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Some limit theorems of intermediate term of a random number of independent random variables

H.M. BARAKAT AND M.A. EL–SHANDIDY

Abstract. Necessary and sufficient conditions are obtained for the weak convergence of the normalized intermediate terms of order statistics, when the sample size is a sequence of positive integer–valued random variables (r.v.).

Keywords: weak convergence, order statistics, intermediate sequences, sample of random size

Classification: 60F99

1. INTRODUCTION

Let $\{X_n\}$ be a sequence of r.v.'s with a common continuous distribution function (d.f.) $F(x)$ and $\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_n^{(n)}$ be its order statistic. The term $\xi_{K_n}^{(n)}$ is called an intermediate term if its rank sequence $\{K_n\}$ is such that $1 \le K_n \le n$ for all n, $K_n \to \infty$ and $K_n/n \to 0$, as $n \to \infty$.

Chibisov (1964) [1] and Wu (1966) [2] both have shown that the normal and log-normal distributions are possible limiting distributions for the intermediate terms.

When the intermediate rank sequence $\{K_n\}$ satisfies the limit relation

$$\lim_{n\to\infty}(\left(\sqrt{K_{n+r_n}} - \sqrt{K_n}\right) = \frac{\alpha\ell\gamma}{2}, \qquad \ell > 0, \tag{1.1}$$

for any sequence of integer values $\{r_n\}$ for which $r_n/n^{1-\frac{\alpha}{2}} \to \gamma$, as $n \to \infty$, ($0 < \alpha < 1$, γ is any arbitrary real number), Chibisov [1] proved that, whenever there exist sequence $a_n > 0$, b_n such that $P\left(\xi_{k_n}^{(n)} < a_n x + b_n\right)$ has a nondegenerate limit, the limiting distribution must be of the form $\Phi(U(x))$, where $\Phi(x)$ is the standard normal distribution and $U(x)$ has the following form, (up to an affine transformation of x)

$$\begin{aligned}
&1)\quad U_1^{(\beta)}(x) = \begin{cases} -\infty, & x \le 0, \\ \beta \ln x, & x > 0; \end{cases} \\
&2)\quad U_2^{(\beta)}(x) = \begin{cases} -\beta \ln |x|, & x \le 0, \\ \infty, & x > 0, \beta > 0; \end{cases} \\
&3)\quad U_3^{(\beta)}(x) = U_3(x) = x, \qquad -\infty < x < \infty.
\end{aligned} \tag{1.2}$$

Here β is some positive constant depending only on α, ℓ and the type of the d.f. $F(x)$.

Wu [2] has shown that the types $\Phi\left(U_i^{(\beta)}(x)\right)$, $i = 1,2,3$ are only the possible limit types of d.f.'s of the normalized term $\xi_{K_n}^n$, When K_n is any nondecreasing intermediate rank sequence. Moreover, the domains of attraction of the limiting forms $\Phi\left(U_i^{(\beta)}(x)\right)$, $i = 1,2,3$ have been obtained in [1], [2] and [3].

The problem of asymptotic behaviour of extreme and central order statistics of random sample sizes was studied by various authors ([4]–[9]) and the following general result was obtained:

Theorem 1.1. *Let ν_n be a positive integer valued r.v. which is distributed independently of X_i, $i = 1,2,\ldots,n$ for each n.*

Suppose that, as $n \to \infty$ and for suitably chosen constants $a_n > 0, b_n$, the following conditions are satisfied:

(I)
$$P\left(\xi_{K_n}^{(n)} < a_n x + b_n\right) \xrightarrow{w} G(U(x)),$$

(II)
$$P(\nu_n/n < x) \xrightarrow{w} A(x), \ A(x) \text{ is a d.f.}$$

Then, as $n \to \infty$,

(III)
$$P\left(\xi_{k\nu_n}^{(\nu_n)} < a_n x + b_n\right) \xrightarrow{w} \Psi(x) = \int_0^\infty G\left(z^j U(x)\right) dA(z), \quad j = 1, \frac{1}{2},$$

where

$$G\left(z^j U(x)\right) = \begin{cases} \Gamma_K\left(z^j U(x)\right), j = 1, & \text{extreme case, i.e.,} K_n = K = \text{constant } ([4]\text{–}[8]), \\ \Phi\left(z^j U(x)\right), j = \frac{1}{2}, & \text{central case with } \lambda\text{-domain of attraction, i.e.,} K_n/n \to \lambda \in (0,1), \\ & \sqrt{n}(K_n/n - \lambda) \to 0, \text{ as } n \to \infty, [9]. \end{cases}$$

where ($\xrightarrow{w}$) denotes the week convergence, Γ_K is an incomplete gamma function and the function $U(x)$ either has only one of the possible limiting forms shown in [10] *and the second part of* [11] *(in the extreme case) or $U(x)$ has only one of the possible limiting forms shown in the first part of* [11] *(in the central case).*

When the intermediate rank sequence $\{K_n\}$ satisfies the Chibisov condition (1.1) and the sample size is a r.v., the sufficient conditions for the weak convergence of the normalized intermediate term and its limit distribution are discussed in [12].

