Zuzana Došlá

Higher monotonicity properties of special functions: application on Bessel case $|\nu| < \frac{1}{2}$

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 2, 233--241

Persistent URL: `http://dml.cz/dmlcz/106853`

Higher monotonicity properties of special functions: application on Bessel case $|\nu| < \frac{1}{2}$

ZUZANA DOŠLÁ

Abstract. Suppose that the function $q(t)$ in the differential equation

$$(*) \qquad y'' + q(t)y = 0$$

is decreasing on $(0, \infty)$. We give conditions on q which ensure that $(*)$ has a pair of solutions $y_1(t)$, $y_2(t)$ such that the n-th derivative $(n \geq 1)$ of the function $p(t) = y_1^2(t) + y_2^2(t)$ has the sign $(-1)^{n+1}$ for sufficiently large t, and that the higher differences of sequences related to zeros of solutions of $(*)$ are ultimately monotonic. In particular, we prove the conjecture of [5] for sufficiently large t.

Keywords: Higher monotonicity properties, ultimate monotonicity, Bessel functions

Classification: 34A40, 34C10

1. INTRODUCTION

The aim of the present paper is to study monotonicity properties of solutions of the second order equation

$$(1) \qquad y'' + q(t)y = 0$$

with the function q *decreasing* to a positive constant. In general, q is *n-time monotonic function*, i.e. sgn $q^{(k)}(t) = (-1)^k$, $k = 0, \dots, n$ on (a, ∞). This investigation is motivated by the following conjecture of L. Lorch and P. Szego [5, p. 51] given on the basis of the numerical evidence and the Sturm comparison theorem.

Conjecture. *Let $c_{\nu k}$ denote k-th positive zeros of any Bessel function C_ν of order $|\nu| < \frac{1}{2}$. Then*

$$(2) \qquad (-1)^n \Delta^n c_{\nu k} > 0 \qquad n = 2, 3, \dots k = 1, 2 \dots ^{*)}$$

If n-th differences have the constant sign and (2) holds for all n or n up to N, we say that the sequence is *completely monotonic* or *N-time monotonic*, respectively.

M. Muldoon [8] proved the validity of (2) for $\frac{1}{3} \leq |\nu| < \frac{1}{2}$ but the method used there cannot be applied to the range $|\nu| < \frac{1}{3}$. We are successful in proving (2) for $|\nu| < \frac{1}{3}$ in the sense of *ultimate monotonicity*, i.e., for each n fixed, (2) holds for all $c_{\nu k}$, $k = l_n, l_n + 1, \dots$ (l_n integer) or, by other words, a finite number (depending

*)The symbol $\Delta^n t_k$ means, as usual, the n-th (forward) differences of the sequence $\{t_k\}$, i.e. $\Delta^0 t_k = t_k$, $\Delta t_k = t_{k+1} - t_k$, $\Delta^n t_k = \Delta(\Delta^{n-1} t_k)$.

on n) of the first members of the sequence $\{c_{\nu k}\}_{k=0}^{\infty}$ must be omitted in (2) (see Corollary 2).

Our approach is based on the following ideas:

(i) to study monotonicity properties of the function $p(t) = y_1^2(t) + y_2^2(t)$, where y_1, y_2 are suitable linearly independent solutions of (1). As was showed in [5,8], this is closely related to the monotonicity properties, e.g. higher differences of zeros, of any solution of (1);

(ii) to investigate certain differential operators on the half-line (a, ∞) for using [1, Theorem 22.1$_n$] and [8, Theorem 2.1]. To this end we make some constructions about the signs and asymptotics of monotonic functions and their quasiderivatives;

(iii) to investigate Bessel functions of order $|\nu| < \frac{1}{2}$ and other Sturm-Liouville functions as solutions of the differential equation of the form (1).

Originally, the idea to study n-time monotonic functions and sequences (as the spacing of zeros of special functions) in the theory of ordinary differential equations, was used in [1] and [5] for the case of $q(t)$ *increasing* (and in general, q' is *n-time monotonic*) with applications on Bessel functions of order $|\nu| \geq \frac{1}{2}$. These results were followed by a lot of papers and this case was in detail resolved, e.g. [6, 7, 8, 9, 10]. It is worth to note that the "nonsymmetry" of both cases of monotonicity q and q' is caused by the properties of the composition of monotonic functions and sequences.

