} \quad \textit{and} \quad |x| \geq r,$$

then

$$G_z(x) \leq G(x) - \frac{r^2}{9}.$$

PROOF : Under the assumptions, using Lemma 3 we obtain

$$\begin{aligned} G_z(x) &= (8|z|+1)\frac{\langle x, z\rangle}{|z|} - 4|z|^2 \\ &\leq |x| + 8|x||z| \leq G(x) - \frac{r^2}{9}. \end{aligned} \qquad \blacksquare$$

Lemma 5. *Let G, G_z be the functions from Example 1 and $w \in \mathbf{R}^2$, $0 < |w| < \frac{1}{16}$. Then*

$$G(w) = \sup\{G_z(w) : z \in M_w\},$$

where

$$M_w = \{z \in M : |z| \le 2|w|,\ \langle w, z\rangle \ge |z||w|(1 - 8|z|)\}\,.$$

PROOF : Choose $z \in M \setminus M_w$. We will distinguish two cases.

(a) Assume that $|z| \ge 2|w|$. Then

$$\begin{aligned} G_z(w) &= (8|z| + 1)\frac{\langle w, z\rangle}{|z|} - 4|z|^2 \\ &\le |w| + 8|w||z| - 4|z|^2 \le |w|\,. \end{aligned}$$

(b) Assume that $|z| \le 2|w|$ and

$$\langle w, z\rangle < (1 - 8|z|)|w||z|\,.$$

We obtain

$$\begin{aligned} G_z(w) &= (1 + 8|z|)\frac{\langle w, z\rangle}{|z|} - 4|z|^2 \\ &\le |w| - 64|z|^2|w| - 4|z|^2 \le |w|\,. \end{aligned}$$

In both cases (a) and (b), using Lemma 3 we conclude that $G_z(w) \le G(w) - |w|^2$. The assertion easily follows. ■

Lemma 6. *Let H be the function from Example 1. Then the zero functional is a strict derivative of H at the origin.*

PROOF : Choose $\varepsilon \in (0, 1/4)$. Let x, y be points of $B(0, \varepsilon^2)$. We will estimate the quantity

$$|H(y) - H(x)|\,.$$

We will distinguish two cases.

(a) Let us assume that $|y - x| \ge \varepsilon(|x| + |y|)$. Then we obtain

$$\begin{aligned} |H(y) - H(x)| &\le |H(y)| + |H(x)| \le 4(|x|^2 + |y|^2) \le 4(|x| + |y|)^2 \\ &\le 4(|x| + |y|)\,\frac{|y - x|}{\varepsilon} \le 8\,\varepsilon|y - x|\,. \end{aligned}$$

(b) Let us assume that $|y - x| \le \varepsilon(|x| + |y|)$. Fix $z \in M$. Denote

$$x^* = \frac{|x|}{|z|}\,z\,, \qquad y^* = \frac{|y|}{|z|}\,z\,.$$

Assume $z \in M_x$ (for the notation see Lemma 5). Then

$$\begin{aligned} |x^* - x|^2 &= \frac{1}{|z|^2}\big||x|z - |z|x\big|^2 = \frac{1}{|z|^2}\,(2|x|^2|z|^2 - 2|x||z|\langle x, z\rangle) \\ &\le 2|x|^2(1 - (1 - 8|z|)) = 16\,|z||x|^2 \le 32\,\varepsilon^2(|x| + |y|)^2 \end{aligned}$$

and

$$|y^* - y| \le |y^* - x^*| + |x^* - x| + |x - y| \le (2\varepsilon + \sqrt{32})\,\varepsilon(|x| + |y|) \le 8\,\varepsilon(|x| + |y|).$$

