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Dedicated to the memory of Zdeněk Frolík

Abstract. Let Y be a tree metric space (i.e. satisfies the four–point condition, e.g., a graph tree with the graph metric). If X is a subset of Y and f is a mapping of X into a Banach space Z with Lipschitz constant L, then f can be extended on the whole Y with Lipschitz constant at most $C \cdot L$, where C is an absolute constant. The extension depends linearly and continuously on f.

Keywords: Lipschitz mapping, extension problem, metric tree, tree metric space

Classification: 54C20, 54E35

1. Introduction and statement of results.

We shall consider the following situation: Let Y be a metric space, X a subset of Y, Z a Banach space and $f\colon X \longrightarrow Z$ a Lipschitz mapping (by $\|f\|_{lip}$ we shall denote the Lipschitz constant of f). The question is now whether f can be extended on Y in such a way that its Lipschitz constant does not grow too much. This problem has received a quite a lot of attention (both in the context of extension problems in general and in Banach space theory).

In the sequel, we shall denote by $e(X,Y,Z)$ the necessary ratio of growth, i.e. the quantity

$$e(X,Y,Z) = \sup\{\inf\{\|\overline{f}\|_{lip}/\|f\|_{lip};\ \overline{f}\colon Y \longrightarrow Z,\ \overline{f}\,|_X = f\};\\ f\colon X \longrightarrow Z,\ \|f\|_{lip} \in (0,\infty)\}\,.$$

A pair (Y,Z) is said to have the *contraction–extension property*, if $e(X,Y,Z) = 1$ for any $X \subseteq Y$. Many results about this property can be found in the book [WW75]. In particular, the spaces Z, such that (Y,Z) has the contraction–extension property for all metric spaces Y have been studied independently by several authors (e.g., Dress [Dre84], Isbell); important examples of such spaces are l^n_∞ (the spaces of n-term real sequences with supremum metric). Another famous result is the theorem of Kirzsbraun, stating that the pair (l_2,l_2) has the contraction–extension property.

In connection with the growing interest in quantitative aspects of Banach space theory several works have appeared, investigating the possibility of extension when certain increase of the Lipschitz constant is allowed. As older results in this direction we can mention e.g. [Grü60], [Lin66]. More recent are papers [JL84] and [JLS86]. The former proves that $e(X,Y,l_2) = O(\sqrt{\log n})$ for any metric space Y and any its

n-point subset X, and shows that this cannot be much improved in general. The paper **[JLS86]** proves that $e(X,Y,Z) = O(\log n)$ for any metric space Y, its n-point subspace Y and any Banach space Z. Moreover if Y is k-dimensional Banach space and $X \subseteq Y$ any its subset, then $e(X,Y,Z) = O(k)$.

In this paper we prove that for a certain class of metric spaces Y, $e(X,Y,Z)$ is a constant independent of Y and Z. Our class of spaces will be so-called tree metric spaces. The study of such spaces has eminently practical motivation (numeric taxonomy), but also interesting theoretical aspects (see e.g. **[Dre84]** for more background information).

A metric space (Y,ρ) is a *tree metric space*, if for any four points $x, y, z, u \in Y$ the following *four-point condition* holds:

$$\rho(x,y) + \rho(u,v) \le \max(\rho(x,u) + \rho(y,v), \rho(x,v) + \rho(y,u)).$$

The following equivalent definition may be more illustrative: A *metric tree* T is a metric space (T,ρ), satisfying the following two axioms:

(i) For every $x, y \in T$, $x \neq y$ there exists a uniquely determined isometry $\phi_{x,y}\colon [0,\rho(x,y)] \longrightarrow T$ with $\phi_{x,y}(0) = x$, $\phi_{x,y}(\rho(x,y)) = y$.

(ii) For every one-to-one continuous mapping $f\colon [0,1] \longrightarrow T$ and for every $t \in [0,1]$ it is

$$\rho(f(0),f(t)) + \rho(f(t),f(1)) = \rho(f(0),f(1)).$$

Now tree metric spaces are exactly subspaces of metric trees (see **[Bun74]**, **[Dre84]**). Finite tree metric spaces can be imagined as subspaces of usual graph-theoretic trees (with nonnegative weights on edges, determining their length).

Let $Lip(X,Y)$ denote the linear space of all Lipschitz mappings from a metric space X to a metric space Y. This space can be considered either as a subspace of $C(X,Y)$ with the supremum norm, or with the pseudonorm $\|\,.\,\|_{lip}$.