In this paper we prove that the conditions given in [12] are necessary. Moreover, the case of general intermediate terms is discussed through an example.

2. Basic results

Throughout this paper the following abbreviations will be used:

$$V^{n_1}_{n_2,n_3} = \left(\xi^{(n_1)}_{K_{n_2}} - b_{n_3}\right)/a_{n_3}\ ,\ G^*_{n_1}(x) = G_{n_1}(a_n x + b_n) = P\left(V^{n_1}_{n_1,n} < x\right)$$
$$\text{and } \Psi^*_n(x) = P\left(V^{\nu_n}_{K\nu_n,n} < x\right).$$

Consider an intermediate rank sequence $\{K_n\}$ that satisfies the Chibisov condition (1.1), i.e., $K_n \sim \ell^2 n^\alpha$ ($\ell > 0$, $0 < \alpha < 1$), and the following three conditions:

(A) $$G^*_n(x) \xrightarrow{w} \Phi\left(U_i^{(\beta)}(x)\right), \quad i \in \{1,2,3\},$$

where $U_i^{(\beta)}(x)$ has only one of the possible limiting forms defined in (1.2).

(B) $$P(\bar{\nu}_n < x) \xrightarrow{w} A(x),$$

where $A(x)$ is a d.f., $\bar{\nu}_n = (\nu_n - n)/n^{1-\frac{\alpha}{2}}$ and ν_n is a positive integer valued r.v. distributed independently of X_i, $i = 1, 2, \ldots, n$ for each n.

(C) $$\Psi^*_n(x) \xrightarrow{w} \Psi(x) = \int_{-\infty}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z).$$

Theorem 2.1. *The relations between the conditions (A), (B) and (C) are following:*

1. $(A) \cap (B) \subset (C)$.
2. $(A) \cap (C) \subset (B)$.
3. *If $A(x)$ is a d.f., which has at least one negative growth point and one positive growth point, then $(B) \cap (C) \subset (A)$.*

Theorem 2.2. *In the part (1) and the part (2) of Theorem 2.1 one can replace the conditions (B) and (C) by other equivalent conditions as follows: for condition (B)*

$$P(\bar{\nu}_n < x) = P\left((\nu_n - n)\sqrt{K_n}/n < x\right) \xrightarrow{w} A(x).$$

And for condition (C)

$$\Psi^*_n(x) \xrightarrow{w} \int_{-\infty}^{\infty} \Phi\left(2U_i^{(\beta)}(x) - U_i^{(\beta)}(a(z)x + b(z))\right) dA(z), \quad i \in \{1,2,3\},$$

where $0 < a(z) < \infty$, $-\infty < b(z) < \infty$ and the coefficients $a(z), b(z)$ can be obtained from the relations

$$\lim_{n\to\infty} \frac{a_{n + [nz/\sqrt{K_n}]}}{a_n} = a(z),$$
$$\lim_{n\to\infty} \frac{b_{n + [nz/\sqrt{K_n}]} - b_n}{a_n} = b(z).$$

3. Proofs

PROOF of $(A)\cap(B)\subset(C)$: It is easy to see that $\Phi\left(U_i^{(\beta)}(x)\right)=\Psi(x)$ if and only if,

$$A(x)=\begin{cases}0, & x\le 0,\\ 1, & x>0.\end{cases}$$

Hence, we consider only the case when $A(x)$ is a nondegenerate d.f.

Now, let $P_{ns}=P(\nu_n=s)$. Then, by using the total probability rule, we get

$$\Psi_n^*(x)=\sum_{s=1}^{\infty}P_{ns}P\left(V_{s,n}^{s}<x\right)=\sum_{s=1}^{\infty}P_{ns}G_s^*(x).$$

If we assume that $A_n(x)=P(\nu_n<x)$, then the latter sum is a Riemann sum of the integral

$$\Psi_n^*(x)=\int_1^{\infty}G_s^*(x)\,dA_n(s). \tag{3.1}$$

By virtue of condition (B) of Theorem 2.1 we get

$$A_n\left(n+n^{1-\frac{\alpha}{2}}z\right)\xrightarrow{w}A(z),\quad \text{as } n\to\infty. \tag{3.2}$$

According to the results of Chibisov [1] and the condition (A), we have

$$\lim_{n\to\infty}G_n^*(x)=\lim_{n\to\infty}\Phi\left(U_{n,n}^{n}(x)\right)=\Phi\left(U_i^{(\beta)}(x)\right),\quad i\in\{1,2,3\}, \tag{3.3}$$

where $U_{n_2,n_3}^{n_1}(x)=(n_1F(a_{n_3}x+b_{n_3})-K_{n_2})/\sqrt{K_{n_2}}$.

