Eduard Feireisl

Forced vibrations in one-dimensional nonlinear thermoelasticity as a local coercive-like problem

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 2, 243--255

Persistent URL: `http://dml.cz/dmlcz/106854`

Comment.Math.Univ.Carolinae 31,2(1990)243-255

Forced vibrations in one–dimensional nonlinear thermoelasticity as a local coercive–like problem

EDUARD FEIREISL

Abstract. The aim of the paper is to establish the existence of smooth time–periodic solutions of the hyperbolic–parabolic system of equations of thermoelasticity. The method employed takes advantage of the topological degree of mapping and does not rely upon any approximate linearization. The function spaces framework comprises the Lebesgue and Sobolev spaces of integrable functions.

Keywords: equations of nonlinear thermoelasticity, time–periodic solution, nonlinear operator equation

Classification: 35B10, 73U5, 35Q20

1. PRELIMINARIES

It is an inherent feature of second order nonlinear hyperbolic equations that in the case of one spatial dimension solutions tend to develop singularities after a finite time, no matter how smooth and small the date are (see e.g. Klainerman [**6**]). In the face of it, many interesting problems of mathematical physics call for solutions determined globally in time. To remove this seemingly unsurmountable stumbling–block, one is often left with the choice of either of the alternative methods of tackling the problems.

The first approach is to generalize the concept of solution and turn to weak, ultimately to measure–valued solutions. Recently, the compensated compactness theory of Murat and Tartar has truly embraced this field of research and a relatively long list of contributions culminated in the remarkable work of DiPerna [**3**] in which the first large–data existence result was established.

In this paper, we pursue the latter course based on allowing for the possibility of a certain dissipative mechanism. The origin of this idea may be traced back to Greenberg, MacCamy, Mizel [4] where, however, the presence of a viscosity term converts the equation into a parabolic one. Nevertheless, other recent treatments of Matsumura [**7**], Dafermos [**1**], Shibata [**10**], this group being rather representative than complete, confirmed the conjecture that certain quasilinear or even fully nonlinear hyperbolic equations with dissipation possess global classical solutions on condition that the data are, roughly speaking, small and smooth enough.

The decay mechanism we shall deal with is represented by the interaction between mechanical and thermal effects in elastic bodies. Under certain circumstances (see Slemrod [**11**]), the purely longitudinal motion of a body which occupies the region $0 \le x \le \ell$ can be modelled by the quasilinear system

$$(S_1) \qquad u_{tt} = \psi_F(u_x + 1, \Theta + T_0)_x + b,$$

$$(S_2) \qquad \varrho(\Theta + T_0)(\psi_T(u_x + 1, \Theta + T_0)_t) = q(\Theta_x)_x,$$

where $u = u(x,t)$, $\Theta = \Theta(x,t)$ are respectively the displacement from equilibrium at the instant t of the longitudinal coordinate and the absolute temperature. The function $\psi = \psi(F,T)$ represents the specific Helmholtz free energy, b is the specific body force, ϱ denotes the density and the symbol q stands for the heat flux.

The faces are traction free, which corresponds to the Neumann boundary conditions

$$(B_1) \qquad u_x(0,t) = u_x(\ell,t) = 0, \qquad t \in R^1.$$

Finally, the boundary temperature is maintained at the reference level T_0, i.e.

$$(B_2) \qquad \Theta(0,t) = \Theta(\ell,t) = 0, \qquad t \in R^1.$$

Our interest is directed to what happens when the body force is ω–periodic with respect to t:

$$(A_1) \qquad b(x,t+\omega) = b(x,t) \quad \text{for all } 0 \le x \le \ell, \qquad t \in R^1.$$

More specifically, the question arises whether the dissipative effect of thermal diffusion may result in the existence of solutions with the same property:

$$(P) \qquad u(x,t+\omega) = u(x,t), \quad \Theta(x,t+\omega) = \Theta(x,t).$$

As stated in Section 2, Theorem 1, we can report an affirmative answer on condition that the function b is small, smooth and satisfies

$$(A_2) \qquad \int_0^\ell \int_0^\omega b(x,t)\,dt\,dx = 0.$$

The paper was motivated by the work of Day [2] where the corresponding linear problem is treated. As to the initial–boundary value problem for (S_1), (S_2), we refer to Slemrod [11], Zheng [12], [13], Zheng, Shen [14], or Racke [8] for relevant results.

Influenced by a work of Kato [5], we make use of a rather nonstandard approach which does not rely on any approximate linearization. Being hyperbolic, the problem in question can not, of course, be coercive not even locally. On the other hand, the underlying idea remains the same as in [5].

