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Two examples of the operators with jumping nonlinearities

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Dedicated to the memory of Svatopluk Fučík

Abstract. In this article the spectra of two operators with jumping nonlinearities are thoroughly investigated.

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Classification: 47H12, 90C33

Introduction

The first results about the operators with jumping nonlinearities (see Definition 1 below) were obtained by AMBROSETTI and PRODI in [1] and [2]. Many other papers by various authors concerning this subject appeared since then. Instead of listing them here, I prefer to mention, that many relevant results and references can be found in the FUČÍK's book [3].

FUČÍK and MILOTA had shown in [4], that the operators with jumping nonlinearities in the finite-dimensional setting naturally appear in the investigation of certain variational inequalities. In this context I'd like to mention the paper [5].

By means of the operators with jumping nonlinearities in the finite-dimensional setting one can also formulate the so-called linear complementarity problem (LCP). From the vast literature concerning the LCP I shall mention only [6], [7], because as far as I know, our point of view is rather different from that one of the authors of the papers about LCP.

In [3] one can find some examples of the operators with jumping nonlinearities. Two of them are investigated rather thoroughly there, namely

$$\begin{aligned} S_{\lambda,\mu}(u) &= u''(x) + \lambda u^+(x) - \mu u^-(x)\,, \quad x \in (0,\pi) \\ u(0) &= u(\pi) = 0 \end{aligned}$$

and

$$\begin{aligned} \widetilde{S}_{\lambda,\mu}(u) &= u''(x) + \lambda u^+(x) - \mu u^-(x)\,, \quad x \in (0,\pi) \\ u(0) &= u(\pi) \\ u'(0) &= u'(\pi) \end{aligned}$$

(of course, a weak formulation is necessary if the requirements of Definition 1 are to be satisfied - see below). The spectrum (see Definition 2) of the above defined operators S and $\widetilde{S}$ turns out to be rather simple.

The main purpose of this article is to show, that the spectrum is not simple in general and that it really can exhibit rather strange behaviour, if the linear operator S is properly chosen. Moreover, this is true even in the finite-dimensional case.

I hope that these examples can improve the insight into the nature of the operators with jumping nonlinearities.

In this article only two examples are investigated. Nevertheless, all the methods, which are used here, can be (at least in principle) easily generalized in order to calculate the spectra of a rather broad class of the operators with jumping nonlinearities. In this class many other interesting examples can be found. (As shown in [8] and [9].)

After preliminarities, which are collected in Section 1, a complete description of the spectra of two operators $S : R^4 \to R^4$ and $T : R^4 \to R^4$ is given in Section 2. (See (2) and (22).) Section 3 together with Section 4 contains a sketch of the corresponding calculations. Of course, to perform all the necessary calculations is a lengthy and tedious, but rather elementary task. Hence, the problem consists in the appropriate choice of the examples and not in the calculations.

Section 5 contains the results about the Brouwer degree

$$deg(S_{\lambda,\mu}, 0, B) \quad \text{resp.} \quad deg(T_{\lambda,\mu}, 0, B)$$

of $S_{\lambda,\mu}$ resp. $T_{\lambda,\mu}$ w.r.t. 0 and a ball B (centered in 0) and about the solvability of the corresponding equation

$$S_{\lambda,\mu}(u) = f \qquad \text{resp.} \qquad T_{\lambda,\mu}(u) = f$$

for various λ and μ and S , T as in Section 2.

In Section 6 some conclusive remarks are collected.

Section 1. Definitions, notation

Let H be a Hilbert space with a cone $\mathcal{K}$ of "non-negative" elements. (I.e., for each $u \in H$ there exist

$$\begin{aligned} u^+ &= \max\{u, 0\} \in \mathcal{K}, \\ u^- &= \max\{-u, 0\} \in \mathcal{K}, \\ u &= u^+ - u^- \text{ .)} \end{aligned}$$

Let the mappings

$$u \longmapsto u^+$$
$$u \longmapsto u^-$$

be continuous. Let $S : H \to H$ be a linear completely continuous self-adjoint operator. Let λ and μ be two real parameters. We define the operators

$$\begin{aligned} &S_{\lambda,\mu} : H \to H, \\ &S_{\lambda,\mu}(u) = u + \lambda S u^+ - \mu S u^-. \end{aligned}$$

Definition 1. Any operator of this type is said to be an operator with jumping nonlinearity and any equation of the form

$$S_{\lambda,\mu}(u) = f \tag{1}$$

is said to be an equation with jumping nonlinearity.

Definition 2. The spectrum of a linear self-adjoint operator $S : H \to H$ is the set $\sigma(S) \subset R^2$ of all the pairs $(\lambda, \mu) \in R^2$, for which the equation

$$S_{\lambda,\mu}(u) = 0$$

has a nonzero solution $u \in H$. Any such solution is said to be an eigenvector of S, corresponding to the eigenvalue $(\lambda, \mu) \in \sigma(S)$.

Remark. This definition is substantially different from the usual definition of the spectrum of a linear operator. Whenever we shall need to speak about the spectrum of an operator S in the usual sense, we shall use the word linear. Hence, we shall speak about the linear spectrum, linear eigenvalues etc.

Notation.

(i) $\bar{n} = \{1, 2, 3, \ldots, n\}$.

(ii) Let $\omega \subset \bar{n}$. The sets $R_\omega = \{u = (u_i)_{i\in\bar{n}} \in R^n \mid u_i \le 0$ for all $i \in \omega$ and $u_i \ge 0$, for all $i \in \bar{n} - \omega\}$ are said to be orthants in R^n.

(iii) Let $\omega \subset \bar{n}$ and $S : R^n \to R^n$ be given. $C_\omega \subset \sigma(S)$ is the set of all the eigenvalues $(\lambda, \mu) \in \sigma(S)$ s.t. at least one corresponding eigenvector $u \in R_\omega$.

(iv) We shall see, that for $\omega \subset \bar{n}$ the set C_ω is a semialgebraic variety, which is defined by a polynomial equation

$$P_\omega(\lambda, \mu) = 0$$

and some inequalities. (P_ω is a polynomial of degree n.) Hence we can define the algebraic variety

$$\bar{C}_\omega = \{(\lambda, \mu) \in R^2 | P_\omega(\lambda, \mu) = 0\},$$

and according to this definition $C_\omega \subset \bar{C}_\omega$.

(v) Let $u = (u_i)_{i\in\bar{n}} \in R^n$ be given. We define $u^+ = (u_i^+)_{i\in\bar{n}} \in R^n$ and $u^- = (u_i^-)_{i\in\bar{n}} \in R^n$ as follows:

$$u_i^+ = \max\{u_i, 0\},$$
$$u_i^- = \max\{-u_i, 0\}$$

for every $i \in \bar{n}$.

(vi) In the sequel we shall use the Brouwer degree $deg(S_{\lambda,\mu}, 0, B)$ of the operator $S_{\lambda,\mu}$ w.r.t. the point 0 and a ball B centred in 0. It can be shown, that in the case of the operators with jumping nonlinearities $deg(S_{\lambda,\mu}, 0, B)$ is independent of the actual choice of B. Hence instead of $deg(S_{\lambda,\mu}, 0, B)$ we shall write only $deg(S_{\lambda,\mu})$. (The details can be found in [10].)

(vii) Let $S_{\lambda,\mu} : R^n \to R^n$ be any operator with jumping nonlinearity. Let

$$k(S_{\lambda,\mu}, f)$$

be the number of distinct solutions to the equation (1). We define

$$k(S_{\lambda,\mu}) = \inf_{f\in R^n} \operatorname{ess} k(S_{\lambda,\mu}, f) = \sup_{\mathcal{M}\in\mathcal{O}} \inf_{f\in R^n-\mathcal{M}} k(S_{\lambda,\mu}, f),$$

where $\mathcal{O}$ is the system of all subsets of R^n, the Lebesgue measure of which is equal to zero.