Let $s=n+\left[n^{1-\frac{\alpha}{2}}z\right]$ ($[\theta]$ means the integer part of θ). It is clear that $r_n(z)=s-n$ are sequences of integer values for which

$$r_n(z)/n^{1-\frac{\alpha}{2}}=\left[n^{1-\frac{\alpha}{2}}z\right]/n^{1-\frac{\alpha}{2}}\to z,\quad \text{as } n\to\infty.$$

Therefore, in view of the limit relation (1.1), we deduce that the sequence $\{K_n\}$ satisfies the following limit relation

$$\lim_{n\to\infty}\left(\sqrt{K_s}-\sqrt{K_n}\right)=\lim_{n\to\infty}\left(\sqrt{K_{n+r_n(z)}}-\sqrt{K_n}\right)=\frac{\alpha\ell z}{2}. \tag{3.4}$$

Furthermore, the following limit relations can easily be verified:

$$\left.\begin{aligned}&\frac{s}{n}\sqrt{\frac{K_n}{K_s}}\to 1,\\ &\frac{sK_n-nK_s}{n\sqrt{K_s}}\to \ell(1-\alpha)z,\quad \text{as } n\to\infty.\end{aligned}\right\} \tag{3.5}$$

On the other hand, by virtue of the results of Chibisov [1], we have

$$\lim_{s\to\infty} G_s^*(x) = \lim_{s\to\infty} \Phi\left(U_{s,n}^s(x)\right). \tag{3.6}$$

But, we can write

$$U_{s,n}^s(x) = \frac{s}{n}\sqrt{\frac{K_n}{K_s}}U_{n,n}^n(x) + \frac{sK_n - nK_s}{n\sqrt{K_s}}. \tag{3.7}$$

Hence, from (3.3), (3.5), (3.6) and (3.7), we deduce that

$$G_s^*(x) \xrightarrow{w} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right), \quad i \in \{1,2,3\}, \quad \text{as } n \to \infty, \tag{3.8}$$

for all finite values of z.

Moreover, from the continuity of the functions $U_i^{(\beta)}(x)$, $i = 1,2,3$, it follows that the convergence in (3.8) is uniform with respect to x.

Now, let c and c' ($c > c'$) be two continuity points of $A(x)$ such that $1 + A(c') - A(c) < \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Then

$$\left(\int_{-\infty}^{c'} + \int_{c}^{\infty}\right)\Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z) \le \left(\int_{-\infty}^{c'} + \int_{c}^{\infty}\right) dA(z) = \tag{3.9}$$
$$= 1 + A(c') - A(c) < \varepsilon.$$

On the order hand, if we consider condition (B) we get, for sufficiently large n

$$\left(\int_{-\infty}^{c'} + \int_{c}^{\infty}\right)G_s^*(x)\, dA_n\left(n + n^{1-\frac{\alpha}{2}}z\right) \le \left(\int_{-\infty}^{c'} + \int_{c}^{\infty}\right) dA_n\left(n + n^{1-\frac{\alpha}{2}}z\right) = \tag{3.10}$$
$$= 1 + A_n\left(n + n^{1-\frac{\alpha}{2}}c'\right) - A_n\left(n + n^{1-\frac{\alpha}{2}}c\right) \le 2(1 + A(c') - A(c)) \le 2\varepsilon.$$

Now, we estimate the difference

$$\int_{c'}^{c} G_s^*(x)\, dA_n\left(n + n^{1-\frac{\alpha}{2}}z\right) - \int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z).$$

By the triangle inequality we have

$$(3.11)\quad \left|\int_{c'}^{c} G_s^*(x)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) - \int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\, dA(z)\right| \leq$$

$$\leq \left|\int_{c'}^{c} G_s^*(x)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) - \int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right)\right| +$$

$$+\left|\int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) - \right.$$

$$\left. - \int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\, dA(z)\right|.$$

Since the convergence in relation (3.8) is uniform over the finite interval $c' \leq z \leq c$. Therefore, for arbitrary $\varepsilon > 0$ and sufficiently large n, we have

$$(3.12)\quad \left|\int_{c'}^{c} \left(G_s^*(x) - \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\right)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right)\right| \leq$$

$$\leq \left(A_n\left(n+n^{1-\frac{\alpha}{2}}c\right) - A_n\left(n+n^{1-\frac{\alpha}{2}}c'\right)\right)\varepsilon \leq \varepsilon.$$

In order to estimate the second difference on the right hand side of (3.11), we consider Riemann sums which are close to the integral there. Let N be a fixed number and $c' = c_0 < c_1 < \cdots < c_N = c$ be continuity points of $A(x)$.