Our starting point will be the standard Galerkin approximation, with the help of which the task reduces to solving a sequence of a finite system of nonlinear equations. The resulting problem can be handled successfully by means of the classical degree theory (see Section 3 for the result and Section 4 for the proof). It is worthwhile to note that this crucial step depends heavily on the existence of suitable a priori estimates. Moreover, the same estimates turn out to be strong enough to ensure a limit passage in the sequence of approximate solutions. To cope with the nonlinear terms, certain standard compactness arguments are used. Eventually, the embedding relations of Sobolev type enable us to complete the proof of the existence of at least one classical solution.

2. Preliminary prerequisities and main results

Throughout the whole text we use the symbol c_i, $i = 1, 2, \ldots$ to denote any strictly positive constant, the denomination h stands for all real functions, continuous on $[0, \infty)$, with the common property $\lim_{r \to 0+} h(r) = 0$.

Seeing that the concrete values of ϱ, ℓ, ω do not matter, we may set $\varrho \equiv 1$, $\ell = \pi$, $\omega = 2\pi$.

As to the function spaces, we will only be interested in those containing time–periodic functions. Accordingly, it seems convenient to introduce the Lebesgue spaces $L_p(Q)$ and the Sobolev spaces $H^k(Q)$ determined respectively as the closure of all smooth (real) functions on the cylinder

$$Q = \{(x,t) | x \in [0,\pi], t \in [0,2\pi] | \{0, 2\pi\}\}$$

with respect to the norm

$$|v|_p = \left(\iint_Q |v|^p \, dx \, dt \right)^{\frac{1}{p}}, \quad \|v\|_k = \max \left\{ |v|_2, \left| D_x^k v \right|_2, \left| D_t^k v \right|_2 \right\}.$$

Here (and always) the symbol D_y^k indicates the k–th derivative with respect to the variable $y = x, t$. To simplify the notation, we postulate $D_y^k v = 0$ whenever $k < 0$.

Besides, $C^k(Q)$ will denote the Banach space of functions having all derivatives up to the order k continuous on Q.

In accordance with its physical interpretation, the function $\psi = \psi(F, T)$ is defined and smooth on a neighbourhood of the point $(1, T_0)$. For the sake of definiteness, we set

$$(A_3) \qquad \psi(1, T_0) = \psi_F(1, T_0) = 0.$$

Similarly as in Slemrod [11], the following hypotheses will be assumed, namely,

$$(A_4) \qquad \psi_{FF}(1,T) > 0, \quad \psi_{TT}(1,T) < 0, \quad \psi_{FT}(1,T) \neq 0 \quad \text{for any } T > 0$$

along with

$$(A_5) \qquad q(0) = 0, \quad q'(z) \leq -c_1 < 0 \quad \text{for all } z.$$

We are in a position to formulate our main result.

Theorem 1.

Let the conditions (A_1)–(A_5) be satisfied.

Then for a given integer $k \geq 3$ there exists a positive number $\varepsilon > 0$ such that the problem (S_i), (B_i), (P), $i = 1, 2$ possesses at least one classical solution pair u, Θ on condition that

$$(2.1) \qquad \sum_{j=1}^{k} \left\| D_t^{j-1} b \right\|_1 < \varepsilon.$$

Moreover, the function u being normalized by

$$\iint_Q u(x,t)\,dx\,dt = 0, \tag{2.2}$$

the estimate

$$\begin{aligned} R(u,\Theta) \stackrel{def}{=} \sum_{j=2}^{k} \left\| D_t^{j-2} u \right\|_3 + \sum_{\ell=0}^{k} \left(\left| D_t^{\ell}\Theta \right|_2 + \left| D_t^{\ell}\Theta_x \right|_2 \right) + \\ + \sum_{\ell=1}^{k} \left| D_t^{\ell-1}\Theta_{xx} \right|_2 < r \end{aligned} \tag{2.3}$$

holds, where $r = h\left(\sum_{j=1}^{k} \left\| D_t^{j-1} b \right\|_1\right)$.

3. The proof of Theorem 1

In accordance with the conditions (B_i), (P), $i = 1,2$, a suitable platform for the Galerkin approximation is formed by the system of functions

$$e_{kj}(x,t) = \begin{cases} \sin jt \cos kx, & j = 1,2,\dots, \qquad k = 0,1,\dots \\ \cos jt \cos kx, & j = 0,-1,\dots, \qquad k = 0,1,\dots \end{cases}$$

$$f_{kj}(x,t) = \begin{cases} \sin jt \sin kx, & j = 1,2,\dots, \qquad k = 1,2,\dots \\ \cos jt \sin kx, & j = 0,-1,\dots, \qquad k = 1,2,\dots \end{cases}$$

Specifically, consider a scale of finite–dimensional spaces

$$\begin{aligned} E_n = \{\ [u,\Theta] | u \in \operatorname{span}\{e_{kj} | k, |j| \le n, |j| + k > 0\}, \\ \Theta \in \operatorname{span}\{f_{kj} | k, |j| \le n\}\ \}, \qquad n = 1,2,\dots \end{aligned}$$

provided with a Hilbert structure by means of the inner product

$$\langle [u^1,\Theta^1],[u^2,\Theta^2]\rangle = (u^1,u^2) + (\Theta^1,\Theta^2),$$

where

$$(v,w) = \iint_Q vw\,dx\,dt.$$

With the help of the functional R defined in (2.3) we set

$$K_n(r) = \{[u,\Theta] | [u,\Theta] \in E_n, R(u,\Theta) < r\}.$$

Observe that K_n is an open bounded convex neighbourhood of the point $0 \in E_n$.