(viii) $(.,.)$ denotes a point in R^2.

(ix) In some cases different mathematical objects are denoted by the same symbols, but it doesn't seem to be misleading and was caused mainly by the fact, that, e.g., in the description of the intersection points of $\sigma(T)$ (see (22)) almost all the English alphabet was needed.

Definition 3. Let Ω be any object (point, line etc.) in the (λ, μ) - plane. Its antiobject $\widetilde{\Omega}$ (antipoint, antiline etc.) in the object, which is symmetric to Ω w.r.t. the axis $\lambda = \mu$. Because the description of $\widetilde{\Omega}$ can be obtained from the description of Ω by interchanging the roles of λ and μ , we can speak about various formulae and antiformulae as well. Also we will use the same notation with tilde in the case of the antiformulae.

In [3] FUČÍK had formulated the

Conjecture. Let B be a ball centred in $0 \in H$. Let the Leray-Schauder degree $deg(S_{\lambda,\mu}, 0, B)$ of $S_{\lambda,\mu}$ w.r.t. the point 0 and the ball B be defined and let

$$deg(S_{\lambda,\mu}, 0, B) = 0.$$

Then there exists some $f \in H$ such that the equation (1) has no solution.

This conjecture is false. The following counterexample was constructed in [10]:

$$S_{1,-1} : R^4 \to R^4,$$

where

$$S = \begin{pmatrix} 3.5 & -1 & -1 & -1 \\ -1 & 3.5 & -1 & -1 \\ -1 & -1 & 3.5 & -1 \\ -1 & -1 & -1 & 2.5 \end{pmatrix}$$

(Here as well as in the sequel we identify the operators S with the corresponding matrix.)

The matrix S has a double linear eigenvalue and we can ask, what happens with $\sigma(S)$, if this eigenvalue splits into two simple ones. In order to answer this question, we should find a self-adjoint operator $T : R^4 \to R^4$ s.t.

(i) T has only simple linear eigenvalues,

(ii) $T_{1,-1}$ gives a counterexample to the FUČÍK's conjecture,
(iii) $T - S$ is small enough (in the usual matrix norm).
(iv) the entries of the matrix of T have not many digits (else we would have to overcome certain unpleasant numerical difficulties),
(v) the components of $R^2 - \sigma(T)$ are not too small (else we couldn't draw any intelligible figure of $\sigma(T)$).

Of course, these requirements are rather contradictory, thus it wasn't easy to find an example, which would satisfy all of them in some reasonable extent. The example

$$T = \begin{pmatrix} 3.8 & -1 & -1 & -1 \\ -1 & 3.5 & -1 & -1 \\ -1 & -1 & 3.2 & -1 \\ -1 & -1 & -1 & 2.5 \end{pmatrix}$$

seems to be near to optimal. We will see, that $\sigma(S)$ and $\sigma(T)$ are fairly similar, on the other hand the differences between $\sigma(S)$ and $\sigma(T)$ are interesting.

Section 2. The description of the spectra

The spectrum $\sigma(S)$ of the operator

$$S = \begin{pmatrix} 3.5 & -1 & -1 & -1 \\ -1 & 3.5 & -1 & -1 \\ -1 & -1 & 3.5 & -1 \\ -1 & -1 & -1 & 2.5 \end{pmatrix} \tag{2}$$

consists precisely of the following curves (see Fig. 1,2):

$C_{\emptyset}$:

$$\lambda = -(8 + 2\sqrt{13})/3\,, \quad \mu \in R;$$

the corresponding eigenvectors u of $R_{\emptyset}$ satisfy the equations

$$u_1 = u_2 = u_3 = u_4(1 + \sqrt{13})/6. \tag{3}$$

$C_{\{1\}} = C_{\{2\}} = C_{\{3\}}$:

$$\mu = -(34\lambda^2 + 40\lambda + 8)/(27\lambda^2 + 116\lambda + 28), \tag{4}$$

$$\lambda \in \left]-(58 + 4\sqrt{163})/27; -2/7\right] \cup [-2/9; +\infty[\,, \qquad \mu \in]-34/27; +\infty[\,; \tag{5}$$

this curve has asymptotes

$$\begin{aligned} &\lambda = -(58 + 4\sqrt{163})/27, \\ &\mu = -34/27; \\ &\mu(-2/7) = -2/9 = \mu(-2/9); \end{aligned} \tag{6}$$

the corresponding eigenvectors u of $R_{\{1\}}$ satisfy the equations

$$(7)\qquad (9\mu+2)u_1=(9\lambda+2)u_2=(9\lambda+2)u_3=(7\lambda+2)u_4\,,\quad \text{w.r.t. (4)}$$
$$(8)\qquad 9\mu+2=-4(7\lambda+2)(9\lambda+2)/(27\lambda^2+116\lambda+28)\,;$$

interchanging the indices 1 and 2, resp. 1 and 3 in the equations (7), we obtain the equations of the eigenvectors u of $R_{\{2\}}$ resp. $R_{\{3\}}$.

$C_{\{4\}}$:

$$(9)\qquad \mu=-(6\lambda+4)/(3\lambda+10)\,,$$
$$(10)\qquad \lambda\in\,]-10/3;+\infty[\,,$$
$$\mu\in\,]-2;+\infty[\,;$$

this curve has asymptotes

$$\lambda=-10/3\,,$$
$$\mu=-2\,,$$
$$\mu(-2/9)=-2/7\,;$$

the corresponding eigenvectors of $R_{\{4\}}$ satisfy the equations

$$(11)\qquad (9\lambda+2)u_1=(9\lambda+2)u_2=(9\lambda+2)u_3=(7\mu+2)u_4\,,\quad \text{w.r.t. (9)}$$
$$(12)\qquad 7\mu+2=-4(9\lambda+2)/(3\lambda+10)\,.$$

$C_{\{1,2\}}=C_{\{1,3\}}=C_{\{2,3\}}$:

$$(13)\qquad \mu=-(62\lambda^2+48\lambda+8)/(27\lambda^2+88\lambda+20)\,,$$
$$(14)\qquad \lambda\in\left]-(44+2\sqrt{349})/27;-2/7\right]\cup[-2/9;+\infty[\,,$$
$$\mu\in\,]-62/27;+\infty[\,;$$

this curve has asymptotes

$$\lambda=-(44+2\sqrt{349})/27\,,$$
$$\mu=-62/27\,,$$
$$\mu(-2/7)=-2/9=\mu(-2/9)\,;$$

the corresponding eigenvectors u of $R_{\{1,2\}}$ satisfy the equations

$$(15)\qquad (9\mu+2)u_1=(9\mu+2)u_2=(9\lambda+2)u_3=(7\lambda+2)u_4\,,\quad \text{w.r.t. (13)}$$
$$(16)\qquad 9\mu+2=-8(7\lambda+2)(9\lambda+2)/(27\lambda^2+88\lambda+20)\,;$$

the corresponding eigenvectors of $R_{\{1,3\}}$, resp. $R_{\{2,3\}}$ satisfy the equations, which can be obtained from (15) by an obvious permutation of the indices 1,2,3.