Our result is now stated as follows:

Theorem. *Let T be a metric tree, $X \subseteq T$ any its subset and Z a Banach space; then $e(X,T,Z) \le C$, where C denotes an absolute constant.*

The mapping $e\colon Lip(X,Z) \longrightarrow Lip(T,Z)$ defined by the above extension is a linear operator (of norm 1 with respect to l_∞ norm and of norm $\le C$ with respect to $\|\,.\,\|_{lip}$ pseudonorm). For every $f \in Lip(X,Z)$ we have $\operatorname{Im} e(f) \subseteq \overline{\operatorname{conv} \operatorname{Im} f}$.

A simple example shows that we cannot take $C = 1$ in the previous theorem: Consider a tree with three vertices of degree one (leaves) and one vertex of degree three, with unit-length edges. The set of leaves is isometrically embedded into the set of vertices of an equilateral triangle with side 2, but no extension of this mapping into the plane on the three-valent vertex is a contraction.

The proof of Theorem is given in the next section. Its heart is a construction of a suitable cover of the metric tree; in this it resembles the method of **[JL84]** and of other extension theorems (e.g., Dugundji theorem – see **[Eng77]**).

2. The proof.

We introduce the following notation: If T is ametric tree, x, $y \in T$, then $\langle x, y\rangle$ will denote the set $\operatorname{Im}\phi_{x,y}$, where $\phi_{x,y}$ is the isometry guaranteed by axiom (i) from the definition of metric tree. The set $\langle x, y\rangle$ is compact and the mapping $\phi_{x,y}$ induces a complete linear ordering on it, so it makes sense to speak about supremum and infimum of a subset (we take $x < y$ in the above notation). Further for $X \subseteq T$ we denote $\langle X\rangle = \overline{\bigcup_{x,y\in X} \langle x, y\rangle}$ (the closure of the union of all $\langle x, y\rangle$).

Lemma. *Let T be a metric tree.*

(i) [Dre84] *Let x, y, $z \in T$. Then there exists exactly one point $t \in T$ such that*

$$\langle x,y\rangle = \langle x,t\rangle \cup \langle t,y\rangle, \qquad \langle x,z\rangle = \langle x,t\rangle \cup \langle t,z\rangle,$$
$$\langle y,z\rangle = \langle y,t\rangle \cup \langle t,z\rangle,$$

where the unions are disjoint upto the point t; t can be expressed as $\sup(\langle x,y\rangle \cap \langle x,z\rangle)$ (similarly with permutations of x, y, z).

(ii) *If $X \subseteq T$, $Y = \langle X\rangle$ and $x' \in T \setminus Y$, then for every $y \in Y$ the point*

$$z = \inf(\langle x,y\rangle \cap Y)$$

is the same, and it is the nearest point of Y to the point x.

PROOF : (ii) is easily obtained from (i): Consider two points y, $y' \in Y$ and take the point t for the points x, y, y' as in (i). Since $\langle y, y'\rangle \subseteq Y$, it is also $t \in Y$ and the point $z = \inf(\langle x,y\rangle \cap Y)$ lies on $\langle x,t\rangle$, and so it coincides with $\inf(\langle x,y'\rangle \cap Y)$. ∎

Now we can begin the proof of Theorem. Let T be given metric tree, $X \subseteq T$ and $f\colon X \longrightarrow Z$ a Lipschitz mapping. First we shall discuss the easy phases of the extension.

Firstly, a Lipschitz mapping defined on some set can be uniquely extended to the closure of this set without an increase of Lipschitz constant; so we shall assume that X is closed in T.

Assume that a Lipschitz mapping f is already defined on the set $Y = \langle X\rangle$. If x is a point of $T \setminus Y$, we put $\overline{f}(x) = f(\inf(\langle x,y\rangle \cap Y))$, where y is any point of Y (according to the Lemma (ii), this value is independent of y).

Let $x, y \in T \setminus Y$. If $\langle x,y\rangle \cap Y = \emptyset$, choose $z \in Y$ and take the point t for x, y, z as in Lemma (ii): $t = \sup(\langle z,x\rangle \cap \langle z,y\rangle)$. Then we have $t \notin Y$ and $\langle t,z\rangle \cap Y = \langle x,z\rangle \cap Y = \langle y,z\rangle \cap Y$, hence $\overline{f}(x) = \overline{f}(y) = f(\inf(\langle t,z\rangle \cap Y))$.

If $\langle x,y\rangle \cap Y \neq \emptyset$, we define $x' = \inf(\langle x,y\rangle \cap Y)$, $y' = \sup(\langle x,y\rangle \cap Y)$. Then $\overline{f}(x) = f(x')$, $\overline{f}(y) = f(y')$ and at the same time $\rho(x,y) > \rho(x',y')$, therefore $\|\overline{f}\|_{lip} = \|f\|_{lip}$.