The main tool we are going to exploit for solving nonlinear equations is the following assertion, an easy consequence of the Poincaré–Bohl theorem.

Proposition 1. [**9** ,Chapter 4, Corollary 1.16].

Let $E, <,>$ be respectively a finite dimensional Hilbert space and the inner product on E. Let $\emptyset \neq K \subset E$ be an open bounded convex neighbourhood of the point $0 \in E$. Finally, let $\mathbf{F}\colon E \longrightarrow E$ and $\mathbf{L}\colon E \longrightarrow E$ be respectively a continuous mapping and a linear mapping satisfying the inequality

$$\langle \mathbf{F}(v), \mathbf{L}v \rangle > 0 \tag{3.1}$$

for all $v \in \partial K$.

Then there is at least one point $v^0 \in K$, a solution of the equation

$$\mathbf{F}(v^0) = 0. \tag{3.2}$$

Turning attention to the original problem, we set

$$F_1(u, \Theta) = u_{tt} - \psi_F(u_x + 1, \Theta + T_0)_x - b,$$
$$F_2(u, \Theta) = \psi_T(u_x + 1, \Theta + T_0)_t - \frac{1}{\Theta + T_0} q(\theta_x)_x$$

along with

$$L_1(u, \Theta) = \sum_{j=0}^{k} (-1)^j D_t^{2j+1} u + \sum_{j=1}^{k} (-1)^j \Big(D_t^{2j-1} u_{xx} + \delta_2 D_t^{2j-2} u_{xx} -$$
$$- \delta_3 D_t^{2j} u \Big) + \delta_4 \sum_{j=2}^{k} (-1)^j D_t^{2j-4} u_{xxxx},$$
$$L_2(u, \Theta) = \sum_{j=0}^{k} (-1)^{j+1} D_t^{2j} \Theta + \sum_{j=1}^{k} (-1)^{j+1} \Big(D_t^{2j-2} \Theta_{xx} + \delta_1 \tilde{s} D_t^{2j-1} u_x \Big),$$

where $\tilde{s} = \operatorname{sgn} \psi_{FT}(1, T_0)$. The numbers $\delta_i > 0$, $i = 1, \ldots, 4$ will be determined later.

Note that the linear operator $\mathbf{L} = (L_1, L_2)$ maps E_n into itself while $\mathbf{F} = (F_1, F_2)$ may be viewed as a nonlinear mapping from E_n into its dual $(E_n)^*$, the latter space being identified with E_n via the Riesz isometry.

What is needed, is an assertion relevant to (3.1):

Lemma 1.

There exist strictly positive constants $\varepsilon, \delta_i > 0$, $i = 1, \ldots, 4$, independent of n, having the following property.

For every b satisfying (2.1) there is a number $r > 0$, $r = h\left(\sum_{j=1}^{k} \left\| D_t^j b \right\|_1\right)$ such that the inequality

$$\langle \mathbf{F}(u, \Theta), \mathbf{L}(u, \Theta) \rangle = (F_1(u, \Theta), L_1(u,, \Theta)) + (F_2(u, \Theta), L_2(u, \Theta)) > 0 \tag{3.3}$$

holds whenever $[u,\Theta] \in \partial K_n(r)$.

What the lemma actually says is that there is a sufficiently large family of a priori estimates related to the small solutions of the problems $F_1 = F_2 = 0$. The proof being rather technical, we postpone it to the next section.

When taken for granted, Lemma 1 in conjunction with Proposition 1 make the proof of Theorem 1 quite elementary.

Indeed, Proposition 1 guarantees the existence of the sequence $\{u^n,\Theta^n\}_{n-1}^{\infty}$ of approximate solutions satisfying

$$(S_1^n) \qquad (F_1(u^n,\Theta^n),w_1) = 0,$$
$$(S_2^n) \qquad (F_2(u^n,\Theta^n),w_2) = 0$$

for any pair $[w_1,w_2] \in E_n$. Observe that, b being small (in $C(Q)$), we have $T_0+\Theta^n \geq c_2 > 0$.

