Direct finiteness of group rings – a simple proof of the Kaplansky's conjecture for finite groups

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Abstract. We present a simple proof of the direct finiteness of the group ring KG in case K is a field and G is a finite group.

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A well–known conjecture of Kaplansky states that for any group G and any field K the group ring KG is directly finite (i.e. $xy = 1 \implies yx = 1$ for all x, $y \in KG$). The conjecture is known to be true provided char$(K) = 0$, but its validity for char$(K) \neq 0$ is still an open problem (see e.g. [**3**, 2.23] or [**4**, p. 38]).

In [**2**], G. Losey proved the conjecture e.g. for any field K and any residually finite group G. In our short note, using another approach, we present a simple proof of the following assertion: if H is any subgroup of a countable direct product of finite groups and F is any field, then FH (and even each full matrix ring $M_n(FH)$, $0 < n < \aleph_0$) is directly finite. In particular, we re–prove the conjecture for any field K and any finite group G.

At first sight, we could except the proof deals with the characterization of left invertible elements of KG showing that they coincide with the right invertible ones. Here, that is not the case. We rather make use of the simple fact that the direct finiteness of a ring is assured by the direct finiteness of any of its over–rings (with the same unit). Of course, we are left to prove the direct finiteness of the ring $R = FP$ where $P = \prod_{n<\aleph_0} G_n$ is a direct product of finite groups $G_n = (G_n, \odot_n)$, card$(G_n) > 1$, and F is a field. The main point of the proof is that R is a subring of a ring S, where S is a reduced product of simple artinian rings. Since S is unit regular (i.e. for each $x \in S$ there is a left and right invertible element $y \in S$ such that $x = xyx$), all full matrix rings over S are directly finite, and the same holds true for R.

In order to simplify our notation, we shall assume that for each $n < \aleph_0$, $G_n = \{0, \dots, p_n - 1\}$, 0 is the null element of the additive (non–commutative) group $(G_n, \odot_n)$, and $p_n = \text{card}(G_n) > 1$. The group operation on P will be denoted by $\odot$. For $n < \aleph_0$, π_n denotes the n–th canonical projection of P onto G_n. Further, put $Q_0 = \{0\}$ and $Q_n = \{(z_0, \dots, z_{n-1})\colon z_i < p_i \,\forall i < n\}$ for $0 < n < \aleph_0$. Let $q_n = \text{card}(Q_n)$, $n < \aleph_0$, i.e. $q_0 = 1$ and $q_n = p_0 \dots p_{n-1}$ for $0 < n < \aleph_0$.

Definition. For each $r \in R$ and each $n < \aleph_0$, define a matrix $A_n(r) \in M_{q_n}(F)$ – called the *n–th approximation matrix* of r – as follows.

We have $r = \sum_{g \in P} f_g g$, the set $\mathrm{Supp}(r) = \{g \in P\colon f_g \neq 0\}$ being finite. If $\mathrm{card}(\mathrm{Supp}(r)) \leq 1$, put $m_r = 0$. If $\mathrm{card}(\mathrm{Supp}(r)) > 1$, let $m_r < \aleph_0$ be the least number such that, for all $g, g' \in \mathrm{Supp}(r)$, there is $i < m_r$ with $\pi_i g \neq \pi_i g'$.

If $n < m_r$, put $A_n(r) = 0$. Let $n \geq m_r$. If $n = 0$, we put $A_n(r) = f_g \in F$ provided $\mathrm{card}(\mathrm{Supp}(r)) = 1$ and $g \in \mathrm{Supp}(r)$, and $A_n(r) = 0$ provided $\mathrm{Supp}(r) = \emptyset$. If $n > 0$, then, for $u = (z_0, \dots, z_{n-1}) \in Q_n$ and $u' = (z'_0, \dots, z'_{n-1}) \in Q_n$, we put $a_{uu'} = f_g$ provided there is a (unique) $g \in \mathrm{Supp}(r)$ such that $\pi_i g \odot_i z'_i = z_i$ for all $i < n$, and $a_{uu'} = 0$ otherwise. Finally, put $A_n(r) = (a_{uu'})_{u,u' \in Q_n}$.

Theorem. *Let W be a filter on $\aleph_0$ such that W contains the Fréchet filter. For each $n < \aleph_0$, denote by ρ_n the n-th canonical projection of the ring $\prod_{n<\aleph_0} M_{q_n}(F)$ onto M_{q_n}. Put $I_W = \{x \in \prod_{n<\aleph_0} M_{q_n}(F)\colon \exists w \in W \forall n \in W\colon \rho_n x = 0\}$. Denote by T the reduced product of the rings $M_{q_n}(F)$, $n < \aleph_0$ by the filter W, i.e. $T = \prod_{n<\aleph_0} M_{q_n}(F) \backslash I_W$. Define a mapping $\varphi(r) = (A_n(r)\colon n < \aleph_0) + I_W$, $A_n(r)$ being the n-th approximation matrix of r, $n < \aleph_0$. Then φ is an injective ring homomorphism.*

PROOF : Clearly, φ is a mapping such that $\varphi(1_R) = 1_T$ and $\varphi(-r) = -\varphi(r)$ for every $r \in R$. Let $r, r' \in R$, $r = \sum_{g \in P} f_g g$ and $r' = \sum_{g \in P} f'_g g$. Denote by D the symmetric difference of the sets $\mathrm{Supp}(r)$ and $\mathrm{Supp}(r')$. Let $m \geq \max(m_r, m_{r'})$ be the least number such that for all $g, g' \in D$ there is $i < m$ with $\pi_i g \neq \pi_i g'$. Then, for all $n \geq m$, $A_n(r) + A_n(r') = A_n(r + r')$, whence $\varphi(r + r') = \varphi(r) + \varphi(r')$.

Let $p \geq \max(m_r, m_{r'})$ be the least number such that for all $g, g' \in \mathrm{Supp}(r)$ and all $h, h' \in \mathrm{Supp}(r')$ with $g \odot h \neq g' \odot h'$, there is $i < p$ such that $\pi_i(g \odot h) \neq \pi_i(g' \odot h')$. Then, for all $n \geq p$, $A_n(r) \cdot A_n(r') = A_n(rr')$, whence $\varphi(rr') = \varphi(r) \cdot \varphi(r')$.

Finally, if $\varphi(r) = 0$ for some $r \in R$, then $A_n(r) = 0$ for infinitely many $n \geq m_r$, and consequently $f_g = 0$ for all $g \in P$. ∎

Corollary.

(i) *The ring $M_n(R)$ is directly finite for all $0 < n < \aleph_0$.*

(ii) *Let F be a field and H be a subgroup of a countable direct product of finite groups. Then the ring $M_n(FH)$ is directly finite for all $0 < n < \aleph_0$.*

(iii) *Let F be a field and H be a finite group. Then the group ring FH is directly finite.*

PROOF :

(i) Since each of the rings $M_{q_n}(F)$, $n < \aleph_0$, is unit regular, so is the ring $T = \prod_{n<\aleph_0} M_{q_n}(F)/I_W$. In particular, for each $0 < n < \aleph_0$, the ring $M_n(T)$ is unit regular (see [1, Corollary 4.7]). Hence, $M_n(T)$ is directly finite, and we can use our theorem.

(ii) Since FH is a subring of $R = FP$, the result follows from (i).

(iii) By (ii).

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