Similarly we have

$$|y^* - y| \le 8\,\varepsilon(|x| + |y|) \qquad \text{and} \qquad |x^* - x| \le 8\,\varepsilon(|x| + |y|)$$

assuming that $z \in M_y$. Now, let $z \in M_x \cup M_y$. Then

$$\begin{aligned} |(G_z(y)-|y|)-(G_z(x)-|x|)| &= \left|(8|z|+1)\frac{\langle y-x, z\rangle}{|z|} - \frac{\langle y, y-x\rangle + \langle x, y-x\rangle}{|x|+|y|}\right| \\ &= \left|8|z|\frac{\langle y-x, z\rangle}{|z|} + \frac{\langle y-x, x^*-x\rangle}{|x|+|y|} + \frac{\langle y-x, y^*-y\rangle}{|x|+|y|}\right| \\ &\le \left(8|z| + \frac{|x^*-x|}{|x|+|y|} + \frac{|y^*-y|}{|x|+|y|}\right)|y-x| \le 24\varepsilon|y-x|. \end{aligned}$$

It easily follows

$$|H(y) - H(x)| \le \sup\{|(G_z(y) - |y|) - (G_z(x) - |x|)| : z \in M_y \cup M_x\} \le 24\,\varepsilon|y - x|.$$

The estimates in (a) and (b) show that the zero functional is a strict derivative of H at the origin. ■

Theorem 7. *The function H from Example 1 is delta-convex on $\mathbf{R}^2$. The function H is strictly differentiable at 0, but none of its control functions is (Fréchet) differentiable at 0.*

PROOF : Most of the required properties is proved in Lemma 2 – Lemma 6. It only remains to show that none of the control functions to H is differentiable at 0. Assume that h is a control function to H which is differentiable at 0. We may assume that $h(0) = h'(0) = 0$. Find r, $0 < r < 1$, such that

$$|h(x)| \le \frac{1}{8}|x| \quad \text{if} \quad |x| \le 3r.$$

Denote

$$\begin{aligned} &\xi(t) = (r, t), \quad t \in [-2r, 2r], \\ &\varphi(t) = |\xi(t)|, \\ &\gamma(t) = G(\xi(t)), \\ &\kappa(t) = h(\xi(t)). \end{aligned}$$

The function γ is piecewise linear, as by Lemma 4

$$G(\xi(t)) = \max\{G_z(\xi(t)) : z \in M, |z| > \frac{r}{9}\}.$$

We find points t_i, $-r = t_0 < t_1 < \cdots < t_m = r$, such that γ is linear on each interval $[t_{i-1}, t_i]$. Then

$$\gamma'_-(t_i) = \gamma'_+(t_{i-1}) \tag{1}$$

for each $i = 1, \dots, m$. Since h is a control function to H, the function $\kappa + \gamma - \varphi$ is convex on $[-2r, 2r]$. Hence for each $i = 1, \dots, m$ we have

$$\kappa'_-(t_i) - \kappa'_+(t_{i-1}) + \gamma'_-(t_i) - \gamma'_+(t_{i-1}) - \varphi'(t_i) + \varphi'(t_{i-1}) \geq 0\,. \tag{2}$$

Using convexity of κ we obtain

$$\begin{aligned} \kappa'_-(r) &\leq \frac{1}{r}(\kappa(2r) - \kappa(r))\,, \\ -\kappa'_+(-r) &\leq \frac{1}{r}(\kappa(-2r) - \kappa(-r))\,. \end{aligned} \tag{3}$$

From (1), (2) and (3) it follows

$$\begin{aligned} \sqrt{2} = \varphi'(r) - \varphi'(-r) &= \sum_{i=1}^{m}(\varphi'(t_i) - \varphi'(t_{i-1})) \\ &\leq \sum_{i=1}^{m}(\kappa'_-(t_i) - \kappa'_+(t_{i-1})) \leq \kappa'_-(r) - \kappa'_+(-r) \\ &\leq \frac{1}{r}(\kappa(2r) - \kappa(r) + \kappa(-2r) - \kappa(-r)) \leq 1\,, \end{aligned}$$

which is a contradiction. ∎

Local and global deltaconvexity.