Furthermore, let N and the c_j, $j = 1, 2, \ldots, N$ be such that

$$\left|\int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) - \right.$$

$$\left. -\sum_{j=1}^{N} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)c_j\right)\left(A_n\left(n+n^{1-\frac{\alpha}{2}}c_j\right) - A_n\left(n+n^{1-\frac{\alpha}{2}}c_{j-1}\right)\right)\right| \leq \varepsilon,$$

and

$$\left|\int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)z\right)\, dA(z) - \right.$$

$$\left. -\sum_{j=1}^{N} \Phi\left(U_i^{(\beta)}(x)+\ell(1-\alpha)c_j\right)\left(A(c_j) - A(c_{j-1})\right)\right| \leq \varepsilon.$$

By virtue of the assumption, $A_n\left(n+n^{1-\frac{\alpha}{2}}c_j\right) \xrightarrow{w} A(c_j)$, $0 \le j \le N$, as $n \to \infty$, the difference between two Riemann sums is less than ε for all sufficiently large n. Thus, once again by the triangle inequality, the absolute value of the difference of the integrals is smaller than 3ε. Combining this fact with (3.1), the left hand side of (3.11) becomes smaller than 4ε for all large n. Therefore, in view of (3.9) and (3.10) we get

$$
\begin{aligned}
(3.13)\quad & |\Psi_n^*(x) - \Psi(x)| \le \\
& \le \left| \int_{c'}^{c} G_s^*(x)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) - \int_{c'}^{c} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z) \right| + \\
& + \int_{-\infty}^{c'} G_s^*(x)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) + \int_{-\infty}^{c'} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z) + \\
& + \int_{c}^{\infty} G_s^*(x)\, dA_n\left(n+n^{1-\frac{\alpha}{2}}z\right) + \int_{c}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z) < 7\varepsilon
\end{aligned}
$$

Finally, from (3.13), (3.8)), (3.2) and (3.1) we obtain

$$\Psi_n^*(x) \xrightarrow{w} \int_{-\infty}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z).$$

■

Proof of $(A) \cap (C) \subset (B)$: We assume that, as $n \to \infty$, conditions (A) and (C) hold. We select a subsequence $\{n'\}$ of the sequence $\{n\}$ for which $P(\bar{\nu}_{n'} < z)$ converges weakly to an extended d.f. $A'(z)$, i.e., $A'(\infty) - A'(-\infty) \le 1$ and such a subsequence exists by the compactness of the d.f.'s. Then, by repeating the proof of the first part of Theorem 2.1 for the subsequence $\{n'\}$, with the exception that we choose c and c' ($c > c'$) so that $(A'(\infty) - A'(-\infty)) - (A'(c) - A'(c')) < \varepsilon$, we get

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA'(z), \quad i \in \{1,2,3\}$$

(the limits $\Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right)$ and $\Psi(x)$ do not depend on the subsequence $\{n'\}$ because the conditions (A) and (C) hold).

Since $\Psi(x)$ is a d.f. we get

$$\Psi(\infty) = 1 = \int_{-\infty}^{\infty} dA'(z) = A'(\infty) - A'(-\infty),$$

that is, $A'(z)$ is a d.f.

Now, if $P(\bar{\nu}_n < z)$ does not converge weakly, then we can select two subsequence $\{n'\}$ and $\{n''\}$ such that

$$P(\bar{\nu}_{n'} < z) \xrightarrow{w} A'(z) \text{ and } P(\bar{\nu}_{n''} < z) \xrightarrow{w} A''(z),$$

and in this case, due to condition (C), we have

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA'(z) = \tag{3.14}$$

$$= \int_{-\infty}^{\infty} \Phi(y + \ell(1-\alpha)z)\, dA''(z), \quad i \in \{1,2,3\}.$$

Let $G_1(y) = \int_{-\infty}^{\infty} \Phi(y + \ell(1-\alpha)z)\, dA'(z)$,

and $G_2(y) = \int_{-\infty}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA''(z)$.