Our method and results cover the case q converging to a positive constant with just the order t^{ϵ}, as it corresponds to the Bessel functions. The case q converging "slowly" to a non-negative constant will be solved in [12].

2. Statement of results.

Let us define the sequence $\{M_i\}_{i=0}^{\infty}$ by

$$M_i = \int_{t_i}^{t_{i+1}} [p(t)]^{-\alpha} |y(t)|^{\lambda} dt \qquad i = 0, 1, \dots, \tag{3}$$

where $y(t)$ is an arbitrary (non-trivial) solution of (1), $\{t_i\}_{i=0}^{\infty}$ denotes any sequence of consecutive zeros of any solution $z(t)$ of (1) which may or may not be linearly independent of $y(t)$, $p(t) = y_1^2(t) + y_2^2(t)$ and $y_1(t)$, $y_2(t)$ are linearly independent solutions of (1), $\lambda > -1$ and $\alpha < 1 + \frac{\lambda}{2}$.

By a special choice of numbers λ, α we get quantities of various geometrical meaning and describing oscillatory properties of any solution of (1), e.g. if $\alpha = \lambda = 0$ then $M_i = \Delta t_i = t_{i+1} - t_i$.

Agreement. Throughout this paper, the symbol $f = \mathcal{O}(t^{-\alpha})$ for $t \to \infty$ means the order properties of the best estimation, i.e. we write $f = \mathcal{O}(t^{-\alpha})$ $t \to \infty$ if

i) $\limsup_{t\to\infty} |f(t)| t^{\alpha} < \infty$, i.e. $f = O(t^{-\alpha})$,

ii) $\lim_{t\to\infty} |f(t)| t^{\alpha+\xi} = \infty$ for every $\xi > 0$.

Theorem. *Let $n \geq 0$ be a fixed integer. Let the function q in (1) satisfy $q(\infty) > 0$ and for $k = 0, 1, \dots, n+2$*

$$(-1)^k q^{(k)}(t) \geq 0, \qquad 0 < t < \infty \tag{4}$$

$$q^{(k)} = \mathcal{O}(t^{-(k+\epsilon)}) \quad t \to \infty, \qquad \epsilon > 0. \tag{5}$$

Then (2) has a pair of solutions $y_1(t)$, $y_2(t)$ such that the function $p(t) = y_1^2(t) + y_2^2(t)$ satisfies $p(t) \to 1$ for $t \to \infty$,

$$(6) \qquad (-1)^k p^{(k+1)}(t) \geq 0, \quad \mu_k < t < \infty, \quad k = 0,\ 1, \ldots, n,$$

where $\{\mu_k\}_1^n$ is a nondecreasing sequence and $\mu_k = \mu_{k+1}$ only if $\mu_k = 0$;

$$(7) \qquad p^{(k)} = \mathcal{O}(t^{-(k+\epsilon)}) \quad t \to \infty, \quad n \geq 4,\ k = 1, 2, \ldots, n-3$$

and the corresponding quantities M_i defined by (3) satisfy

$$(8) \qquad (-1)^k \Delta^{k+1} M_i \geq 0 \qquad k = 0, \ldots, n-3,\ i = l_k,\ l_k + 1, \ldots,$$

where $l_k = l(k)$ is integer, $0 = l_0 \leq l_1 \leq \cdots \leq l_{n-3}$ and $l_k = l_{k+1}$ only if $l_k = 0$.

In particular, we have for the sequence $\{t_i\}_{i=l(k)}^{\infty}$ of positive zeros of any solution of (1)

$$(-1)^k \Delta^{k+2} t_i \geq 0 \qquad k = 0, \ldots, n-3,\ n \geq 3,\ i = l_k,\ l_k + 1, \ldots$$

Remark 1. If the function q satisfies (4) (i.e. q is monotonic of order $n+2$), then $q^{(k)} = O(t^{-k})$, $k = 1, \ldots, n+1$ (see [11]), but (5) need not hold. For example $h(t) = 1 + 1/\lg t$ is completely monotonic (i.e. (4) holds for $k = 0, 1, \ldots$) and $\lim_{t\to\infty} h' t^{1+\epsilon} = -\infty$ for any $\epsilon > 0$.