Is every locally delta-convex function delta-convex ? The answer is positive in the finite-dimensional case ([**2**]). In this section we will study various related questions in case of infinite dimension. We present several "negative" results, which show why the final result cannot be stronger.

Let us introduce a notation: if $x \in l^2$, then x^j stands for the j-th coordinate of x. We denote by e_i the element of l^2 with i-th coordinate 1 and remaining coordinates 0.

Lemma 8. *Let X be a normed linear space and $R > 0$. Let H be a function on $B(0,R)$. Suppose that there exists a bounded control function h to H on $B(0,R)$ Then H, $h - H$ and $h + H$ are bounded on $B(0,R)$.*

Proof : Denote $F = h - H$, $G = h + H$. From the convexity of F and G we obtain existence of continuous linear functionals f, g on X such that for each $x \in A$ we have $\langle f, x\rangle \leq F(x) - F(0)$ and $\langle g, x\rangle \leq G(x) - G(0)$. Then, of course, f and g are bounded on $B(0,R)$ and for each $x \in B(0,R)$ we estimate

$$F(0) + \langle f, x\rangle \leq F(x) = 2h(x) - G(x) \leq 2h(x) - G(0) - \langle g, x\rangle\,.$$

Similarly we conclude that G and $H = \frac{1}{2}(G - F)$ are bounded on $B(0,R)$. ∎

Lemma 9. *Let X be a normed linear space and $R > 0$. Let H be a function on $B(0, 2R)$. Suppose that there exists a bounded control function h to H on $B(0, 2R)$ Then H is Lipschitz-continuous on $B(0, R)$.*

PROOF : It is well known (see e.g. [**3**, Lemma 1.9]) that any bounded convex function on $B(0, 2R)$ is Lipschitz-continuous on $B(0, R)$. If we apply this result to the functions $h + H$ and $h - H$ (which are convex by the definition of a control function and bounded on $B(0, 2R)$ by the preceding lemma), we deduce that $H = \frac{1}{2}\big((h + H) - (h - H)\big)$ is Lipschitz-continuous on $B(0, R)$. ∎

Lemma 10. *There exists a bounded nonnegative delta-convex function H on l^2 such that $H(x) = 0$ if $|x| \geq 1$ and none of the control functions to H is bounded on $B(0, 1)$.*

PROOF : If $x \in l^2$, we define

$$F(x) = \sup\{|x|, x^1 - \frac{1}{4}, 2(x^2 - \frac{1}{4}), 3(x^3 - \frac{1}{4}), \dots)\}$$

and

$$G(x) = \max\{1, F(x)\}\,.$$

The functions F, G are convex on l^2. If $y \in l^2$, then

$$F(x) = \max\{|x|, x^1 - \frac{1}{4}, 2(x^2 - \frac{1}{4}), 3(x^3 - \frac{1}{4}), \dots, m(x^m - \frac{1}{4})\}$$

holds for all $x \in B(y, \frac{1}{8})$ and $m = \max\{j : y^j \geq \frac{1}{8}\}$. This proves the continuity of F and G. Hence the function $H = G - F$ is delta-convex. Obviously H is bounded and $H = 0$ outside $B(0, 1)$. Denote

$$u_k = \frac{1}{4}\, e_k\,, \quad v_k = (\frac{1}{4} + \frac{1}{k})\, e_k\,, \quad k = 5, 6, 7, \dots\,.$$

Then

$$F(v_k) = 1\,, \quad F(u_k) = \frac{1}{4}\,,$$

and thus

$$|H(v_k) - H(u_k)| = \frac{3}{4} \geq \frac{k}{2}|v_k - u_k|\,.$$

It follows that H is not Lipschitz-continuous on $B(0, \frac{1}{2})$, and thus, by Lemma 9, none of the control functions to H is bounded on $B(0, 1)$. ∎

Example 11. Let $\Omega \subset l^2$ be an open convex set. We will construct a function H on l^2 such that H is locally delta-convex on l^2, but it is not delta-convex on Ω.