If the functions $G_1(y)$ and $G_2(y)$ are determined for some interval $y_1 < y < y_2$, then in this interval both functions $G_1(y)$ and $G_2(y)$ will be analytic functions. By differentiating $G_1(y)$ and $G_2(y)$ with respect to y and from (3.14) we get

$$\int_{-\infty}^{\infty} e^{-\ell(1-\alpha)yz}\left(e^{-\frac{1}{2}\ell^2(1-\alpha)^2z^2}\, dA'(z)\right) =$$

$$= \int_{-\infty}^{\infty} e^{-\ell(1-\alpha)yz}\left(e^{-\frac{1}{2}\ell^2(1-\alpha)^2z^2}\, dA''(z)\right).$$

Let $\ell(1-\alpha)y = \varrho$. Then

$$\int_{-\infty}^{\infty} e^{-\varrho z}\left(e^{-\frac{1}{2}\ell^2(1-\alpha)^2z^2}\, dA'(z)\right) = \int_{-\infty}^{\infty} e^{-\varrho z}\left(e^{-\frac{1}{2}\ell^2(1-\alpha)^2z^2}\, dA''(z)\right).$$

Since the Laplace transformations with respect to the measures $\left\{e^{-\frac{1}{2}\ell^2(1-\alpha)^2z^2}\, dA'(z)\right\}$ and $\left\{e^{-\frac{1}{2}\ell^2(1-\alpha)^2z^2}\, dA''(z)\right\}$ coincide to each other, we deduce that $A'(z) = A''(z)$.

This completes the proof of the second part of Theorem 2.1. ∎

PROOF of $(B)\cap(C)\subset(A)$: We first prove that the sequence $\{V_{n,n}^n\}$ is stochastically bounded (see [**13**] p. 247). Assuming the converse, i.e., we assume that the sequence $\{V_{n,n}^n\}$ is stochastically unbounded, then we can find $\varepsilon_0>0$ such that for each x we have $\overline{\lim\limits_{n\to\infty}} P\left(\left|V_{n,n}^n\right|\geq x\right)\geq\varepsilon_0$, which is equivalent to the following assertion:

There exists $\varepsilon_1>0$ such that for all $x>0$ we have

$$\text{(a)}\qquad \overline{\lim_{n\to\infty}} P\left(V_{n,n}^n\geq x\right)\geq\frac{1}{2}\varepsilon_1,$$

or

$$\text{(b)}\qquad \overline{\lim_{n\to\infty}} P\left(V_{n,n}^n<-x\right)\geq\frac{1}{2}\varepsilon_1.$$

The assertions (a) and (b) are equivalent to the stochastically unboundedness of the sequence $\{V_{n,n}^n\}$ at ∞ (from the right) and $-\infty$ (from the left) respectively.

Now, assuming the condition that there exists a negative growth point of $A(x)$ (i.e., there exists a finite point $z_0<0$ such that $A(z_0+\varepsilon)-A(z_0-\varepsilon)>0$, for all $\varepsilon>0$) we can find two finite numbers a_1, a_2 such that $a_2<a_1<0$, $a_1<\alpha a_2$ and a positive finite number $\delta>0$ such that

$$P\left(a_2\leq\bar{\nu}_n<a_1\right)\geq\delta>0. \tag{3.15}$$

By using the total probability rule, we can write

$$P\left(V_{\nu_n,n}^{\nu_n}\geq x\right)=\sum_{j=1}^{\infty}P\left(\nu_n=j\right)P\left(V_{j,n}^j\geq x\right)\geq$$
$$\geq\sum_{n+z_{n,a_2}\leq j\leq n+z_{n,a_1}}P\left(\nu_n=j\right)P\left(V_{j,n}^j\geq x\right),$$

where $z_{n,a_i}=\left[na_i/\sqrt{K_n}\right],\quad i=1,2.$

In view of the well known relation

$$V_{j',n}^{j''}\geq V_{j,n}^{j''}\geq V_{j,n}^{j}\geq V_{j,n}^{j'}\geq V_{j'',n}^{j'},\qquad\text{for all } j'\geq j\geq j'', \tag{3.16}$$

we get

$$P\left(V_{\nu_n,n}^{\nu_n}\geq x\right)\geq P\left(V_{n+z_{n,a_2},n}^{n+z_{n,a_1}}\geq x\right)\sum_{n+z_{n,a_2}\leq j\leq n+z_{n,a_1}}P\left(\nu_n=j\right).$$

Turning to the relation (3.15) we have

$$\overline{\lim_{n\to\infty}} P\left(V_{\nu_n,n}^{\nu_n}\geq x\right)\geq\delta\,\overline{\lim_{n\to\infty}} P\left(V_{n+z_{n,a_2},n}^{n+z_{n,a_1}}\geq x\right). \tag{3.17}$$