In the sequel we may thus assume that $T = \langle X\rangle$. We shall extend the mapping f, defined on a closed subset $X \subsetneq T$, on each arcwise connected component of $\overline{T \setminus X}$ separately. For every component K we thus obtain a Lipschitz mapping $\overline{f_K}$ on $K \cup X$, extending f. If x, $y \in T \setminus X$ are two points of distinct components K, L, then $\langle x, y\rangle$ contains points of X. If x' is the first such point and y' the last such point, we have $\rho(x,y) = \rho(x,x') + \rho(x',y') + \rho(y',y)$, so we may use the estimate

$$\|\overline{f_K}(x) - \overline{f_L}(y)\| \le \|\overline{f_K}(x) - f(x')\| + \|f(x') - f(y)\| + \|f(y') - \overline{f_L}(y)\|,$$

and then the Lipschitz property of f, $\overline{f_K}$ and $\overline{f_L}$.

Let K be a component of arcwise connectedness of $T \setminus X$. The base of our proof will be a construction of a certain cover of K by metric trees.

We choose any point $r \in K$; we call it the root of K. If $x \in K$ and d is a positive real number, we define the sets

$$S(x,d) = \{y \in K;\ x \in \langle r,y\rangle,\ \rho(x,y) < d\},$$
$$BS(x,d) = \{y \in K;\ x \in \langle r,y\rangle,\ \rho(x,y) = d\}.$$

We introduce the following notation and terminology: The point x is called the *root* of the set $S = S(x,d)$ (notation $x = \mathrm{root}(S)$), and the number d the *height* of S ($d = \mathrm{ht}(S)$). The points of $BS(x,d)$ are called *leaves* of S.

We shall define set systems $C_0, C_1 \ldots$ and auxiliary sets $R_0, R_1, \ldots$ by induction. We put

$$C_0 = \{S(r, \rho(r,X)/2)\}, \qquad R_0 = BS(r, \rho(r,X)/2).$$

If the set R_{j-1} has already been defined, we put

$$C_j = \{S(x, \rho(x,X)/2);\ x \in R_{j-1}\},$$
$$R_j = \cup\{BS(x, \rho(x,X)/2);\ x \in R_{j-1}\}.$$

Finally we have

$$C = \bigcup_{j=0}^{\infty} C_j.$$

In the sequel, we shall prove the following properties of the system C:

(i) The sets of C are disjoint
(ii) The sets of C cover K
(iii) For every $S \in C$, $\mathrm{ht}(S) = \rho(\mathrm{root}(S), X)/2$
(iv) For every $S \in C_j$, $j \ge 1$ there exists exactly one $S' \in C_{j-1}$ such that the root of S is a leaf of S'.

If S, S' are as in (iv), we define $\mathrm{pred}(S) = S'$. For the (single) $S \in C_0$ we set $\mathrm{pred}(S) = S$.

(v) For every $S \in C$, $\mathrm{ht}(S)/2 \le \mathrm{ht}(\mathrm{pred}(S)) \le 2\,\mathrm{ht}(S)$.

Using these properties, we shall now define the extension $\overline{f}$ of the mapping f on $X \cup K$ and we shall estimate its Lipschitz constant. The proof of (i)–(v) is postponed to the end of the section.

For every $S \in C$ we choose a point $\mathrm{prox}(S) \in X$, such that $\rho(\mathrm{root}(S), X) = \rho(\mathrm{root}(S), \mathrm{prox}(S))$ (this is possible in a metric tree, but also $\rho(\mathrm{root}(S), \mathrm{prox}(S)) \le 2\rho(\mathrm{root}(S), X)$ would suffice).

For $x \in X$ we shall have $\overline{f}(x) = f(x)$. If $x \in K$, By (i), (ii) there exists a unique $S \in C$ with $x \in S$. We put

$$S' = \mathrm{pred}(S),$$
$$\rho_x = \rho(x, \mathrm{root}(S)),$$
$$h = \mathrm{ht}(S),$$
$$f_1 = f(\mathrm{prox}(S)),$$
$$f_2 = f(\mathrm{prox}(S')), \text{ and we define}$$
$$(*) \qquad \overline{f}(x) = \frac{\rho_x f_2 + (h - \rho_x) f_1}{h}$$

We shall estimate the norm of the difference $\overline{f}(x) - \overline{f}(y)$ for x, $y \in X \cup K$. It is easily seen that it suffices to take x, y with $x \in \langle r, y\rangle$ (for general x, y consider the point $z = \sup(\langle r, x\rangle \cap \langle r, y\rangle) \in \langle x, y\rangle$ and use $\rho(x, y) = \rho(x, z) + \rho(z, y)$).