Without loss of generality we may assume that $0 \in \Omega$. Denote $U = \{\frac{1}{2}x : x \in \Omega\}$. Let us find a sequence $\{x_k\}$ of points of U and $\delta > 0$ such that the balls $B(x_k, 2\delta)$ are contained in U and pairwise disjoint. By a slight modification Lemma 10 we obtain for every $k \in \mathbf{N}$ a delta-convex function H_k such that $H_k = 0$ outside $B(x_k, \frac{1}{k}\delta)$ and none of the control functions to H_k is bounded on $B(x_k, \frac{1}{k}\delta)$. Set

$$H = \sum_{k=2}^{\infty} H_k .$$

Obviously H is locally delta-convex on l^2. Assume that there is a control function h to H on Ω. Using the unboundedness property, we find $u_k \in U$ such that $|u_k| \le \frac{1}{k}\delta$ and $h(x_k + u_k) \ge \max\{h(2x_k), k\}$. From convexity of h on the line connecting $2u_k$, $u_k + x_k$ and $2x_k$ we get $h(2u_k) \ge k$. Since $u_k \to 0$, the function h is not continuous at 0, which is a contradiction.

Theorem 12. *There exists a function H on l^2 which has the following properties: With every point $z \in l^2$ we can associate a continuous convex function h_z on l^2 such that h_z is bounded on each bounded subset of l^2 and controls H on a neighborhood of z. Nevertheless, H is not delta-convex on $B(0,3)$.*

PROOF : For $i, j = 1, 2, \dots$ we denote

$$z_{ij} = e_i + 2^{-i-1} e_j ,$$
$$B_i = B(e_i, 2^{-i-1}) ,$$
$$B_{ij} = B(z_{ij}, 2^{-i-2}) .$$

Let

$$H(x) = \begin{cases} j\left(1 - \dfrac{|x - z_{ij}|}{2^{-i-2}}\right) & x \in B_{ij} , \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that there are control functions h_z to H with the required properties. (Notice that the function $\max(|x| - 1, 2|x| - 2)$ controls $\max(1 - |x|, 0)$.) We will show that H is not delta-convex on $B(0,3)$. Let us assume that there is a control function h to H on $B(0,3)$. By Lemma 8, h is unbounded on each ball B_i. Now, for every $i \in \mathbf{N}$ we find a point $x_i \in B_i$, so that $h(x_i) \ge \max\{h(2e_i), i\}$. Set $y_i = 2(x_i - e_i)$. From the convexity of the function h we obtain $h(y_i) \ge i$ (the point x_i belongs to the segment connecting y_i and $2e_i$). Since $|y_i| \le 2^{-i}$, we have

$$\lim_{i \to \infty} y_i = 0 \text{ and } \lim_{i \to \infty} h(y_i) = \infty,$$

which contradicts the continuity of h at the point 0. ∎

Remarks 13. 1. If we do not require h_z to be bounded on bounded sets, then the assertion easily follows from Example 11.

2. A similar example as in Theorem 12 can be constructed in each infinite-dimensional normed linear space.

Theorem 14. *There exists a function H on l^2 which is delta-convex on each bounded convex subset of l^2, but it is not delta-convex on l^2.*

PROOF : Let us specify Example 11 so that $\Omega = l^2$ and $\lim |x_j| = \infty$. We obtain a function H, which is not delta-convex on l^2. Nevertheless, H is delta-convex on an arbitrary bounded convex set $M \subset l^2$, as H coincides on M with a sum of a finite family of delta-convex functions. ■

Lemma 15. *Let H, h be functions on an open convex subset A of a normed linear space X. Suppose that h is convex and continuous. If every point of A has a neighborhood U such that h controls H on U, then h controls H on A.*

PROOF : It is an obvious consequence of the fact that every locally convex function is convex. ■

The following theorem is an infinite-dimensional generalization of Hartman's result on locally delta-convex functions.