On the other hand, we have

$$U^{n+z_{n,a_1}}_{n+z_{n,a_2},n}(x) =$$

$$= \frac{n+z_{n,a_1}}{n}\sqrt{\frac{K_n}{K_{n+z_{n,a_2}}}}\left(U^n_{n,n}(x)+\frac{(n+z_{n,a_1})K_n - nK_{n+z_{n,a_2}}}{(n+z_{n,a_1})\sqrt{K_n}}\right) =$$

$$= \frac{n+z_{n,a_1}}{n}\sqrt{\frac{K_n}{K_{n+z_{n,a_2}}}}\left(U^n_{n,n}(x)+\frac{(n+z_{n,a_1})K_n - nK_{n+z_{n,a_1}}}{(n+z_{n,a_1})\sqrt{K_n}}\right) +$$

$$+\sqrt{\frac{K_n}{K_{n+z_{n,a_2}}}}\left(\sqrt{\frac{K_{n+z_{n,a_1}}}{K_n}}+\sqrt{\frac{K_{n+z_{n,a_2}}}{K_n}}\right)\left(\left(\sqrt{K_{n+z_{n,a_1}}}-\sqrt{K_n}\right) - \right.$$

$$\left. - \left(\sqrt{K_{n+z_{n,a_2}}}-\sqrt{K_n}\right)\right).$$

Using the following relations

$$\frac{n+z_{n,a_1}}{n}\sqrt{\frac{K_n}{K_{n+z_{n,a_2}}}} \to 1,$$

$$\sqrt{\frac{K_n}{K_{n+z_{n,a_2}}}}\left(\sqrt{\frac{K_{n+z_{n,a_1}}}{K_n}}+\sqrt{\frac{K_{n+z_{n,a_2}}}{K_n}}\right) \to 2,$$

$$\frac{(n+z_{n,a_1})K_n - nK_{n+z_{n,a_1}}}{(n+z_{n,a_1})\sqrt{K_n}} \to a_1(1-\alpha)$$

$$\text{and } \sqrt{K_{n+z_{n,a_i}}}-\sqrt{K_n} \to \frac{\alpha a_i}{2}, \quad i=1,2, \qquad \text{as } n\to\infty,$$

we get

$$\overline{\lim_{n\to\infty}}\, U^{n+z_{n,a_1}}_{n+z_{n,a_2},n}(x) \le \overline{\lim_{n\to\infty}}\, U^n_{n,n}(x) + (a_1-\alpha a_2) \le \overline{\lim_{n\to\infty}}\, U^n_{n,n}(x), \quad \forall x>0.$$

Combining the last inequality with relations (3.17) and (3.3), we get

$$\overline{\lim_{n\to\infty}}\, P\left(V^{\nu_n}_{\nu_n,n}\ge x\right) \ge \delta\, \overline{\lim_{n\to\infty}}\, P\left(V^n_{n,n}\ge x\right) \ge \delta\frac{\varepsilon_1}{2} > 0,$$

which contradicts the stochastic boundedness of the sequence $\left\{V^{\nu n}_{\nu_n,n}\right\}$ (the stochastic boundedness of the sequence $\left\{V^{\nu_n}_{\nu_n,n}\right\}$ follows from the condition (C)).

Now, considering the case (b) and satisfying the condition that $A(x)$ has a positive finite growth point (i.e., there exists a finite point $z'_0>0$ such that $A(z'_0+\varepsilon)-A(z'_0-\varepsilon)>0$ for all $\varepsilon>0$), we can find two positive finite numbers a'_1, a'_2 such that $0<a'_2<a'_1$, $\alpha a'_1<a'_2$ and a positive finite value $\delta'>0$ such that

$$P\left(a'_2\le \bar{\nu}_n < a'_1\right) \ge \delta' > 0. \tag{3.18}$$

By similar discussion as the case (a), we can prove that

$$\overline{\lim_{n\to\infty}} P\left(V_{\nu_n,n}^{\nu_n} < -x\right) \geq \delta' \overline{\lim_{n\to\infty}} P\left(V_{n+z_{n,a_1'},n}^{n+z_{n,a_2'}} < -x\right),$$

and

$$\overline{\lim_{n\to\infty}} U_{n+z_{n,a_1'},n}^{n+z_{n,a_2'}}(-x) \geq \overline{\lim_{n\to\infty}} U_{n,n}^{n}(-x) + (a_2' - \alpha a_1') \geq \overline{\lim_{n\to\infty}} U_{n,n}^{n}(-x).$$

Hence, combining the last inequality with (3.18) and (3.3), we get

$$\overline{\lim_{n\to\infty}} P\left(V_{\nu_n,n}^{\nu_n} < -x\right) \geq \delta' \overline{\lim_{n\to\infty}} P\left(V_{n,n}^{n} < -x\right) \geq \delta' \frac{\varepsilon_1}{2} > 0,$$

which again contradicts the stochastic boundedness of the sequence $\left\{V_{\nu_n,n}^{\nu_n}\right\}$.