A typical example where our theorem is applicable, yields the equation

$$(9) \qquad y'' + [1 + \beta t^{-\gamma}] y = 0, \qquad \beta,\ \gamma > 0.$$

P. Hartman [2, Theorem 11.2] proved that (9) has the solutions $y_1(t)$, $y_2(t)$ such that $p(t) = y_1^2 + y_2^2 \to 1$ as $t \to \infty$ and

$$(10) \qquad (-1)^k p^{(k+1)}(t) > 0 \qquad k = 0,\ 1, \ldots$$

for $0 < \gamma \leq 1$ and $\gamma = 2$ (the case of Bessel functions), p' is not completely monotonic on $(0, \infty)$ for $\gamma > 2$ (i.e. (10) does not hold) and the case $1 < \gamma < 2$ is not completely settled. Therefore this result of Hartman is partially completed by the following

Corollary 1. *Equation (9) has a pair of solutions $y_1(t)$, $y_2(t)$ such that $p(t) = y_1^2 + y_2^2 \to 1$ as $t \to \infty$ and*

$$(-1)^k p^{(k+1)}(t) > 0 \qquad \begin{cases} 0 < t < \infty & \text{for } 0 < \gamma \leq 1 \text{ or } \gamma = 2 \\ \mu_k < t < \infty & \text{otherwise} \end{cases}$$

where $\{\mu_k\}_{k=1}^{\infty}$ is a nondecreasing sequence and $\mu_k = \mu_{k+1}$ only if $\mu_k = 0$. In particular,

$$p' > 0 \text{ and } p'' < 0 \text{ for } t > \left[\frac{\beta(9\gamma - 2)}{\gamma + 2}\right]^{\frac{1}{\gamma}}.$$

Moreover, for the sequence $\{t_i\}_{i=l(k)}^{\infty}$ *of zeros of any solution of* (9), *the result* (8) *holds.*

In the sequel, we adopt the usual notation for Bessel functions $C_\nu(t) = AJ_\nu(t) + BY_\nu(t)$ and its positive zeros $c_{\nu k}$ $(k = 1, 2, \dots)$.

If we consider the generalized Airy equation (cf. [8])

$$w'' + \frac{1}{(2\nu)^2} t^{\frac{1}{\nu}-2} w = 0 \qquad (\frac{1}{\nu} - 2 > 1) \tag{11}$$

having a pair of solutions $w(t) = t^{1/2} J_\nu(2\nu t^{1/(2\nu)}),\ t^{1/2} Y_\nu(2\nu t^{1/(2\nu)})$, then the derivative of the carrier $\frac{1}{(2\nu)^2} t^{\frac{1}{\nu}-2}$ of (11) is completely monotonic for $\frac{1}{3} \le \nu < \frac{1}{2}$.

If we reduce the Bessel equation

$$y'' + \frac{1}{t} y' + (1 - \frac{\nu^2}{t^2}) y = 0$$

to the equation

$$z'' + (1 + \frac{\frac{1}{4} - \nu^2}{t^2}) z = 0 \tag{12}$$

having a pair of solutions $z(t) = t^{1/2} J_\nu(t),\ t^{1/2} Y_\nu(t)$ then the carrier of (12) is completely monotonic for $|\nu| < \frac{1}{2}$. Thus Theorem can be applied and the conjecture (2) is proved in the case of ultimate monotonicity.

Corollary 2. *If* $\lambda > -1$ *and* $|\nu| < \frac{1}{2}$, *then*

$$(-1)^n \Delta^n \left\{ \int_{c_{\nu k}}^{c_{\nu k+1}} |t^{\frac{1}{2}} C_\nu(t)|^\lambda dt \right\} \ge 0 \quad n = 0, 1, \dots;\ k = l_n,\ l_n + 1, \dots \tag{13}$$

In particular, (2) *holds for fixed* $n \ge 2$ *and* $k = l_n,\ l_n + 1, \dots$, *where* $\{l_n\}_{n=2}^{\infty}$ *is a nondecreasing sequence of integer numbers and* $l_n = l_{n+1}$ *only if* $l_n = 0$.