Theorem 16. *Let H_α be a family of functions on an open convex subset A of a normed linear space X. Suppose that for every bounded closed convex set $E \subset A$ there is a bounded continuous convex function h_E on E which controls each H_α on E. Then there is a continuous convex function h on X which controls each H_α on A.*

PROOF : We may suppose that $0 \in A$. Let p be the Minkowski's functional of A, defined by

$$p(x) = \inf\{\lambda \in (0, +\infty) : \lambda^{-1}x \in A\}.$$

and

$$q(x) = |x| + \frac{p(x)}{1 - p(x)}$$

Then q is a continuous convex function on A, as the function $y \mapsto \dfrac{y}{1-y}$ is increasing and convex on $[0, 1)$. Set

$$E_k = \{x \in A : q(x) \leq k\}.$$

The sets E_k are obviously bounded and closed subsets of X and

$$A = \bigcup E_k .$$

Fix $k \in \{0, 1, 2, \dots\}$. Denote

$$\begin{aligned} h_k &= h_{E_k}, \\ M_k &= \sup_{E_{k+3}} h_{k+3}, \\ m_k &= \inf_{E_{k+3}} h_{k+3}, \\ c_k &= M_k + (k-2)(M_k - m_k) = m_k + (k-1)(M_k - m_k), \\ f_k(x) &= h_{k+3}(x) + (M_k - m_k)q(x) - c_k, \\ w_k(x) &= 5(M_k - m_k)(q(x) - (k+1)). \end{aligned}$$

Then f_k is a convex function on E_{k+3}. We will estimate $f_k(x)$ for some positions of x: If $x \in E_{k+3} \setminus E_{k+2}$, then

$$f_k(x) \le M_k + (k+3)(M_k - m_k) - c_k \le 5(M_k - m_k) \le w_k(x),$$

if $x \in E_{k-2}$, then

$$f_k(x) \le M_k + (k-2)(M_k - m_k) - c_k = 0$$

and if $x \in E_{k+1} \setminus E_{k-1}$, then

$$f_k(x) \ge m_k + (k-1)(M_k - m_k) - c_k = 0 \ge w_k(x).$$

Set

$$g_k(x) = \begin{cases} \max\{0, f_k(x), w_k(x)\} & \text{if } x \in E_{k+3}, \\ w_k(x) & \text{if } x \in A \setminus E_{k+3}. \end{cases}$$

Then $g_k = 0$ on E_{k-2}, $g_k = f_k$ on $E_{k+1} \setminus E_{k-1}$ and $g_k = w_k$ on $A \setminus E_{k+2}$. It follows that g_k is a continuous convex function on A which controls each H_α on each convex subset of $E_{k+1} \setminus E_{k-1}$. Set

$$h = \sum_{k=0}^{\infty} g_k .$$

Since for every bounded set K we can find $n \in \mathbf{N}$ such that $g_k = 0$ on K if $k \ge n$, we deduce that h is a continuous convex function on A. By Lemma 15, h controls each H_α on A. ∎

Remark 17. Let X and Y be normed linear spaces. Let H be a mapping of an open convex set $A \subset X$ into Y. Following [**3**], we say that H is delta-convex, if there is a convex continuous function h which controls H, this means that h controls every function $H_\alpha : x \mapsto \langle f_\alpha, H(x)\rangle$, where $\{f_\alpha\}$ is the collection of all linear functionals on Y with $\|f_\alpha\| \le 1$. From Theorem 16 we immediately see that the following result is true:

Corollary 18. *Let X and Y be normed linear spaces. Let A be an open convex subset of X and $H : A \to Y$ be a mapping. Suppose that for every bounded closed convex set $E \subset A$ there is a bounded continuous convex function h_E on E which controls H on E. Then H is delta-convex on X.*

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