Therefore, the sequence $\left\{V_{n,n}^{n}\right\}$ is stochastically bounded.

Now, if the d.f. of $V_{n,n}^{n}$ $(G_n^*(x) = G_n(a_n x + b_n))$ did not converge weakly, then we could select two subsequences $\{n'\}$ and $\{n''\}$ of n such that $G_{n'}(a_{n'}x + b_{n'})$ would converge weakly to $\Phi\left(U_i^{(\beta)}(x)\right)$ and $G_{n''}(a_{n''}x + b_{n''})$ to another d.f. $\Phi\left(U_{i'}^{(\beta')}(x)\right)$, $i, i' \in \{1,2,3\}$. Appealing to the condition (C), we get

$$\Psi(x) = \int_{-\infty}^{\infty} \Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) dA(z) = \tag{3.19}$$

$$= \int_{-\infty}^{\infty} \Phi\left(U_{i'}^{(\beta')}(x) + \ell(1-\alpha)z\right) dA(z), \quad i, i' \in \{1,2,3\}.$$

If there exists x_0 such that $U_i^{(\beta)}(x_0) \neq U_{i'}^{(\beta')}(x_0)$, then ,without loss of generality, we can assume that $U_i^{(\beta)}(x_0) \geq U_{i'}^{(\beta')}(x_0)$. Consider the difference

$$R(x) = \int_{-\infty}^{\infty} \left(\Phi\left(U_i^{(\beta)}(x) + \ell(1-\alpha)z\right) - \Phi\left(U_{i'}^{(\beta')}(x) + \ell(1-\alpha)z\right)\right) dA(z).$$

And, in particular,

$$R(x_0) = \int_{-\infty}^{\infty} \int_{U_{i'}^{(\beta')}(x)+\ell(1-\alpha)z}^{U_i^{(\beta)}(x)+\ell(1-\alpha)z} e^{-t^2/2}\, dt\, dA(z) > 0,$$

which contradicts (3.19) ((3.19) means that $R(x) = 0$ for all x). ■

PROOF of Theorem 2.2: The proof follows from Theorem 2.1 and the following theorem of [1]:

The necessary condition for the limit relation $G_n^*(x) \xrightarrow{w} \Phi(U(x))$, as $n \to \infty$, where $U(x) = U_i^{(\beta)}(x)$, $i \in \{1,2,3\}$ is the existence of coefficients $a(\gamma)$ and $b(\gamma)$ for which

$$\lim_{n\to\infty} \frac{a_{n+[n\gamma/K_n]}}{a_n} = a(\gamma) \text{ and } \lim_{n\to\infty} \frac{b_{n+[n\gamma/K_n]} - b_n}{a_n} = b(\gamma),$$

where $0 < a(\gamma) < \infty$, $-\infty < b(\gamma) < \infty$ and the functions $U(x)$ satisfies the following condition

$$U(a(\gamma)x + b(\gamma)) + (1-\alpha)\gamma = U(x), \qquad \text{for all } x.$$

■

4. Discussion about a general intermediate term

We consider in this section the general intermediate term $\xi_{K_n}^n$ with nondecreasing rank sequence $\{K_n\}$ in n, such that $K_n \to \infty$ and $K_n/n \to 0$, as $n \to \infty$. And we shall show that the d.f. $\Psi_n^*(x)$ does not converge weakly to any nondegenerate d.f. under conditions (I) and (II) of Theorem 1.1 (i.e., when $G_n^*(x)$ converges weakly to a nondegenerate limit d.f. and $P\left(\frac{\nu_n}{n} < x\right) = A_n(nx) \xrightarrow{w} A(x)$, $A(x)$ is a d.f.).