Remark 2. If $\frac{1}{3} \le \nu < \frac{1}{2}$, $-1 < \lambda \le 2$, then (13) holds for $k = 1, 2, \dots$ (see Corollary 4.2.[8]).

3. On differential operators: signs and asymptotics.

We start with additivity of $\mathcal{O}$-symbols.

Lemma 1. *Let* $f = f_1 + f_2$, $f_1 = \mathcal{O}(t^{-\alpha})$, $f_2 = \mathcal{O}(t^{-\beta})$, $\alpha < \beta$. *Then there exists* T *such that* $sgn f(t) = sgn f_1(t)$ *for* $t \ge T$ *and* $f = \mathcal{O}(t^{-\alpha})$ *for* $t \to \infty$.

PROOF : It holds

$$\mathrm{sgn} f = \mathrm{sgn}[t^{-\beta}(f_1 t^\alpha t^{\beta-\alpha} + f_2 t^\beta)] = \mathrm{sgn}(f_1 t^\alpha t^{\beta-\alpha} + f_2 t^\beta) = \mathrm{sgn} f_1,$$

since $\beta > \alpha > 0$ and thus $f_1 t^\alpha t^{\beta-\alpha} \to \infty$ and $f_2 t^\beta < \infty$. Further, $f = O(t^{-\alpha})$ and for $0 < \xi \le \beta - \alpha$ $\lim_{t\to\infty} |f| t^{\alpha+\xi} \ge \lim_{t\to\infty}(|f_1| t^{\alpha+\xi} - |f_2| t^{\alpha+\xi}) = \infty$. ∎

In what follows, we describe the sign and asymptotics of quasiderivatives of two functions. Note that this does not hold for $\epsilon = \delta = 0$, corresponding to a slow convergence of $\lim_{t\to\infty} q(t)$ in (1).

The notation $(fD)^k(g)$ means that the differential operator $f(t)\frac{d}{dt}$ is applied k-times.

Lemma 2. *Let $n \geq 1$ be a fixed integer. Let the functions f, g be such that $f(t) > 0$, $f^{(k)} = \mathcal{O}(t^{-(k+\epsilon)})$, $g^{(k)} = \mathcal{O}(t^{-(k+\delta)})$ as $t \to \infty$, $\epsilon > 0$, $\delta > 0$, $k = 1, \ldots, n$. Then $(fD)^k(g) = \mathcal{O}(t^{-(k+\delta)})$ and there exists a nondecreasing sequence $\{T_k\}_1^n$ such that $T_k = T_{k+1}$ only if $T_k = 0$ and $sgn(fD)^k(g) = sgn g^{(k)}$ on (T_k, ∞).*

PROOF : Note that $f' = \mathcal{O}(t^{-(1+\epsilon)})$ implies f bounded and the conclusion for $n = 1$ is obvious.

Let $n \geq 2$. By [5, pp.57–58] it holds for $k = 1, \ldots, n$

$$(fD)^k(g) = f^k g^{(k)} + \sum \Phi(k,t) g^{(\beta)} f^{\gamma}, \tag{14}$$

where $\Phi(k,t)$ is a homogeneous form in $f', \ldots, f^{(k-1)}$ whose typical term is

$$\text{const}(f')^{\alpha_1} \ldots (f^{(k-1)})^{\alpha_{k-1}} \tag{15}$$

with $1 \leq \alpha_1, \beta, \gamma \leq k-1$, $\sum_1^{k-1} i\alpha_i + \beta = k$ and $0 \leq \alpha_i \leq k-i$ for $i = 2, \ldots, k-i+1$.

Let us investigate asymptotic properties of functions in the right-hand side of (14). Since $(f^{(i)})^{\alpha_i} = \mathcal{O}(t^{-\alpha_i(i+\epsilon)})$ and $\sum_1^{k-1} \alpha_i \geq 1$, we have $(fD)^k(g) = \mathcal{O}(t^{-(k+\delta)}) + \mathcal{O}(t^{-\mu})$, where $\mu = \sum_1^{k-1} i\alpha_i + \epsilon \sum_1^{k-1} \alpha_i + \beta + \delta = k + \epsilon \sum_i^{k-1} \alpha_i + \delta \geq k + \epsilon + \delta > k + \delta$.