First we consider the case $x \in S \in K$, $y \in X$. The value of $\overline{f}(x)$ is a weighted average of some values of $f(y')$, $y' \in X$, where

$$\begin{aligned}&\rho(x, y') \le \\ &\le \max(\rho(\mathrm{root}(S), X) + \mathrm{ht}(S), \quad \rho(\mathrm{root}(\mathrm{pred}(S)), X) + \mathrm{ht}(S) + \mathrm{ht}(\mathrm{pred}(S))) = \\ &\qquad = O(\mathrm{ht}(S)) = O(\rho(x, X)) = O(\rho(x, y)),\end{aligned}$$

so in this case the Lipschitz constant grows in a bounded ratio only.

Further if $S \in C$, $S' = \mathrm{pred}(S)$, then $\overline{f}(\mathrm{root}(S)) = f(\mathrm{prox}(S'))$. If the point $x \in S'$ approaches $y = \mathrm{root}(S)$, then $\rho(x, \mathrm{root}(S'))$ tends to the value (S'), and so $\overline{f}(x)$ has limit $\overline{f}(y)$. Therefore it suffices to consider the case when both x and y are in the same $S \in C$, $x \in \langle r, y\rangle$. In the notation of $(*)$ we have

$$\begin{aligned}&\|\overline{f}(x) - \overline{f}(y)\| = \|\rho_x f_1 + (h - \rho_x) f_2 - \rho_y f_1 - (h - \rho_y) f_2\|/h = \\ &|\rho_x - \rho_y| \cdot \|f_1 - f_2\|/h = \rho(x, y)\|f\|_{lip} \cdot \rho(\mathrm{prox}(S), \mathrm{prox}(S'))/h.\end{aligned}$$

However, it is

$$\begin{aligned}&\rho(\mathrm{prox}(S), \mathrm{prox}(S')) \le \rho(\mathrm{prox}(S), \mathrm{root}(S)) + \rho(\mathrm{root}(S), \mathrm{root}(S')) + \\ &\rho(\mathrm{root}(S'), \mathrm{prox}(S')) = \rho(\mathrm{root}(S), X) + \mathrm{ht}(S') + \rho(\mathrm{root}(S'), X) = O(\mathrm{ht}(S))\end{aligned}$$

(by (iii) and (v)), so $\|\overline{f}(x) - \overline{f}(y)\| = \rho(x, y) \cdot O(\|f\|_{lip})$.

The statements in the second part of the theorem are easily seen from the above (the linearity of the extensor e is immediate, the statement about the norm of e with respect to $\|\cdot\|_\infty$ is a reformulation of the bound on the Lipschitz constant of $\overline{f}$).

It remains to prove (i)–(v):

(i) On K we can define a partial order by the relation $x \le y$ iff $\langle r, x\rangle \subseteq \langle r, y\rangle$ (the correctness of this definition is seen from Lemma, part (i)). Then the construction implies that the elements of every set R_i are pairwise incomparable, and if $y \in$

$S(x,d)$, $x \in R_i$, then $y \geq x$, and y is incomparable to all other elements of R_i – this is proved by induction.

(ii) For a contradiction, we assume that some $z \in K$ is not covered by any set $S \in C$. Let y be the supremum of the set of all points in $\langle r, z\rangle$ which are covered. The set of covered points on $\langle r, z\rangle$ is open, so y is not covered. Since $y \notin X$ and X is closed, it is $\rho(y, X) = \varepsilon > 0$. The point y is necessarily a limit of points $x_i \in R_i$, lying on $\langle r, y\rangle$; so let us take an index i such that $x_i \in \langle r, y\rangle$ and $\rho(x_i, y) < \varepsilon/4$. But then $\rho(x_i, X) \geq \rho(y, X) - \rho(y, x_i) \geq 3\varepsilon/4$, so in $(i+1)$-th step of the construction the covered part of $\langle r, z\rangle$ should have been extended by at least $3\varepsilon/8$, which would have covered also y – a contradiction.

(iii) is obvious from the construction.

(iv) is easily seen from the considerations in the proof of (i).

(v) Let us denote $S' = \text{pred}(S)$. We have $\text{ht}(S') = \rho(\text{root}(S'), X)/2 \leq \rho(\text{root}(S), X)/2 + \rho(\text{root}(S'), \text{root}(S))/2 = \text{ht}(S) + \text{ht}(S')/2$, hence $\text{ht}(S') \leq 2\,\text{ht}(S)$. Similarly it is $\text{ht}(S) = \rho(\text{root}(S), X)/2 \leq \rho(\text{root}(S'), X)/2 + \rho(\text{root}(S), \text{root}(S'))/2 = \text{ht}(S') + \text{ht}(S')$. ■

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