The remaining parts of $\sigma(S)$ are the anticurves of the curves in the above list, because

$$C_\omega = \widetilde{C}_{\bar{4}-\omega}. \tag{17}$$

The above written description of the eigenvectors is exact in all but three points of $\sigma(S)$. The exceptional points are C, D and its antipoint $\widetilde{D}$ (see (21)).
C: The corresponding eigenvectors are just those $u \neq 0$, which satisfy the equations

$$u_1 + u_2 + u_3 = 0, \tag{18}$$
$$u_4 = 0. \tag{19}$$

Such vectors can be found in any orthant of R^4 except of $R_\emptyset, R_{\{4\}}, R_3$ and $R_{\bar{4}}$.
D: All the corresponding eigenvectors are in $R_{\{4\}}$ and each of them satisfies the equation

$$u_1 + u_2 + u_3 = -9u_4/7. \tag{20}$$

The curves $C_\omega, \omega \subset \bar{4}$ intersect themselves just in the points

$$\begin{aligned} A &= (-(8+2\sqrt{13})/3, -(8+2\sqrt{13})/3),\\ B &= ((-8+2\sqrt{13})/3, (-8+2\sqrt{13})/3),\\ C &= (-2/9, -2/9),\\ D &= (-2/9, -2/7),\\ E &= (0, -2/5),\\ F &= (-2/11, -2/5) \end{aligned} \tag{21}$$

and in their antipoints. All the eigenvectors corresponding to the points A, B, E, F, $\widetilde{E}$, $\widetilde{F}$ can be obtained from the relevant ones of the equations (3), (7), (11), (15) etc.

For the convenience of the reader let us note, that the line $\mu = 1, \lambda \in R$ inserts (with increasing λ) the curves $C_\omega, \omega \subset \bar{4}$ in the order: $C_\emptyset$, $C_{\{1\}} = C_{\{2\}} = C_{\{3\}}$, $C_{\{4\}}$, $C_{\{1,2\}} = C_{\{1,3\}} = C_{\{2,3\}}$, $C_{\{1,4\}} = C_{\{2,4\}} = C_{\{3,4\}}$, $C_{\{1,2,3\}}$, $C_{\{1,2,4\}} = C_{\{1,3,4\}} = C_{\{2,3,4\}}$. By means of (17) we obtain the order of C_ω for the line $\lambda = 1, \mu \in R$.

The spectrum $\sigma(T)$ of the operator

$$T = \begin{pmatrix} 3.8 & -1 & -1 & -1\\ -1 & 3.5 & -1 & -1\\ -1 & -1 & 3.2 & -1\\ -1 & -1 & -1 & 2.5 \end{pmatrix} \tag{22}$$

consists precisely of the following curves (see Fig. 3,4,5):

$C_\emptyset$:

$$\lambda = -5.3103926\,, \quad \mu \in R;$$

the corresponding eigenvectors u of $R_\emptyset$ satisfy the equations

$$(23) \qquad -24.489884u_1 = -22.896767u_2 = -21.303649u_3 = -17.586374u_4$$

$C_{\{1\}}$:

$$(24) \qquad \begin{aligned} &\mu = -\frac{16.8\lambda^3 + 24.95\lambda^2 + 9.2\lambda + 1}{14.49\lambda^3 + 70.41\lambda^2 + 31.96\lambda + 3.8}\,, \\ &\lambda \in \,]-4.3679998; -1/3.5] \cup [-1/4.5; +\infty[\,, \\ &\mu \in \,]-1.1594203; +\infty[\,; \end{aligned}$$

this curve has asymptotes

$$\begin{aligned} &\lambda = -4.3679998, \\ &\mu = -1.1594203; \\ &\mu(-1/3.5) = -1/4.8 = \mu(-1/4.5); \end{aligned}$$

the corresponding eigenvectors u of $R_{\{1\}}$ satisfy the equations

$$(25) \qquad \begin{aligned} &(4.8\mu + 1)u_1 = (4.5\lambda + 1)u_2 = (4.2\lambda + 1)u_3 = (3.5\lambda + 1)u_4\,, \\ &\text{w.r.t. (24)} \\ &4.8\mu + 1 = -\frac{(4.5\lambda + 1)(4.2\lambda + 1)(3.5\lambda + 1)}{14.49\lambda^3 + 70.41\lambda^2 + 31.96\lambda + 3.8}. \end{aligned}$$

$C_{\{2\}}$:

$$(26) \qquad \begin{aligned} &\mu = -\frac{18.9\lambda^3 + 26.66\lambda^2 + 9.5\lambda + 1}{14.49\lambda^3 + 68.31\lambda^2 + 30.25\lambda + 3.5}\,, \\ &\lambda \in \,]-4.2347782; -1/3.5] \cup [-1/4.8; +\infty[\,, \\ &\mu \in \,]-1.3043478; +\infty[\,; \end{aligned}$$

this curve has asymptotes

$$\begin{aligned} &\lambda = -4.2347782, \\ &\mu = -1.3043478, \\ &\mu(-1/3.5) = -1/4.5 = \mu(-1/4.8); \end{aligned}$$

the corresponding eigenvectors u of $R_{\{2\}}$ satisfy the equations

$$(27) \qquad \begin{aligned} &(4.8\lambda + 1)u_1 = (4.5\mu + 1)u_2 = (4.2\lambda + 1)u_3 = (3.5\lambda + 1)u_4\,, \\ &\text{w.r.t. (26)} \\ &4.5\mu + 1 = -\frac{(4.8\lambda + 1)(4.2\lambda + 1)(3.5\lambda + 1)}{14.49\lambda^3 + 68.31\lambda^2 + 30.25\lambda + 3.5}. \end{aligned}$$

$C_{\{3\}}$:

(28)
$$\begin{aligned}&\mu = -\frac{21.45\lambda^3 + 28.55\lambda^2 + 9.8\lambda + 1}{14.49\lambda^3 + 65.76\lambda^2 + 28.36\lambda + 3.2},\\ &\lambda \in]-4.0708406; -1/3.5] \cup [-1/4.8; +\infty[,\\ &\mu \in]-1.4803313; +\infty[;\end{aligned}$$

this curve has asymptotes

$$\begin{aligned}&\lambda = -4.0708406,\\ &\mu = -1.4803313;\\ &\mu(-1/3.5) = -1/4.2 = \mu(-1/4.8);\end{aligned}$$

the corresponding eigenvectors u of $R_{\{3\}}$ satisfy the equations

$$\begin{aligned}&(4.8\lambda + 1)u_1 = (4.5\lambda + 1)u_2 = (4.2\mu + 1)u_3 = (3.5\lambda + 1)u_4, \quad \text{w.r.t. (28)}\\ &4.2\mu + 1 = -(4.8\lambda + 1)(4.5\lambda + 1)(3.5 + 1)/(14.49\lambda^3 + 65.76\lambda^2 + 28.36\lambda + 3.2).\end{aligned}$$

$C_{\{4\}}$:

(29)
$$\begin{aligned}&\mu = -\frac{30.06\lambda^3 + 33.66\lambda^2 + 10.5\lambda + 1}{14.49\lambda^3 + 57.15\lambda^2 + 23.25\lambda + 2.5},\\ &\lambda \in]-3.4997029; -1/4.2] \cup [-1/4.8; +\infty[,\\ &\mu \in]-2.0745342; +\infty[;\end{aligned}$$

this curve has asymptotes

$$\begin{aligned}&\lambda = -3.4997029,\\ &\mu = -2.0745342;\\ &\mu(-1/4.2) = -1/3.5 = \mu(-1/4.8);\end{aligned}$$

the corresponding eigenvectors u of $R_{\{4\}}$ satisfy the equations

$$\begin{aligned}&(4.8\lambda + 1)u_1 = (4.5\lambda + 1)u_2 = (4.2\lambda + 1)u_3 = (3.5\mu + 1)u_4, \quad \text{w.r.t. (29)}\\ &3.5\mu + 1 = -(4.8\lambda + 1)(4.5\lambda + 1)(4.2\lambda + 1)/(14.49\lambda^3 + 57.15\lambda^2 + 23.25\lambda + 2.5)\end{aligned}$$

$C_{\{1,4\}}$: This curve is defined by the equation

(30)
$$\mu = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

where

(31)
$$\begin{aligned}&a = 14.49\lambda^2 + 40.35\lambda + 8.5,\\ &b = 46.86\lambda^2 + 38.21\lambda + 6.3,\\ &c = 10.2\lambda^2 + 6.7\lambda + 1,\end{aligned}$$

and

$$\lambda \in]-2.5550940; -1/4.2] \cup [-1/4.5; +\infty[,$$
$$\mu \in]-2.9992513; 1/3.5] \cup [-1/4.5; +\infty[; \tag{32}$$

this curve has asymptotes

$$\begin{aligned} &\lambda = -2.5550940, \\ &\mu = -2.9992513, \\ &\mu(-1/4.2) = -1/4.8, \quad \mu(-1/4.5) = -1/3.5; \end{aligned}$$

the corresponding eigenvectors u of $R_{\{1,4\}}$ satisfy the equations

$$(4.8\mu + 1)u_1 = (4.5\lambda + 1)u_2 = (4.2\lambda + 1)u_3 = (3.5\mu + 1)u_4. \tag{33}$$

(where μ is given by (30), of course).