Hence, applying Lemma 1 in (14), we get the existence of a sequence $\{T_k\}_1^n$ such that $\text{sgn}(fD)^k(g) = \text{sgn} g^{(k)}$ on (T_k, ∞).

Note that if the function g is n-times monotonic on $(0, \infty)$ and $g \not\equiv$ const, then $g^{(k)}(t) \neq 0$ for $t \in (0, \infty)$, $k = 1, \ldots, n-1$ (see e.g. [9, Lemma 0.3]).

Let

$$T_k = \min\{T : \text{sgn}(fD)^k(g(t)) = \text{sgn} g^{(k)}(t) \text{ for } t > T\}, \quad k = 1, \ldots, n.$$

Suppose, by contradiction, that there exists $k \in \{1, \ldots, n-1\}$ such that $T_{k+1} > T_k$ or $T_k = T_{k+1} > 0$. Without loss of generality suppose $g^{(k+1)}(t) < 0$ and $g^{(k)}(t) > 0$ for $t \in (0, \infty)$, $k = 1, \ldots, n-2$. Putting $F_k = (fD)^k(g)$, $k = 1, \ldots, n-1$, we have $F_{k+1} = fDF_k$. If $k \in \{1, \ldots, n-2\}$ it holds

$$DF_k < 0 \text{ for } t > T_{k+1}, \quad F_k > 0 \text{ for } t > T_k.$$

Thus with respect to the continuity of F_k and definition of T_k we get $F_k(T_k) = 0$ and $F_k(t)$ is decreasing for $t > T_k \geq T_{k+1}$, i.e., $F_k(t) < 0$ for $t > T_k$, which is a contradiction with the definition of T_k.

Similarly, if $k = n-1$, then $F_{n-1}(T_{n-1}) = 0$, $F_{n-1}(t) < 0$ for $t > T_{n-1}$, $DF_{n-1}(t) \geq 0$ for $t > T_n \geq T_{n-1}$, which is the same contradiction as above. ∎

The last auxiliary result concerns the composition of monotonic functions, in particular, if f is an n-times monotonic function, then f^λ is ultimately monotonic. It should be compared with [3, Theorems 5 and 8].

Lemma 3. *Let $n \geq 1$ be an integer and $\lambda \neq 0$ a real number. Let the function $f(t)$ satisfy $0 < f(t) < \infty$ and $f^{(k)} = \mathcal{O}(t^{-(k+\epsilon)})$ as $t \to \infty$, $\epsilon > 0$, $k = 1, \dots, n$. Then $(f^\lambda)^{(k)} = \mathcal{O}(t^{-(k+\epsilon)})$, $k = 1, \dots, n$ and there exists a non-decreasing sequence $\{T_k\}_1^n$ such that $T_k = T_{k+1}$ only if $T_k = 0$ and $sgn[f^\lambda(t)]^{(k)} = sgn\lambda sgn f^{(k)}(t)$ for $T_k < t < \infty$.*

PROOF : The statement obviously holds for $n = 1$. Let $n \geq 2$. We will prove by induction that for $k = 1, \dots, n$

$$(f^\lambda)^{(k)} = \lambda f^{\lambda-1} f^{(k)} + \sum \Phi(k,t) f^\gamma \tag{16}$$

holds, where $\gamma = \lambda - 2, \dots, \lambda - k$ and $\Phi(k,t)$ is of the form (15) with $1 \leq \alpha_1 \leq k$, $\sum_1^{k-1} i\alpha_i = k$, $0 \leq \alpha_i \leq k - i$ for $i = 2, \dots, k - i + 1$.

Suppose the validity of (16) for k. Then $(f^\lambda)^{(k+1)} = [(f^\lambda)^{(k)}]' = \lambda f^{\lambda-1} f^{(k+1)} + \lambda(\lambda-1) f^{\lambda-2} f' f^{(k)} + \sum \tilde{\Phi}(k+1,t) f^\gamma$, where $\gamma = \lambda - 2, \dots, \lambda - k - 1$ and $\tilde{\Phi}(k+1,t)$ is a homogeneous form in $f', \dots, f^{(k)}$ whose typical term is const$(f')^{\beta_1} \dots (f^{(k)})^{\beta_k}$, where $1 \leq \beta_1 \leq k$, $\sum_1^k i\beta_i = k+1$, $0 \leq \beta_i \leq k - i + 1$ for $i = 2, \dots, k - i + 1$. Thus (16) holds for $k + 1$.