Next, in accordance with the definition of E_n, we get

$$(3.4) \qquad \iint\limits_Q u^n(x,t)\,dx\,dt = 0, \qquad n = 1,2,\ldots$$

Eventually, the most important piece of information is contained in the fact that

$$(3.5) \qquad [u^n,\Theta^n] \in K_n(r), \qquad n = 1,2,\ldots$$

To see this, consider an arbitrary pair $[u,\Theta] \in K_n(r)$, the actual value of n being irrelevant. Since $k \geq 3$, the terms u_{xx}, u_{xxt}, viewed as vector functions of the variable t (which will be sometimes the case), belong to a bounded subset of the space $L_2((0,2\pi);H^1(0,\pi))$, which yields the boundedness of u_{xx} in $C^{\frac{1}{2}}\left([0,2\pi],H^1(0,\pi)\right)$.

As the embedding $H^1(0,\pi) \circlearrowleft C[0,\pi]$ is compact, a generalized version of the Arzelà–Ascoli theorem implies the compactness of u_{xx} in $C(Q)$. Following the line of the same arguments we deduce that

(3.6) $\{u^n\}, \{\Theta^n\}_{n=1}^{\infty}$ belong respectively to a compact subset of $C^2(Q), C^1(Q)$.

Consequently, the only term lying beyond the scope of the above considerations is Θ_{xx}^n which is bounded in $L_2(Q)$ only. Passing to subsequences if necessary, we may suppose that

(3.7) $u^n \to u$ strongly in $C^2(Q)$,
$\Theta^n \to \Theta$ strongly in $C^1(Q)$, $\Theta^n \to \Theta$ weakly in $H^2(Q)$.

By virtue of (S_1^n), (S_2^n), we obtain

$$(3.8) \qquad (F_1(u,\Theta),w_1) = 0, \quad (F_2(u,\Theta),w_2) = 0$$

for any pair $[w_1,w_2] \in \bigcup_{n=1}^{\infty} E_n$. Thus in view of (A_2), we infer that $[u,\Theta]$ is a (strong) solution of the problem (S_i), (B_i), (P), $i = 1,2$.

To conclude with, the term Θ_{xx} can be expressed in (S_2) in order to demonstrate $\Theta_{xx} \in C(Q)$.

Theorem 1 has been proved.

4. THE PROOF OF LEMMA 1

To begin with, suppose that

$$(4.1)\qquad [u,\Theta]\in\overline{K_n(r)}, r>0,\quad \sum_{j=1}^{k}\left\|D_t^{j-1}b\right\|_1=\beta.$$

The proof of Lemma 1 splits into a number of auxiliary steps, we now proceed to assemble.

4.1. $\left\|D_t^{j-2}u_{xx}\right\|_1, \left\|D_t^{j-1}u_x\right\|_1, \left\|D_t^{j-1}\Theta_x\right\|_1, \left\|D_t^{j-1}\Theta\right\|_1 \leq c_3 r$ *for all* $j\leq k$.

PROOF 4.1: An immediate consequence of (4.1). ■

4.2. *For any* $i=1,\dots,m$ *let* v_i *be one of the quantities appearing in the* H^1*-norm on the left-hand side of 4.1.*

Then the product $\prod_{i=1}^m v_i$ *belongs to* $L_2(Q)$ *and*

$$\left|\prod_{i=1}^m v_i\right|_2\leq c_4(m)r^m.$$

PROOF 4.2: A straightforward application of the Hölder inequality in conjunction with the embedding relation $H^1(Q)\circlearrowleft L_q(Q)$, $q<\infty$ arbitrary. ■

4.3. $\|u\|_{C^2(Q)}, \|\Theta\|_{C^1(Q)}, \left\|D_t^{j-2}\Theta_x\right\|_{C(Q)}\leq c_5 r,\quad j\leq k.$

PROOF 4.3: As to the first and second term, we refer to Section 3.

Next, we have $D_t^{j-2}\Theta_x\in L_2((0,2\pi),H^1(0,\pi))$, $(D_t^{j-2}\Theta_x)_t \in L_2((0,2\pi),H^1(0,\pi))$, which, along with the relation $H^1(0,\pi)\circlearrowleft C[0,\pi]$, completes the proof. ■

4.4 The chain rule :. *Let the functions* $u=u(x,t)$, $v=v(x,t)$, $f=f(u,v)$ *have the prerequisite derivatives for the analysis to be valid. Then we have*

$$D_t^jD_x^sf(u,v)=\sum D_u^{\alpha_1}D_v^{\alpha_2}f(u,v)\left(D_t^{j_1}D_x^{k_1}u\right)^{\ell_1}\dots\left(D_t^{j_n}D_x^{k_n}u\right)^{\ell_n}(D_t^{p_1}D_x^{q_1}v)^{z_1}\dots$$
$$\dots(D_t^{p_m}D_x^{q_m}v)^{z_m}$$

where $\sum_{i=1}^n\ell_i=\alpha_1$, $\sum_{i=1}^m z_i=\alpha_2, \sum_{i=1}^n j_i+\sum_{i=1}^m p_i\leq j$, $\sum_{i=1}^n k_i+\sum_{i=1}^m q_i\leq s$, $\sum_{i=1}^n(j_i+k_i)\ell_i+\sum_{i=1}^m(p_i+q_i)z_i=j+s$.