$C_{\{2,4\}}$: This curve is defined by (30), where now

$$\begin{aligned} a &= 14.49\lambda^2 + 38.25\lambda + 7.75, \\ b &= 48.96\lambda^2 + 38\lambda + 6, \\ c &= 11.16\lambda^2 + 7\lambda + 1, \end{aligned}$$

further

$$\begin{aligned} &\lambda \in]-2.4186116; -1/4.2] \cup [-1/4.8; +\infty[, \\ &\mu \in]-3.1330561; -1/3.5] \cup [-1/4.5; +\infty[; \end{aligned}$$

this curve has asymptotes

$$\begin{aligned} &\lambda = -2.4186116, \\ &\mu = -3.1330561; \\ &\mu(-1/4.2) = -1/4.5, \quad \mu(-1/4.8) = -1/3.5; \end{aligned}$$

the corresponding eigenvectors u of $R_{\{2,4\}}$ satisfy the equations

$$(4.8\lambda + 1)u_1 = (4.5\mu + 1)u_2 = (4.2\lambda + 1)u_3 = (3.5\mu + 1)u_4.$$

$C_{\{3,4\}}$: This curve is defined by (30), where now

$$\begin{aligned} a &= 14.49\lambda^2 + 35.7\lambda + 7, \\ b &= 51.51\lambda^2 + 37.61\lambda + 5.7, \\ c &= 12.3\lambda^2 + 7.3\lambda + 1, \end{aligned}$$

further

$$\lambda \in]-2.2489615; -1/4.5] \cup [-1/4.8; +\infty[,$$
$$\mu \in]-3.2974346; -1/3.5] \cup [-1/4.8; +\infty[;$$

this curve has asymptotes

$$\lambda = -2.2489615,$$
$$\mu = -3.2974346;$$
$$\mu(-1/4.5) = -1/4.2, \qquad \mu(-1/4.8) = -1/3.5;$$

the corresponding eigenvectors u of $R_{\{3,4\}}$ satisfy the equations

$$(4.8\lambda + 1)u_1 = (4.5\lambda + 1)u_2 = (4.2\mu + 1)u_3 = (3.5\mu + 1)u_4.$$

W.r.t. (17) the remaining parts of $\sigma(T)$ are the anticurves of the curves in the above list.

The above written description of eigenvectors is exact except of the points E, F, G, H, I, J, and their antipoints. The coordinates of E, F, G, H, I, J are written in (34) and in these points of $\sigma(T)$ the eigenvectors are exactly the following ones:
E: All eigenvectors are in $R_{\{3\}} \cap R_{\{1,2,3\}}$, they satisfy the equations

$$u_1 = u_2 = 0, \qquad u_3/4.2 + u_4/3.5 = 0.$$

F: All eigenvectors are in $R_{\{2\}} \cap R_{\{1,2,4\}}$, they satisfy the equations

$$u_1 = u_4 = 0, \qquad u_2/4.5 + u_3/4.2 = 0.$$

G: All eigenvectors are in $R_{\{1\}} \cap R_{\{1,3,4\}}$, they satisfy the equations

$$u_3 = u_4 = 0, \qquad u_1/4.8 + u_2/4.5 = 0.$$

H: All eigenvectors are in $R_{\{2\}} \cap R_{\{1,2,3\}}$, they satisfy the equations

$$u_1 = u_3 = 0, \qquad u_1/4.5 + u_4/3.5 = 0.$$

I: All eigenvectors are in $R_{\{1\}} \cap R_{\{1,2,4\}}$, they satisfy the equations

$$u_2 = u_4 = 0, \qquad u_1/4.8 + u_3/4.2 = 0.$$

J: All eigenvectors are in $R_{\{1\}} \cap R_{\{1,2,3\}}$, they satisfy the equations

$$u_2 = u_3 = 0, \qquad u_1/4.8 + u_4/3.5 = 0.$$

In all other intersection points of the curves $C_\omega, \omega \subset \bar{4}$ the description of the eigenvectors can be obtained from the related ones of the equations (23), (25), (27) etc.

The curves $C_\omega, \omega \subset \bar{4}$ intersect themselves just in the points

$$
(34) \qquad
\begin{aligned}
A &= (-5.3103926, -5.3103926),\\
C &= (-0.2299105, -0.2299105),\\
D &= (-0.2131986, -0.2131986),\\
E &= (-1/3.5, -1/4.2),\\
F &= (-1/4.2, -1/4.5),\\
G &= (-1/4.5, -1/4.8),\\
H &= (-1/3.5, -1/4.5),\\
I &= (-1/4.2, -1/4.8),\\
J &= (-1/3.5, -1/4.8),\\
K &= (-0.5584973, -0.1331650),\\
L &= (-0.5313592, -0.1404882),\\
M &= (-0.5109042, -0.1483088),\\
N &= (-0,4275674, -0,1677518),\\
P &= (-0.4071712, -0.1803472),\\
Q &= (-0.3789583, -0.1975821),\\
R &= (-0.3567177, -0.1539252),\\
S &= (-0.3469854, -0.1634512),\\
T &= (-0.3293075, -0.1765747),\\
U &= (-0.3117133, -0.1985050),\\
V &= (-0.5150901, 0.2578774),\\
W &= (-0.3798825, -0.0413857),\\
X &= (-0.3496113, -0.1016550),\\
Y &= (-0.2854421, -0.1725130),\\
Z &= (-0.2576014, -0.1953875)
\end{aligned}
$$

and their antipoints.

The line $\mu = 1$, $\lambda \in R$ intersects the curves C_ω, $\omega \subset \bar{4}$ in the following order of ω: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,4\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$.

Section 3. The calculation of the spectra

Spectrum of operator S. Let us notice, that w.r.t. the properties of u^+ and u^- we can give an alternative definition of $\sigma(S)$:
The point $(\lambda, \mu) \in \sigma(S)$ iff there exists a set $\omega \subset \bar{4}$ and a vector $u \in R_\omega$, $u \neq 0$ such that

$$
(35) \qquad M_\omega u = \begin{pmatrix} 3.5\delta_1 + 1 & -\delta_2 & -\delta_3 & -\delta_4 \\ -\delta_1 & 3.5\delta_2 + 1 & -\delta_3 & -\delta_4 \\ -\delta_1 & -\delta_2 & 3.5\delta_3 + 1 & -\delta_4 \\ -\delta_1 & -\delta_2 & -\delta_3 & 2.5\delta_4 + 1 \end{pmatrix} u = 0,
$$

where

$$\delta_i = \mu \qquad \text{if} \quad i \in \omega,$$
$$\delta_i = \lambda \qquad \text{if} \quad i \in \bar{4} - \omega.$$