Now according to (16) it holds $(f^\lambda)^{(k)} = \mathcal{O}(t^{-(k+\epsilon)}) + \mathcal{O}(t^{-\rho})$ where $\rho = k + \epsilon \sum_1^{k-1} \alpha_i \geq k + 2\epsilon$. (The equation $\sum_1^{k-1} i\alpha_i = k$ shows that at least two of the α's must be ≥ 1.) Hence we can apply Lemma 1 to the right-hand side of (16). The rest of the proof is analogous to that one of Lemma 2. ■

4. PROOF OF THE THEOREM.

The idea of the proof is based on Lemma 2 jointly with the following results

Theorem A. [1, Theorem 22.1_n] *Let $n \geq 0$. Let $q(t)$ be non-increasing, $q(\infty) > 0$ and*

$$(-1)^k (q^{-1}D)^k (-2q'q^{-3}) \geq 0 \qquad k = 0, \dots, n+1. \tag{17}$$

Then (1) has a pair of solutions $y_1(t)$, $y_2(t)$ such that $p(t) = y_1^2 + y_2^2 \to 1$ as $t \to \infty$ and

$$(-1)^k (q^{-1}D)^k (p') \geq 0, \qquad k = 0, 1, \dots, n. \tag{18}$$

Theorem B. [8, Theorem 2.1 where $W(t) = 1$]. *Let $y_1(t)$, $y_2(t)$ be the independent solutions of (1) on (a,b) and $p(t) = y_1^2(t) + y_2^2(t)$. Suppose that for $k = 0, 1, \dots, n$*

$$(pD)^k (p^{1+\frac{1}{2}\lambda-\alpha}) \tag{19}$$

has a constant sign ϵ_k $(= \pm 1)$ on (a,b), where $\lambda > -1$, $\alpha < 1 + \lambda/2$. Then we have $sgn\Delta^k M_i = \epsilon_k$ $(k = 0, 1, \dots, n,\ i = 1, 2, \dots)$.

PROOF of the Theorem in Section 2: consists of the following steps:

$$(4),(5) \xrightarrow{(a)} (18) \xrightarrow{(b)} (6),(7) \xrightarrow{(c)} (19) \to (8)$$

Step (a). Let the function q satisfy the assumptions of Theorem. Then by Lemma 3 the functions $f = q^{-1}$, $g = (q^{-2})' = -2q'q^{-3}$ satisfy the assumptions of Lemma 2, from where $\operatorname{sgn}(q^{-1}D)^k(-2q'q^{-3}) = \operatorname{sgn}(-2q'q^{-3})^{(k)} = \operatorname{sgn}[(q^{-2})^{k+1}] =$
$= \operatorname{sgn}(-q^{(k+1)}) = \operatorname{sgn} q^{(k)} = (-1)^k$ on (T_k, ∞), $T_k \le T_{k+1}$ and $T_k = T_{k+1}$ only if $T_k = 0$.

Hence, by applying Theorem A on (T_k, ∞) we get the validity of (18) on (T_k, ∞).

Step (b). We use the result proved in (a) and show that (6) holds. Consider the functions $v = \int_a^t q ds = f(t)$ and $\tilde{p}(v) = p'(f^{-1}(v))$, for which $f'(t) = q(t)$ and $\tilde{p}(v)$ is n-times monotonic function of t and v, respectively. By the rule for the composition of monotonic functions [6, pp. 1241–1242] it holds that $\tilde{p}(f(t)) = p'(t)$ is the n-time monotonic function of t.