4.5. *Given an integer* $j\in[0,k]$, *we have the inequality*

$$\left(F_1(u,\Theta),(-1)^j\left(D_t^{2j+1}u+D_t^{2j-1}u_{xx}\right)\right)\geq$$
$$\geq-\left(\psi_{FT}D_t^{j+1}\Theta,D_t^ju_x\right)-\left(\psi_{FT}D_t^j\Theta_x,D_t^{j-1}u_{xx}\right)-h(r)r^2-\beta r$$

(recall our agreement that $D^\ell v \equiv 0$ whenever $\ell < 0$).

PROOF 4.5: With (B_i), (P), $i = 1,2$ in mind, we are allowed to carry out a series of by-parts integrations in order to rewrite the left-hand side of the above inequality in the form

$$-\left(D_t^{j+1}\psi_F, D_t^j u_x\right) - \left(D_t^j D_x \psi_F, D_t^{j-1} u_{xx}\right) - \left(D_t^j b, D_t^{j+1} u + D_t^{j-1} u_{xx}\right).$$

Making use of the estimates 4.1–4.3 in conjunction with the chain rule we get

$$-\left(D_t^{j+1}\psi_F, D_t^j u_x\right) \geq -\left(\psi_{FF} D_t^{j+1} u_x, D_t^j u_x\right) - \left(\psi_{FT} D_t^{j+1}\Theta, D_t^j u_x\right) - h(r)r^2.$$

Moreover, we have

$$-\left(\psi_{FF} D_t^{j+1} u_x, D_t^j u_x\right) = -\left(\psi_{FF}, \frac{1}{2} D_t \left(D_t^j u_x\right)^2\right) = \left(D_t \psi_{FF}, \frac{1}{2}\left(D_t^j u_x\right)^2\right) \geq$$
$$\geq -h(r)r^2.$$

Since the same arguments apply to the term $\left(D_t^j D_x \psi_F, D_t^{j-1} u_{xx}\right)$, the Hölder inequality completes the proof. ■

4.6.

$$\left(F_2(u,\Theta), (-1)^{j+1}\left(D_t^{2j}\Theta + D_t^{2j-2}\Theta_{xx}\right)\right) \geq -\left(\psi_{FT} D_t^{j+1} u_x, D_t^j \Theta\right) -$$
$$-\left(\psi_{FT} D_t^j u_{xx}, D_t^{j-1}\Theta_x\right) + c_6\left(\left|D_t^j \Theta_x\right|_2^2 + \left|D_t^{j-1}\Theta_{xx}\right|_2^2\right) - h(r)r^2$$

for all $0 \leq j \leq k$.

PROOF 4.6: Bounding the left-hand side as in 4.5 we arrive at the inequality

$$\left(D_t \psi_T, (-1)^{j+1}\left(D_t^{2j}\Theta + D_t^{2j-2}\Theta_{xx}\right)\right) = -\left(D_t^{j+1}\psi_T, D_t^j\Theta\right) - \tag{1}$$
$$-\left(D_t^j D_x \psi_T, D_t^{j-1}\Theta_x\right) \geq$$
$$\geq -\left(\psi_{FT} D_t^{j+1} u_x, D_t^j\Theta\right) - \left(\psi_{FT} D_t^j u_{xx}, D_t^{j-1}\Theta_x\right) - h(r)r^2.$$

Next, we have to cope with the term

$$-\left(\frac{1}{T_0+\Theta} q(\Theta_x)_x, (-1)^{j+1} D_t^{2j}\Theta\right) = -\left(D_t^j\left(\frac{1}{T_0+\Theta} q(\Theta_x)\right), D_t^j \Theta_x\right) + \tag{2}$$
$$+\left(D_t^j\left(\frac{\Theta_x}{(T_0+\Theta)^2} q(\Theta_x)\right), D_t^j\Theta\right) \geq$$

(according to 4.1–4.4)

$$\geq -\left(\frac{1}{T_0+\Theta} q'(\Theta_x) D_t^j \Theta_x, D_t^j \Theta_x\right) - h(r)r^2 \geq$$

(by virtue of (A_5))

$$\geq c_7 \left|D_t^j \Theta_x\right|_2^2 - h(r)_r^2.$$

Finally, we are left with the most difficult term:

$$(3) \quad -\left(\frac{1}{T_0+\Theta}q'(\Theta_x)\Theta_{xx},(-1)^{j+1}D_t^{2j-2}\Theta_{xx}\right)=$$
$$=-\left(D_t^{j-1}\left(\frac{1}{T_0+\Theta}q'(\Theta_x)\Theta_{xx}\right),D_t^{j-1}\Theta_{xx}\right)\geq$$
$$\text{(in view of } (A_5)\text{, 4.3)}$$
$$\geq c_8\left|D_t^{j-1}\Theta_{xx}\right|_2^2-\left(\frac{1}{T_0+\Theta}q'(\Theta_x)D_t^{j-1}\Theta_x\Theta_{xx},D_t^{j-1}\Theta_{xx}\right)-h(r)r^2.$$