In view of the results of [2], condition (I), for any general nondecreasing intermediate rank sequence, can be written in the form

$$\text{(I')} \qquad G_n^*(x) \xrightarrow{w} G(U(x)) = \Phi\left(U_i^{(\beta)}(x)\right), \quad i \in \{1,2,3\}, \quad \text{as } n \to \infty.$$

Also the condition (II) can be deduced from the general condition

$$\text{(II')} \qquad P(\nu_n/n^\rho < x) \xrightarrow{w} A(x), \quad \rho \geq 1, \quad \text{as } n \to \infty.$$

It can be shown that d.f. $\Psi_n^*(x)$ does not converge weakly to any nondegenerate d.f. under the two conditions (I') and (II'). In fact, by using the total probability rule and by virtue of the results of [2], we can write

$$\Psi_n^*(x) = \int_1^\infty G_s(x)\, dA_n(s),$$

where $\lim_{s\to\infty} G_s^*(x) = \lim_{s\to\infty} \Phi\left(U_{s,n}^s(x)\right)$.

Let $s = zn^\rho$. We get, for large n

$$\Psi_n^*(x) \sim \int_0^\infty \Phi\left(zn^{\rho-1}\sqrt{\frac{K_n}{K_{zn^\rho}}}U_{n,n}^n(x) + \frac{zn^\rho}{\sqrt{K_{zn^\rho}}}\left(\frac{K_n}{n} - \frac{K_{zn^\rho}}{zn^\rho}\right)\right) dA_n(zn^\rho),$$

where $U_{n,n}^n(x) \xrightarrow{w} U_i^{(\beta)}(x)$, $A_n(zn^\rho) \xrightarrow{w} A(z)$, as $n \to \infty$ (due to the conditions (I') and (II')).

Now, let us consider $\Psi^*_{n_r}(x)$, where $\{n_r\}$ is a subsequence of the sequence $\{n\}$, for which $\frac{n_{r+1}}{n_r} \to R > 1$ as $r \to \infty$ and R is any integer number. It is easy to show that

$$\frac{zn_r^\rho}{\sqrt{K_{zn_r^\rho}}}\left(\frac{K_{n_r}}{n_r} - \frac{K_{zn_r^\rho}}{zn_r^\rho}\right) \geq \frac{n_{r+1}}{\sqrt{K_{n_{r+1}}}}\left(\frac{K_{n_r}}{n_r} - \frac{K_{n_{r+1}}}{n_{r+1}}\right), \quad \text{for all } z \geq R.$$

Turning to the result of Wu [2], we see that the sequence $\frac{n_{r+1}}{\sqrt{K_{n_{r+1}}}}\left(\frac{K_{n_r}}{n_r} - \frac{K_{n_{r+1}}}{n_{r+1}}\right)$ has an infinite limit point. Therefore, $\overline{\lim\limits_{n\to\infty}} \frac{zn_r^\rho}{\sqrt{K_{zn_r^\rho}}}\left(\frac{K_{n_r}}{n_r} - \frac{K_{zn_r^\rho}}{zn_r^\rho}\right) = \infty$, $z \geq R$.

Since R is arbitrarily chosen, then $\Psi^*_{n_r}(x)$ does not converge weakly to any nondegenerate d.f. Hence, the result.

It is well known that the asymptotic behaviour of the intermediate terms is independent of that of the extreme or of the central terms, for fixed sample size. The following remark provides some alternatives, for random sample size.

Remark 4.1. If we have a sample of random size and an intermediate term with rank sequence satisfying Chibisov condition (1.1) such that the conditions (A) and (B) hold with nondegenerate d.f.'s $\Phi\left(U_i^{(\beta)}(x)\right)$, $i \in \{1,2,3\}$ and $A(x)$, then the condition (C) is also satisfied with nondegenerate d.f. $\Psi(x)$. Moreover, the d.f.'s of all extreme or central terms can only converge weakly to the d.f.'s $\Psi(x) = G(U(x))$, where $G(x)$ and $U(x)$ are defined for both extreme and central terms in Theorem 1.1. On the other hand, the convergence of the d.f.'s of extreme or central terms to nondegenerate d.f.'s such that the conditions (I) and (II) are satisfied in Theorem 1.1 with nondegenerate d.f.'s leads to the nonconvergence of the d.f.'s of the intermediate terms.

PROOF : Assume that we have a random sample size ν_n for which $P\left(\frac{\nu_n - n}{n^{1-\frac{\alpha}{2}}} < x\right) \xrightarrow{w} A(x)$, $A(x)$ is nondegenerate d.f. and $0 < \alpha < 1$. Since $n/n^{1-\frac{\alpha}{2}} \to \infty$, then by using Lemma 4.1.1 of [14] we get

$$P(\nu_n/n < x) \to \begin{cases} 0, & x \leq 1, \\ 1, & x > 1. \end{cases} \tag{4.1}$$

It is easy to show that the condition (4.1) is the necessary and sufficient condition to get the equality $\Psi(x) = G(U(x))$ in Theorem 1.1. The second part of Remark 4.1 follows from Theorems 1.1, 2.1 and the above example. ∎

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