We show next that for $k = 1, \dots, n-3$

$$\lim_{t\to\infty} |p^{(k)}| t^{k+\epsilon} < \infty, \tag{20}$$

$$\lim_{t\to\infty} |p^{(k)}| t^{k+\epsilon+\xi} = \infty \quad \text{for every } \xi > 0. \tag{21}$$

To this end, recall that the function p satisfies the Appell equation

$$p''' + 4qp' + 2q'p = 0. \tag{22}$$

Since $p''' > 0$ for sufficiently large t, we get $-q'p \ge 2qp'$ and in view of (5) ($k = 1$) and the fact that p, $q \to$ const, we have the validity of (20) for $k = 1$. Supposing the validity of (20) for k let us prove (20) for $k+1$. By differentiating (22) k-times, $k \le n-2$, we have

$$\begin{aligned} p^{(k+3)} &= -4\sum_{i=0}^{k}\binom{k}{i} q^{(i)}p^{(k+1-i)} - 2\sum_{j=0}^{k}\binom{k}{j} q^{(j+1)}p^{(k-j)} = \\ &= -4qp^{(k+1)} - 4\sum_{i=1}^{k}\binom{k}{i} q^{(i)}p^{(k+1-i)} - 2q^{(k+1)}p - \\ &\quad - 2\sum_{j=0}^{k-1}\binom{k}{j} q^{(j+1)}p^{(k-j)}. \end{aligned}$$

It follows from the induction assumption and from (5) that

$$\sum_{i=1}^{k} q^{(i)}p^{(k+1-i)} = O(t^{-(k+1+2\epsilon)}) = \sum_{j=0}^{k-1} q^{(j+1)}p^{(k-j)},$$

thus

$$p^{(k+3)} = -4qp^{(k+1)} - 2q^{(k+1)}p + O(t^{-(k+1+2\epsilon)}). \tag{23}$$

Since $q^{(k+1)}p = O(t^{-(k+1+\epsilon)})$, $\text{sgn}p^{(k+3)} = \text{sgn}p^{(k+1)}$ for sufficiently large t, (20) is valid for $k+1$.

Finally, if $\xi \leq \min\{2,\epsilon\}$, then multiplying (23) by $t^{k+1+\epsilon+\xi}$, we get

$$4\lim_{t\to\infty} q|p^{(k+1)}|t^{k+1+\epsilon+\xi} = 2\lim_{t\to\infty} p|q^{(k+1)}|t^{k+1+\epsilon+\xi} = \infty,$$

because

$$\lim_{t\to\infty}\sup|p^{(k+3)}|t^{k+1+\epsilon+\xi} \leq \lim_{t\to\infty}\sup|p^{(k+3)}|t^{k+3+\epsilon} < \infty.$$

The proof is complete.

Step (c). In applying Lemmas 2 and 3 to obtain the last conclusion, we set $f = p$ $(p(t) > 0)$, $g = p^a$, $a > 0$ real number. Then

$$\text{sgn}(pD)^k(p^a) = \text{sgn}(p^a)^{(k)} = \text{sgn}p^{(k)} = (-1)^{k+1} \text{ on } \mu_k < t < \infty,$$

and the sequence $\{\mu_k\}_1^n$ is nondecreasing such that $\mu_k = \mu_{k+1}$ only if $\mu_k = 0$. Let $l_k = l(k)$ be the smallest integer such that l_k-th zero $t_{l(k)} \geq \mu_k$. Applying Theorem B on (μ_k,∞) we get $\text{sgn}\Delta^k M_i = (-1)^{k+1}$ for $k = 1,\dots,n-3$, $i = l_k, l_k+1,\dots$ and $\text{sgn}M_i = \text{sgn}p' = 1$ for $i = 0,1,\dots$. The monotonicity property of $\{l_k\}_0^{n-3}$ follows from that one of $\{\mu_k\}$.

If $\lambda = \alpha = 0$ then $M_i = \Delta t_i$. The proof is complete. ∎

Proof of Corollary 1: Let $q(t) = 1 + \beta/t^\gamma$, $\beta > 0$, $\gamma > 1$, $\gamma \neq 2$ (otherwise see Theorem 11.2 of [2]). By a routine computation we get from Theorem A, $i = 1,2$, $3q'^2 \leq qq''$ for $t^\gamma > \beta(2\gamma-1)/(\gamma+1)$ and $10q''q'q - 15q'^3 - q'''q^2 \geq 0$ which is satisfied if $10q''q' - q'''q > 0$. This inequality holds for $t^\gamma > \beta(9\gamma-2)/(\gamma+2)$, hence the first one holds for the same t. Since $\frac{2\gamma-1}{\gamma+1} < \frac{9\gamma-2}{\gamma+2}$, we have the conclusion. ∎

5. Concluding remarks

(i) We comment here our attempts to finish the proof of (2) on the *whole* interval $(0,\infty)$.