Consequently, all we need to prove 4.6, is the estimate

$$(4) \qquad \left|D_t^{j-1}\Theta_x\Theta_{xx}\right|_2\leq c_9r^2.$$

To this end, observe that 4.1 yields $D_t^{j-1}\Theta_x \in L_2((0,2\pi);H^1(0,\pi))$, $\Theta_{xx}\in C([0,2\pi],L_2(0,\pi))$. Since $H^1(0,\pi)\circlearrowleft C[0,\pi]$, (4) follows. ■

4.7.

$$\left(F_1(u,\Theta),\sum_{j=0}^{k}(-1)^j\left(D_t^{2j+1}u+D_t^{2j-1}u_{xx}\right)\right)+$$
$$+\left(F_2(u,\Theta),\sum_{j=0}^{k}(-1)^{j+1}\left(D_t^{2j}\Theta+D_t^{2j-2}\Theta_{xx}\right)\right)\geq$$
$$\geq c_{10}\left(\sum_{j=0}^{k}\left|D_t^j\Theta\right|_2^2+\left|D_t^j\Theta_x\right|_2^2+\left|D_t^{j-1}\Theta_{xx}\right|_2^2\right)-c_{11}\beta r-h(r)r^2.$$

PROOF 4.7: The relation 4.7 follows from 4.5, 4.6. To see this, we have to integrate by parts

$$-\left(\psi_{FT}D_t^{j+1}u_x,D_t^j\Theta\right)-\left(\psi_{FT}D_t^{j+1}\Theta,D_t^ju_x\right)=$$
$$=-\left(\psi_{FT},D_t\left(D_t^ju_xD_t^j\Theta\right)\right)=\left(D_t\psi_{FT}D_t^ju_x,D_t^j\Theta\right)\geq-h(r)r^2 \qquad \text{etc.}$$

Moreover, recall the Poincaré inequality $\left|D_t^j\Theta_x\right|_2\geq c_{12}\left|D_t^j\Theta\right|_2$, which completes the proof. ■

4.8.

$$\left(F_2(u,\Theta),(-1)^{j+1}\tilde{s}D_t^{2j-1}u_x\right)\geq c_{13}\left|D_t^ju_x\right|_2^2-$$
$$-c_{14}\left(\left|D_t^j\Theta\right|_2^2+\left|D_t^j\Theta_x\right|_2^2+\left|D_t^{j-1}\Theta_{xx}\right|_2^2\right)-h(r)r^2$$

for any $1 \leq j \leq k$.

PROOF 4.8: We may decompose $\left(F_2(u,\Theta),(-1)^{j+1}\tilde{s}D_t^{2j-1}u_x\right) = A_1 + A_2$, where

(1) $A_1 = \tilde{s}\left(D_t^j\psi_T, D_t^j u_x\right) \geq$

(according to (A_4), 4.1)

$$\geq c_{15}\left|D_t^j u_x\right|_2^2 + \tilde{s}\left(\psi_{TT}D_t^j\Theta, D_t^j u_x\right) - h(r)r^2 \geq$$

$$\geq c_{16}\left|D_t^j u_x\right|_2^2 - c_{17}\left|D_t^j\Theta\right|_2^2 - h(r)r^2;$$

(2) $A_2 = -\tilde{s}\left(D_t^{j-1}\left(\frac{1}{T_0+\Theta}q'(\Theta_x)\Theta_{xx}\right), D_t^j u_x\right) \geq$

(in view of 4.3)

$$\geq -\tilde{s}\left(\frac{1}{T_0+\Theta}q'(\Theta_x)D_t^{j-1}\Theta_{xx}, D_t^j u_x\right) - \tilde{s}\left(D_t^{j-1}\left(\frac{1}{T_0+\Theta}q'(\Theta_x)\right)\Theta_{xx}, D_t^j u_x\right) - $$
$$- h(r)r^2.$$

Seeing that the second term on the right–hand side has been treated in 4.7, we conclude with the estimate

$$A_2 \geq -\frac{c_{16}}{2}\left|D_t^j u_x\right|_2^2 - c_{18}\left|D_t^{j-1}\Theta_{xx}\right|_2^2 - h(r)r^2. \tag{3}$$

Adding (1), (3) we obtain 4.8. ∎

4.9.

$$\left(F_1(u,\Theta),(-1)^j D_t^{2j-2}u_{xx}\right) \geq c_{19}\left|D_t^{j-1}u_{xx}\right|_2^2 - \left|D_t^j u_x\right|_2^2 - c_{20}\left|D_t^{j-1}\Theta_x\right|_2^2 -$$
$$- h(r)r^2 - c_{21}\beta r$$

for any $1 \leq j \leq k$.