We can subtract the last row of the matrix M_ω from all the preceding ones. This way we obtain the equivalent matrix M'_ω with the same determinant:

$$M'_\omega = \begin{pmatrix} 4.5\delta_1 + 1 & 0 & 0 & -(3.5\delta_4 + 1) \\ 0 & 4.5\delta_2 + 1 & 0 & -(3.5\delta_4 + 1) \\ 0 & 0 & 4.5\delta_3 + 1 & -(3.5\delta_4 + 1) \\ -\delta_1 & -\delta_2 & -\delta_3 & 2.5\delta_4 + 1 \end{pmatrix}$$

Now it follows easily:

(i) The determinant of the matrix M_ω

$$\begin{aligned} P_\omega(\lambda,\mu) = {} & (4.5\delta_1 + 1)(4.5\delta_2 + 1)(4.5\delta_3 + 1)(3.5\delta_4 + 1) - \\ & -\delta_1 \quad (4.5\delta_2 + 1)(4.5\delta_3 + 1)(3.5\delta_4 + 1) - \\ & -(4.5\delta_1 + 1) \quad \delta_2 \quad (4.5\delta_3 + 1)(3.5\delta_4 + 1) - \\ & -(4.5\delta_1 + 1)(4.5\delta_2 + 1) \quad \delta_3 \quad (3.5\delta_4 + 1) - \\ & -(4.5\delta_1 + 1)(4.5\delta_2 + 1)(4.5\delta_3 + 1) \quad \delta_4 \end{aligned} \tag{36}$$

(ii) The formula (36) implies: If $P_\omega(\lambda,\mu) = 0$, then either none or least two of the equations

$$\begin{aligned} 4.5\delta_1 + 1 &= 0, \\ 4.5\delta_2 + 1 &= 0, \\ 4.5\delta_3 + 1 &= 0, \\ 3.5\delta_4 + 1 &= 0 \end{aligned} \tag{37}$$

are fulfilled.

(iii) The eigenvector u satisfies (35) iff

$$(4.5\delta_1 + 1)u_1 = (4.5\delta_2 + 1)u_2 = (4.5\delta_3 + 1)u_3 = (3.5\delta_4 + 1)u_4 \tag{38}$$

and

$$\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 = (2.5\delta_4 + 1)u_4. \tag{39}$$

Because $u \neq 0$, the equations (38) and (39) are not independent. If none of the equations (37) holds and u is an eigenvector, then (39) follows from (38). Thus according to (ii) we have to take the equation (39) into account if at least two of the equations (37) hold. But this can happen only on the lines $\lambda = -2/9$ and $\mu = -2/9$. In any other point of $\sigma(S)$ we do not need to care about (39).

We have to distinguish 16 cases, because there are 16 orthants in R^4. But if we take into account that in any pair of orthants R_ω and $R_{\bar{4}\omega}$, $\omega \subset \bar{4}$ there are the vectors with opposite signs of corresponding coordinates, we can easily deduce that we need to investigate only 8 of the 16 cases. The results about the other 8 cases can then be obtained by interchanging the roles of λ and μ, ω and $\bar{4} - \omega$. Because of the symmetry of S in the first 3 coordinates, the number of cases, which we really have to treat separately, further reduces to four of them.

1. The eigenvector u is in $R_\emptyset$.

Then $\delta_i = \lambda$ for all $i \in \bar{4}$ and according to (36)

$$P_\emptyset(\lambda) = (4.5\lambda + 1)^2(0.75\lambda^2 + 4\lambda + 1).$$

(We write only $P_\emptyset(\lambda)$, because $P_\emptyset$ is independent of μ.) We are seeking for a solution $u \neq 0$ to (35), hence $P_\emptyset(\lambda)$ must be zero. Thus either

$$\begin{aligned} \lambda &= -2/9 \quad \text{or} \\ \lambda &= (-8 + 2\sqrt{13})/3 \quad \text{or} \\ \lambda &= -(8 + 2\sqrt{13})/3. \end{aligned}$$

For $\lambda = -2/9$ the equations (38) and (39) reduce to (18) and (19). But with the exception of the zero vector, none of the solutions to (18), (19) is in $R_\emptyset$, hence the line $\lambda = -2/9$ doesn't contain any point of $\sigma(S)$ s.t. the corresponding eigenvectors would be in $R_\emptyset$. The case $\lambda = (-8 + 2\sqrt{13})/3$ is completely analogous.

In the case $\lambda = -(8 + 2\sqrt{13})/3$ the solutions to (35) are all the vectors $u \in R^4$, which satisfy (38). Among them one can easily find nonzero vectors of $R_\emptyset$. Hence $C_\emptyset$ is in $\sigma(S)$. The equations (3) follow from (38).

2. The eigenvector u is in $R_{\{1\}}$ or $R_{\{2\}}$ or $R_{\{3\}}$.

According to (36)

$$\begin{aligned} &P_{\{1\}}(\lambda,\mu) = P_{\{2\}}(\lambda,\mu) = P_{\{3\}}(\lambda,\mu) = \\ &= (4.5\lambda + 1)((3.375\lambda^2 + 14.5\lambda + 3.5)\mu + 4.25\lambda^2 + 5\lambda + 1) \end{aligned}$$

Hence either $\lambda = -2/9$ or (4) holds.

Let, e.g., $u \in R_{\{1\}}$.

If $\lambda = -2/9$ and $\mu \neq -2/9$, then the equations (38) and (39) imply

$$u_1 = u_4 = u_2 + u_3 = 0,$$

but $u \in R_{\{1\}}$, hence $u = 0$, which is a contradiction, because u is an eigenvector. The case $\lambda = -2/9, \mu = -2/9$ corresponds to the point C , it will be investigated separately.

Let (4) hold. From (38) we obtain (7).

But $u \in R_{\{1\}}$, thus either

(40) $\quad 4.5\mu + 1 \geq 0, \quad 4.5\lambda + 1 \leq 0 \quad \text{and} \quad 3.5\lambda + 1 \leq 0$

or

(41) $\quad 4.5\mu + 1 \leq 0, \quad 4.5\lambda + 1 \geq 0 \quad$ and$3.5\lambda + 1 \geq 0$

or

(42) $\quad u_i = 0 \quad \text{for some} \quad i \in \bar{4}.$

Let (4) hold. $\mu \neq 0$, hence according to (42) and (7) at least one the equations (37) must be fulfilled. But then at least two of the equations (37) are fulfilled, thus either $\lambda = -2/9$ or $\mu - 2/9$ and these cases will be investigated separately. Hence we can assume that either (40) or (41) holds.

Now from (4) follows (8), which together with (40) and (41) implies (5). According to (6) (which can be easily calculated) the cases, in which (39) must be taken into account correspond to the points C and $\widetilde{D}$ and will be treated separately. The remaining calculations concerning $C_{\{1\}} = C_{\{2\}} = C_{\{3\}}$ are very simple.

3.The eigenvector u is in $R_{\{4\}}$.

$$P_{\{4\}}(\lambda, \mu) = (4.5\lambda + 1)^2((0.75\lambda + 2.5)\mu + 1.5\lambda + 1),$$

hence either $\lambda = -2/9$ or (9) holds.

If $\lambda = -2/9$ and $\mu \neq -2/7$, the equations (38) and (39) imply (18) and (19). But only $u = 0$ is in $R_{\{4\}}$ and satisfies (18) and (19). The case $\lambda = -2/9$, $\mu = -2/7$ will be investigated separately.

Let (9) hold. From (38) we obtain (11) and $u \in R_{\{4\}}$ implies that either

$$4.5\lambda + 1 \geq 0 \quad \text{and} \quad 3.5\mu + 1 \leq 0 \tag{43}$$

or

$$4.5\lambda + 1 \leq 0 \quad \text{and} \quad 3.5\mu + 1 \geq 0. \tag{44}$$

From (9) follows (12). (43), (44) and (12) imply (10). (39) should be taken into account only in the point $D \in C_{\{4\}}$ because $\mu(-2/9) = -2/7$ in this case. This will be done later.