The first one*) consists in investigating $\Delta^n c_{\nu k}$ as a function of order ν for each fixed k, $n = 1,2,\dots$, as was done for $|\nu| > \frac{1}{2}$ in [4]. As it has been emphasized in [4, p. 95], some "balancing" in differential expression for $[f(g(t))]^{(n)}$ – similar to (14) – may still leave that expression of an appropriate sign without every term individually being of that sign. A similar idea was used de facto in Lemmas 2 and 3 for sufficiently large t and leads to the ultimate monotonicity in the general case. This is the reason why we were not successful to resolve the whole interval $(0,\infty)$ in the case of Bessel functions even if knowing here explicitly the function $p(t)$ and the fact that p' is completely monotonic on $(0,\infty)$.

The second approach to the resolving (2) is based on the fact that every completely monotonic function and sequence can be expressed in the form of Laplace-Stieltjes integral (see e.g. [2, 3, 11]). Taking into account the properties of $\{\mu_k\}_1^n$

*) proposed to the author by Professor L. Lorch under personal communication.

in Theorem and the result of [2]**) this may turn out to be useful in proving (19) on the whole $(0, \infty)$.

(ii) We call attention to some further application of the method and results used in Section 3. In [5], in addition to the conjecture (2), conjectures concerning positive zeros of Legendre polynomials $P_n(\cos\theta)$, Hermite and Laguerre polynomials are given by making numerical checks. In the same manner, by Theorem B and Lemma 2, similar results may be established for these conjectures – that all differences of the zeros are non-negative.

Acknowledgement. The author wishes to thank Professors L. Lorch and M. E. Muldoon for their helpful comments.

REFERENCES

[1] P. Hartman, *On differential equations and the function* $J_\mu^2 + Y_\mu^2$, Amer. J. Math **83** (1961), 154–188.

[2] P. Hartman, *On differential equations, Volterra equations and the functions* $J_\mu^2 + Y_\mu^2$, Amer. J. Math **95** (1973), 552–593.

[3] L. Lorch, D. J. Neuman, *On the composition of completely monotonic functions and completely monotonic sequences and related questions*, J. London Math. Soc. **(2)**, **28** (1983), 31–45.

[4] L.Lorch, P. Szego, *Monotonicity of the differences of zeros of Bessel functions as a function of order*, Proc. Amer. Math. Soc. **15** (1964), 91–96.

[5] L.Lorch, P. Szego, *Higher monotonicity properties of certain Sturm-Liouville functions*, Acta Math. **109** (1963), 55–73.

[6] L.Lorch, M.E. Muldoon, P. Szego, *Higher monotonicity properties of certain Sturm-Liouville functions, III.*, Canad. J. Math. **22** (1970), 1238–1265.

[7] L.Lorch, M.E. Muldoon, P. Szego, *Higher monotonicity properties of certain Sturm-Liouville functions, IV.*, Canad. J. Math. **24** (1972), 349–368.

[8] M.E. Muldoon, *Higher monotonicity properties of certain Sturm-Liouville functions*, Proceedings of the Royal Society of Edinburgh **77A** (1977), 23–37.

[9] J. Vosmanský, *Monotonic properties of zeros and extremants of the differential equation* $y'' + q(t)y = 0$, Arch. Match. (Brno) **6** (1970), 37–74.

[10] J. Vosmanský, *Certain higher monotonicity properties of i-th derivatives of solutions of* $y'' + a(t)y' + b(t)y = 0$, Arch. Math. (Brno) **10** (1974), 87–102.

[11] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press 1941.

[12] Z. Došlá, M. Háčik, M. E. Muldoon, *Further higher monotonicity properties of Sturm-Liouville functions*, to appear.

Masaryk University, Department of Mathematics, Janáčkovo nám. 2a, 662 95 Brno, Czechoslovakia

(*Received* November 2,1989)

**) Here the problem if q completely monotonic implies p' completely monotonic leads to the question of the nonnegativity of solution of a certain Volterra integral equation for *small* $t > 0$.