PROOF 4.9: Taking advantage of the standard arguments we arrive at the inequality:

$$\left(F_1(u,\Theta),(-1)^j D_t^{2j-2}u_{xx}\right) \geq -\left|D_t^j u_x\right|_2^2 + \left(D_t^{j-1}D_x\psi_F, D_t^{j-1}u_{xx}\right) - c_{21}\beta r$$

(according to 4.1–4.4)

$$\geq -\left|D_t^j u_x\right|_2^2 + \left(\psi_{FF}D_t^{j-1}u_{xx}, D_t^{j-1}u_{xx}\right) + \left(\psi_{FT}D_t^{j-1}\Theta_x, D_t^{j-1}u_{xx}\right) - h(r)r^2 -$$
$$- c_{21}\beta r \geq \text{(with the help of } (A_4))$$
$$\geq -\left|D_t^j u_x\right|_2^2 + c_{19}\left|D_t^{j-1}u_{xx}\right|_2^2 - c_{20}\left|D_t^{j-1}\Theta_x\right|_2^2 - h(r)r^2 - c_{21}\beta r.$$

∎

4.10.

$$\left(F_1(u,\Theta),(-1)^{j+1}D_t^{2j}u\right) \geq \left|D_t^{j+1}u\right|_2^2 - c_{22}\left(\left|D_t^{j-1}u_{xx}\right|_2^2 + \left|D_t^{j-1}\Theta_x\right|_2^2\right) - \\ - h(r)r^2 - c_{23}\beta r$$

for any $1 \leq j \leq k$.

PROOF 4.10: Omitted. We could argue similarly as in 4.9. ∎

4.11.

$$\left(F_1(u,\Theta),(-1)^{j}D_t^{2j-2}u_{xxxx}\right) \geq c_{24}\left|D_t^{j-2}u_{xxx}\right|_2^2 - c_{25}\left|D_t^{j}u_x\right|_2^2 - \\ - c_{26}\left|D_t^{j-2}\Theta_{xx}\right|_2^2 - c_{27}\beta r - h(r)r^2$$

for any $2 \leq j \leq k$.

PROOF 4.11: Seeing that $[u,\Theta] \in E_n$ we have $u_{xxx}(0,t) = u_{xxx}(\pi,t) = 0$, which justifies the following integration

$$\left(F_1(u,\Theta),(-1)^{j}D_t^{2j-4}u_{xxxx}\right) \geq -\left(D_t^{j}u_x, D_t^{j-2}u_{xxx}\right) + \\ + \left(D_t^{j-2}D_x^2\psi_F, D_t^{j-2}u_{xxx}\right) - c_{27}\beta r \geq \\ \text{(according to } (A_4)) \\ \geq c_{24}\left|D_t^{j-2}u_{xxx}\right|_2^2 - c_{25}\left|D_t^{j}u_x\right|_2^2 - c_{26}\left|D_t^{j-2}\Theta_{xx}\right|_2^2 - c_{27}\beta r + A.$$

The quantity A is of the form:

$$A = \left(\sum D^{\alpha}\psi\,(D^{p_1}u_x)^{z_1}\dots(D^{p_n}u_x)^{z_n}\,(D^{q_1}\Theta)^{s_1}\dots(D^{q_m}\Theta)^{s_m}, D_t^{j-2}u_{xxx}\right),$$

$$\text{where } \sum_{i=1}^{n} p_i z_i + \sum_{i=1}^{m} q_i s_i = j, \quad p_i, q_i \leq j-1.$$

The most difficult terms to estimate are those where $D^{p_i} = D_t^{\ell-3}D_x^2$ or $D^{q_i} = D_t^{\ell-3}D_x^2$ for at least one p_i or q_i. Observe that if it is not the case, we can use 4.1–4.3 in order to estimate the remaining part of A with the help of $h(r)r^2$.

On the other hand, if the term contains $D_t^{\ell-3}D_x^2\Theta$ or $D_t^{\ell-3}D_x^3u$, the remaining multipliers are bound to be of the form $D_t^{\tilde{j}-2}u_x$, $D_t^{\tilde{j}-2}\Theta$, $\tilde{j} \leq k$ and hence bounded in $C(Q)$. Indeed, (4.1) implies $D_t^{\tilde{j}-2}u_x$, $D_t^{\tilde{j}-2}\Theta \in H^2(Q) \circlearrowleft C(Q)$.