4. The eigenvector u is in $R_{\{1,2\}}$ or $R_{\{1,3\}}$ or $R_{\{2,3\}}$.

$$P_{\{1,2\}}(\lambda, \mu) = P_{\{1,3\}}(\lambda, \mu) = P_{\{2,3\}}(\lambda, \mu) =$$
$$(4.5\mu + 1)((3.375\lambda^2 + 11\lambda + 2.5)\mu + 7.75\lambda^2 + 6\lambda + 1),$$

hence either $\mu = -2/9$ or (13) holds.

Let, e.g., $u \in R_{\{1,2\}}$.

If $\mu = -2/9$, $\lambda \neq -2/7$, we obtain from (38) and (39)

$$u_1 + u_2 = u_3 = u_4 = 0$$

and $u \in R_{\{1,2\}}$ implies, that $u = 0$. The cases $(\lambda, \mu) = C$ and $(\lambda, \mu) = \widetilde{D}$ will be investigated separately.

Let (13) hold. The equations (38) together with $u \in R_{\{1,2\}}$ imply, that either (40) or (41) holds. From (16), (40) and (41) follows (14). The rest of the description of $C_{\{1,2\}}$ is a matter of simple calculations.

Now we only have to investigate the eigenvectors in the points C and D. Because $C = (-2/9, -2/9)$, (38) and (39) imply (18) and (19) and vice-versa. The investigation of the point D is somewhat more complicated. From the above calculations follows, that $D \in C_{\{4\}}$, $D \in C_{\{1,4\}} = C_{\{2,4\}} = C_{\{3,4\}}$ and $D \in C_{\{1,2,4\}} = C_{\{1,3,4\}} = C_{\{2,3,4\}}$. Because $D \in C_{\{4\}}$, there must exist corresponding eigenvectors in $R_{\{4\}}$, which must satisfy (38) and (39). But in this case (38) are fulfilled by any vector and from (39) follows (20). Because $D \in C_{\{1,4\}} = C_{\{2,4\}} = C_{\{3,4\}}$, we have to examine also the eigenvectors of $R_{\{1,4\}}, R_{\{2,4\}}$ and $R_{\{3,4\}}$. But, e.g. (38) and (39) together with $u \in R_{\{1,4\}}$ imply (20) and

$$u_1 = 0.$$

But if $u_1 = 0$ and $u \in R_{\{1,4\}}$, then $u \in R_{\{4\}}$ and we see that these eigenvectors are contained in the set of eigenvectors, which we have just found. Similarly one can show, that the investigation of the vectors of $R_{\{2,4\}}, R_{\{3,4\}}, R_{\{1,2,4\}}$ etc. does not provide any eigenvector, which would not be in $R_{\{4\}}$ or would not satisfy (20).

Spectrum of operator T.
As the investigation of $\sigma(T)$ is very similar to that of $\sigma(S)$, we shall point out only the differences between the calculations of $\sigma(S)$ and $\sigma(T)$.

First of all, in the case of T the curves $C_{\{1\}}$, $C_{\{2\}}$, $C_{\{3\}}$ and $C_{\{1,2\}}$, $C_{\{1,3\}}$, $C_{\{2,3\}}$ do not coincide. Hence, we have to distinguish 8 cases instead of four as in the case of S.

Secondly, the polynomials $P_\omega(\lambda, \mu)$ have a common factor in the case of S. This is not true any more in the case of T. Hence, the description of $\sigma(T)$ is more complicated than the corresponding formulae in the description of $\sigma(S)$.

The third difference in the arguments consists in the investigation of the equations

$$\begin{aligned} 4.8\delta_1 + 1 &= 0, \\ 4.5\delta_2 + 1 &= 0, \\ 4.2\delta_3 + 1 &= 0, \\ 3.5\delta_4 + 1 &= 0, \end{aligned} \tag{45}$$

which correspond to (37). Namely, the equations

$$(4.8\delta_1 + 1)u_1 = (4.5\delta_2 + 1)u_2 = (4.2\delta_3 + 1)u_3 = (3.5\delta_4 + 1)u_4,$$

which correspond to (38), don't give a complete description of the eigenvectors iff (at least) two of the equations (45) hold. This happens only in the points E, F, G, H, I, J and their antipoints. In these points the eigenvectors must be examined more carefully, we have to take into account also the equation

$$\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 = (2.5\delta_4 + 1)u_4$$

(which coincides with (39) because of the special choice of T).

The fourth (and most important) difference in the arguments is to be performed in the investigation of C_ω with *card* $\omega = 2$.

Let , e.g., $\omega = \{1,4\}$. One can easily obtain the formula, which corresponds to (36), hence we have

$$P_{\{1,4\}}(\lambda,\mu) = a\mu^2 + b\mu + c,$$

a, b, c are given in (31). $P_{\{1,4\}}$ is quadratic in μ and its discriminant is exactly

$$\Delta(\lambda) = b^2 - 4ac = 1604.6676\lambda^4 + 1546.4292\lambda^3 + 564.3001\lambda^2 + 92.246\lambda + 5.69,$$
$$\Delta'(\lambda) = 6418.6704\lambda^3 + 4639.2876\lambda^2 + 1128.6002\lambda + 92.246,$$
$$\Delta''(\lambda) = 19256.0112\lambda_2 + 9278.5752\lambda + 1128.6002.$$

The discriminant of $\Delta''(\lambda)$ is

$$-837394.62403392,$$

which is a negative value. Thus

$$\Delta''(\lambda) > 0$$

for all $\lambda \in R$ and Δ is strictly convex. Hence, Δ attains just one minimum in a point λ_0 and

$$\Delta'(\lambda_0) = 0. \tag{46}$$

We can calculate

$$\Delta'(-0.22927256) = 1.4\,.\,10^{-7},$$
$$\Delta'(-0.22927258) = -9\,.\,10^{-8},$$

hence

$$\lambda_0 = -0.22927257$$

is the point, where (46) holds with an error less than 10^{-7}. Further

$$\Delta(\lambda_0) = 5.537\,.\,10^{-6}$$

hence

$$\Delta(\lambda) > 0$$

for any $\lambda \in R$.

It follows, that the equation

$$P_{\{1,4\}}(\lambda,\mu) = 0 \tag{47}$$

has for every $\lambda \in R$ just 2 solutions

$$\mu(\lambda) = (-b \pm \sqrt{b^2 - 4ac})/2a,$$

the only exceptions being those $\lambda \in R$, for which

(48) $$a = 0.$$

In these values (47) has only 1 solution and by this way we obtain simultaneously the asymptotes of $\bar{C}_{\{1,4\}}$. The solutions of (48) are

$$\lambda = -0.2295851$$

and

$$\lambda = -2.5550940.$$

Similarly one can calculate the asymptotes

$$\mu = -0.2347032$$

and

$$\mu = -2.9992513.$$

Performing some more standard calculations, we can conclude, that $\bar{C}_{\{1,4\}}$ looks like in Fig. 6,7.

But the eigenvector $u \in R_{\{1,4\}}$ must satisfy the equations (33). Hence either

(49) $$\mu \geq -1/4.8 \quad \text{and} \quad \lambda \leq -1/4.2$$

or

(50) $$\mu \leq -1/3.5 \quad \text{and} \quad \lambda \geq -1/4.5.$$

In these regions of R^2 only a part of $C_{\{1,4\}}$ is contained, namely (30) with (32). The curve

$$\mu = (-b + \sqrt{b^2 - 4ac})/2a$$

lies completely outside of (49) and (50) (see Fig.6,7). We have obtained (30), (32), the rest of the description of $C_{\{1,4\}}$ is a matter of standard calculations.