Consequently, we infer that $A \geq -h(r)r^2$. ∎

4.12. *There are strictly positive constants δ_i, $i = 1, \ldots, 4$ such that*

$$\langle \mathbf{F}(u,\Theta), \mathbf{L}(u,\Theta)\rangle \geq c_{28} r^2(u,\Theta) - c_{29}\beta r - h(r) r^2$$

for any pair $[u,\Theta]$ and any b satisfying (4.1).

PROOF 4.12: Since u satisfies (2.2), we deduce the estimate

$$|u|_2 \leq c_{30}\left(|u_t|_2 + |u_x|_2\right) \leq c_{31}\left(|u_{tt}|_2 + |u_{xt}|_2 + |u_{xx}|_2\right). \tag{1}$$

Next, we set successively

$$\delta_1 = \frac{c_{10}}{8c_{14}k}, \quad \delta_2 = \min\left\{\frac{\delta_1 c_{13}}{8}, \frac{c_{10}}{8c_{20}k}\right\},$$
$$\delta_3 = \min\left\{\delta_2\left(\frac{c_{19}}{8c_{22}}\right), \frac{c_{10}}{8c_{22}k}\right\}, \quad \delta_4 = \min\left\{\delta_1\left(\frac{c_{13}}{8c_{25}}\right), \frac{c_{10}}{8c_{26}k}\right\}.$$

Multiplying respectively 4.8–4.11 by δ_1–δ_4, summing up the results for $j = 0, \ldots, k$ and adding the resulting expression to 4.7 we then obtain 4.12. Note that, of course, we have exploited the estimate (1). ■

Having shown 4.12, we turn to the proof of Lemma 1. For $[u,\Theta] \in \partial K_n(r)$, 4.12 yields the inequality

$$\langle \mathbf{F}(u,\Theta), \mathbf{L}(u,\Theta)\rangle \geq c_{28} r^2 - h(r) r^2 - c_{29}\beta r.$$

Consequently, r being small, say $r \in (0, r_1)$, $r_1 > 0$, we have $c_{28} r^2 - h(r) r^2 > 0$. Thus there is $\varepsilon > 0$ having all the properties required in Lemma 1. Moreover, if β is small, the number r can be chosen close to zero.

Lemma 1 has been proved.

REFERENCES

[1] Dafermos, C.M., *Global smooth solutions to the initial–boundary value problem for the equations of one–dimensional nonlinear thermoviscoelasticity*, SIAM J. Math. Anal. **13** (1982), 397–408.

[2] Day, W.A., *Steady forced vibrations in coupled thermoelasticity*, Arch. Rational Mech. Anal. **93** (1986), 323–334.

[3] DiPerna, R.J., *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal. **82** (1983), 27–70.

[4] Greenberg, J.M., MacCamy, R.C., Mizel, V.J., *On the existence, uniqueness and stability of solutions of the equation $\varrho\chi_{tt} = E(\chi_x)\chi_{xx} + \lambda\chi_{xxt}$*, J. Math. Mech. **17** (1968), 707–728.

[5] Kato, T., *Locally coercive nonlinear equations, with applications to some periodic solutions*, Duke Math. J. **51** (1984), 923–936.

[6] Klainerman, S., *Global existence for nonlinear wave equation*, Comm. Pure Appl. Math. **33** (1980), 43–101.

[7] Matsumura, A., *Global existence and asymptotics of the solutions of the second order quasilinear hyperbolic equations with the first order dissipation*, Publ. Res. Inst. Math. Soc. **13** (1977), 349–379.

[8] Racke, R., *Initial boundary value problems in one–dimensional non–linear thermoelasticity*, Math. Meth. Appl. Sci. **10** (1988), 517–529.

[9] Rothe, E.H., *Introduction to various aspects of degree theory in Banach spaces*, Providence AMS, 1986.

[10] Shibata, Y., *On the global existence of classical solutions of mixed problem for some second order nonlinear hyperbolic operators with dissipative term in the interior domain*, Funkcialaj Ekvacioj **25** (1982), 303–345.

[11] Slemrod, M., *Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one–dimensional nonlinear thermoelasticity*, Arch. Rational Mech. Anal. **76** (1981), 97–134.

[12] Zheng, S., *Initial boundary value problems for quasilinear hyperbolic–parabolic coupled systems in higher dimensional spaces*, Chinese Ann. of Math. **4B(4)** (1983), 443–462.

[13] Zheng, S., *Global solutions and applications to a class of quasilinear hyperbolic–parabolic coupled system*, Scienta Sinica, Ser. A **27** (1984), 1274–1286.

[14] Zheng, S., Shen, W., *Global solutions to the Cauchy problem of quasilinear hyperbolic–parabolic coupled system*, Scienta Sinica, Ser. A **10** (1987), 1133–1149.

Institute of Mathematics of the Czechoslovak Academy of Sciences, Žitná 25, 115 67 Praha 1, Czechoslovakia

(*Received* August 3, 1989)