Section 4. The multiple points of the spectra

We have to calculate the intersection points of C_ξ and C_ζ for various pairs of different subset $\xi \subset \bar{4}$ and $\zeta \subset \bar{4}$. Any such point is also an intersection point of $\bar{C}_\xi$ and $\bar{C}_\zeta$, hence we shall seek for the solutions to various systems of equations of the form

(51) $$\begin{aligned} P_\xi(\lambda, \mu) &= 0 \\ P_\zeta(\lambda, \mu) &= 0 \end{aligned}$$

If we find all the solutions to (51), we only have to decide, which of them are contained in $C_\xi \cap C_\zeta$.

Let us assume, that neither ξ nor ζ is neither $\emptyset$ nor $\bar{4}$. Else the problem (51) is rather simple.

First of all, let us recall, that $P_\omega(\lambda,\mu)$ is the determinant of M_ω (see(35)). If $\mu=\lambda$, then $M_\omega = M_\emptyset$, hence

$$P_\omega(\lambda,\lambda) = P_\emptyset(\lambda)$$

for every $\omega \subset \bar{4}$. Thus especially

$$P_\xi(\lambda,\lambda) = P_\emptyset(\lambda), \tag{52}$$

$$P_\zeta(\lambda,\lambda) = P_\emptyset(\lambda), \tag{53}$$

and for $\mu=\lambda$ the system (51) is equivalent to the equation

$$P_\emptyset(\lambda) = 0.$$

This is a fourth order equation, hence it has 4 solutions (counting the multiplicities, of course). Each of these solutions λ gives a solution (λ,λ) to the system (51). Thus from now on, we are interested only in those solutions (λ,μ) to (51), for which $\lambda \neq \mu$.

Now P_ξ and P_ζ can be treated as polynomials in μ with coefficients depending on a parameter λ. The system (51) is equivalent to

$$\begin{aligned} P_\xi(\lambda,\mu) &= 0, \\ P_\zeta(\lambda,\mu) - P_\xi(\lambda,\mu) &= 0. \end{aligned} \tag{54}$$

But according to (52) and (53)

$$P_\zeta(\lambda,\lambda) - P_\xi(\lambda,\lambda) = 0,$$

hence $\mu=\lambda$ is a root of the polynomial $P_\zeta - P_\xi$ and

$$P_\zeta(\lambda,\mu) - P_\xi(\lambda,\mu) = (\mu-\lambda)V_{\xi,\zeta}(\lambda,\mu)$$

where $V_{\zeta,\xi}$ is a suitable polynomial. Because we are interested only in the solutions $\mu \neq \lambda$ to (51) now, we can investigate the system

$$\begin{aligned} P_\xi(\lambda,\mu) &= 0 \\ V_{\xi,\zeta}(\lambda,\mu) &= 0 \end{aligned} \tag{55}$$

instead of (54) (or(51)).

The polynomials P_ξ and $V_{\xi,\zeta}$ (in μ,λ is a parameter) have a common solution μ iff their resultant

$$res(P_\xi(\lambda,\mu), V_{\xi\zeta}(\lambda,\mu)) \equiv W_{\xi,\zeta}(\lambda)$$

(which is a polynomial in the coefficients of P_ξ and $V_{\xi,\zeta}$, hence a polynomial in λ only) is equal to zero. (See, e.g., [11] for the definition of the resultant.) Hence instead of (51) we can solve the equation

$$W_{\xi,\zeta}(\lambda) = 0. \tag{56}$$

If λ is a solution to (56), the corresponding value $\mu(\lambda)$ can be calculated from (one of) the equations (51) (or from (55)).

In the investigation of the spectra of S and T four special formulae are important:

(i) If $card\,\xi = card\,\zeta = 1$ and

$$P_\xi(\lambda,\mu) = \alpha_1\mu + \alpha_0,$$
$$P_\zeta(\lambda,\mu) = \beta_1\mu + \beta_0,$$

then

$$W_{\xi,\zeta}(\lambda) = \beta_1 - \alpha_1 \tag{57}$$

and $W_{\xi,\zeta}$ is quadratic.

(ii) If $card\,\xi = 2$, $card\,\zeta = 1$ and

$$P_\xi(\lambda,\mu) = \alpha_2\mu^2 + \alpha_1\mu + \alpha_0,$$
$$P_\zeta(\lambda,\mu) = \beta_1\mu + \beta_0$$

then

$$W_{\xi,\zeta}(\lambda) = ((\beta_0 - \alpha_0)/\lambda)^2 - \alpha_1(\beta_0 - \alpha_0)/\lambda + \alpha + \alpha_0\alpha_2 \tag{58}$$

is a fourth degree polynomial.

(iii) If $card\,\xi = card\,\zeta = 2$ and

$$P_\xi(\lambda,\mu) = \alpha_2\mu^2 + \alpha_1\mu + \alpha_0,$$
$$P_\zeta(\lambda,\mu) = \beta_2\mu^2 + \beta_1\mu + \beta_0,$$

then

$$W_{\xi,\zeta}(\lambda) = \alpha_2((\beta_0 - \alpha_0)/\lambda)^2 + \alpha_1(\beta_2 - \alpha_2)(\beta_0 - \lambda_0)/\lambda + \alpha_0(\beta_2 - \alpha_2)^2 \tag{59}$$

is a fourth degree polynomial.

(iv) If $card\,\xi = 3$, $card\,\zeta = 1$ and

$$P_\xi(\lambda,\mu) = \alpha_3\mu^3 + \alpha_2\mu^2 + \alpha_1\mu + \alpha_0,$$
$$P_\zeta(\lambda,\mu) = \beta_1\mu + \beta_0,$$

then

$$W_{\xi,\zeta}(\lambda) = \beta_1(\beta_0(\lambda\alpha_3 + \lambda_2) - \beta_1(\beta_0 - \alpha_0)/\lambda) - \beta_0^2\alpha_3 \tag{60}$$

is a sixth degree polynomial.

The formulae (57), (58), (59), (60) can be obtained easily from the definition of the resultant, (52) and (53).

We can apply them to the matrix S and we obtain the following list of $W_{\xi,\zeta}$:

$$\begin{aligned}
W_{\{1\},\{4\}} &= 11.25\lambda^2 + 7\lambda + 1 = (4.5\lambda + 1)(2.5\lambda + 1)\,,\\
W_{\{1\},\{1,2\}} &= 248.625\lambda^4 + 252\lambda^3 + 95.5\lambda^2 + 16\lambda + 1 =\\
&= (4.5\lambda + 1)^2(3.5\lambda + 1)^2,\\
W_{\{1\},\{1,4\}} &= 410.0625\lambda^4 + 364.5\lambda^3 + 121.5\lambda^2 + 18\lambda + 1 = (4.5\lambda + 1)^4,\\
W_{\{1\},\{1,2,3\}} &= 7813.96875\lambda^6 + 11907\lambda^5 + 7536.375\lambda^4 + 2536\lambda^3 +\\
&+ 478.5\lambda^2 + 48\lambda + 2 = 2(3.5\lambda + 1)^3(4.5\lambda + 1)^3,\\
W_{\{1\},\{1,2,4\}} &= 10126.265625\lambda^6 + 13471.3125\lambda^5 + +7392.9375\lambda^4 +\\
&+ 2139.5\lambda^3 + 343.75\lambda^2 + 29\lambda + 1 = (3.5\lambda + 1)(4.5\lambda + 1)^3 \times\\
&\times (31.75\lambda^2 + 12\lambda + 1),\\
W_{\{4\},\{1,2\}} &= 121.5\lambda^4 + 94.5\lambda^3 + 24\lambda^2 + 2\lambda = 2(4.5\lambda + 1)^2(3\lambda + 1)\lambda,\\
W_{\{4\},\{1,4\}} &= 410.0625\lambda^4 + 364.5\lambda^3 + 121.5\lambda^2 + 18\lambda + 1 = (4.5\lambda + 1)^4,\\
W_{\{4\},\{1,2,3\}} &= 14762.25\lambda^6 + 22963.5\lambda^5 + 14762.25\lambda^4 +\\
&+ 5022\lambda^3 + 954\lambda^2 + 96\lambda + 4 = 4(4.5\lambda + 1)^4(3\lambda + 1)^2,\\
W_{\{1,2\},\{1,4\}} &= 339.1875\lambda^4 + 312.75\lambda^3 + 109\lambda^2 + 17\lambda + 1 =\\
&= (4.5\lambda + 1)^2(16.75\lambda^2 + 8\lambda + 1).
\end{aligned}$$

Many other $W_{\xi,\zeta}$ coincide with the listed ones, because $C_{\{1\}} = C_{\{2\}} = C_{\{3\}}$ etc. The remaining $W_{\xi,\zeta}(\lambda)$ are hard to calculate. But we can also work with $P_\omega(\lambda,\mu)$ as polynomials in λ with coefficients depending on μ. Then we can define analogously the resultants $\widehat{W}_{\xi,\zeta}(\mu)$. Of course,

$$\widehat{W}_{\xi,\zeta} = \widetilde{W}_{\bar{4}-\xi,\bar{4}-\zeta}.$$

Hence, the above list gives the resultants $\widehat{W}_{\xi,\zeta}$ in the cases, when $W_{\xi,\zeta}$ is complicated.

In the case of T , the analogous list of $W_{\xi,\zeta}$ contains 49 polynomials, because there are no multiple curves in $\sigma(T)$. Nevertheless, the resultants $W_{\xi,\zeta}$ can be calculated. Further, if $\xi \cap \zeta \neq \emptyset$, the polynomial $W_{\xi,\zeta}$ always has at least one root equal to one of the values

$$(61) \qquad -1/4.8\,, -1/4.5\,, -1/4.2\,, -1/3.5$$

and any such root is a double root of $W_{\xi,\zeta}$. Hence twenty four of the polynomials $W_{\xi,\zeta}$ can be further decomposed. Nevertheless, there remains enough work to be done, if we want to find all the roots of all these polynomials. If we find them, in the end we obtain the exact list (34) of all the multiple eigenvalues of T.

Section 5. The Brouwer degree and the number of solutions

In [8],[9] the following special class of the operators with jumping nonlinearities was investigated:

$$S = \begin{pmatrix} -1+a_1 & -1 & -1 & \dots & -1 \\ -1 & -1+a_2 & -1 & \dots & -1 \\ -1 & -1 & -1+a_3 & \dots & -1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -1 & -1 & -1 & \dots & -1+a_n \end{pmatrix} \tag{62}$$

$$\lambda = -\mu, \tag{63}$$

$$a_i + 1/\lambda > 0 \quad \text{and} \quad a_i + 1/\mu > 0 \quad \text{for every } i \in \bar{n}. \tag{64}$$

For any such operator $S_{\lambda,\mu}$ one can define a corresponding hyperplane $\rho(S_{\lambda,\mu}) \subset R^n$ and one can show, that $deg(S_{\lambda,\mu})$ as well as $k(S_{\lambda,\mu})$ can be calculated from the intersection properties of a fixed n-dimensional cube $C^n \subset R^n$ and the hyperplane $\rho(S_{\lambda,\mu})$.

Of course, the operators $S_{\lambda,\mu}$ and $T_{\lambda,\mu}$ (with S and T as in (2) and (22)) do not satisfy (63) and (64) in general. But in the last section of [8] one can find a discussion of the assumptions (62), (63) and (64), which results into the statement, that these assumptions can be weakened and we only need to assume that

$$S = R + D, \tag{65}$$

where R is a matrix with rank $R = 1$ and D is a diagonal matrix. Hence we can generalize the procedures, which have been developed in [8] , thereafter we can rather easily calculate $deg(S_{\lambda,\mu})$ and $k(S_{\lambda,\mu})$ as well as $deg(T_{\lambda,\mu})$ and $k(T_{\lambda,\mu})$ for S and T given in (2) and (22) and any λ and μ. Such a generalization of [8] in the case of T is very easy, if (λ, μ) is either in

$$]-\infty; -1/3.5[\times]-1/4.8; +\infty[\tag{66}$$

or in

$$]-1/4.8; +\infty[\times]-\infty; -1/3.5[, \tag{67}$$

because in any of these two regions all the terms

$$4.8\lambda + 1, \quad 4.5\lambda + 1, \quad 4.2\lambda + 1, \quad 3.5\lambda + 1$$

have the same sign and all the terms

$$4.8\mu + 1, \quad 4.5\mu + 1, \quad 4.2\mu + 1, \quad 3.5\mu + 1$$

have just the opposite sign. This fact plays the same role in the case of $T_{\lambda,\mu}$ as the assumption (64) in the case (62), (63). What concerns the remaining points of the (λ, μ) -plane, the generalization of the results of [8] does not seem to be straightforward, nevertheless it is possible.

$deg(T_{\lambda,\mu})$ is constant in every component of $R^2 - \sigma(T)$, hence we can speak about the degree of a component of $R^2 - \sigma(T)$. A little bit surprising may be the fact, that $k(T_{\lambda,\mu})$ is constant in every component of $R^2 - \sigma(T)$ as well. The proof of this assertion is rather complicated, because first of all one has to generalize the results of [8]. But the main idea of the proof is simple.

Let

$$C = \{(\lambda(t), \mu(t)) \mid t \in [0; 1]\}$$

be a continuous curve, which is completely contained in one component of $R^2 - \sigma(T)$. We can define the hyperplanes $\rho(T_{\lambda(t),\mu(t)})$, $t \in [0,1]$ and these hyperplanes depend continuously on t. Let

$$k(T_{\lambda(t_1)\ \mu(t_1)}) \neq k(T_{\lambda(t_2),\mu(t_2)})$$

for some $t_1, t_2 \in [0,1]$. $k(T_{\lambda,\mu})$ is uniquely determined by a set of 1-dimensional edges of C^n, which are intersected by $\rho(T_{\lambda,\mu})$, hence

$$\rho(T_{\lambda(t_1),\mu(t_1)}) \quad \text{and} \quad \rho(T_{\lambda(t_2),\mu(t_2)})$$

intersect different sets of 1 -dimensional edges of C^n. But then there must exist a value $t_0 \in [t_1, t_2]$ such that $\rho(T_{\lambda(t_0),\mu(t_0)})$ contains an end-point of such an edge. Thus

$$deg(T_{\lambda(t_0),\mu(t_0)})$$

in not defined. This in turn implies, that

$$(\lambda(t_0), \mu(t_0)) \in \sigma(T)$$

which is a contradiction.

Further it can be shown, that if (λ, μ) is neither in (66) nor in (67), then the problem of determining $deg(T_{\lambda,\mu})$ and $k(T_{\lambda,\mu})$ can be reduced to a dimension $n < 4$. But if $S_{\lambda,\mu}$ is any operator with jumping nonlinearity in a dimension $n < 4$, then always

$$k(S_{\lambda,\mu}) = |deg(S_{\lambda,\mu})|.$$

For the operators with jumping nonlinearities in the dimension $n = 4$ we have the following result (at least if S satisfies (65)):

$$|deg(S_{\lambda,\mu})| \leq 3$$

and either

$$k(S_{\lambda,\mu}) = |deg(S_{\lambda,\mu})| \tag{68}$$

or

$$(69) \qquad deg(S_{\lambda,\mu}) = 0 \quad \text{and} \quad k(S_{\lambda,\mu}) = 2.$$

Thus is order to determine the degree of any component κ of $R^2 - \sigma(T)$ it is sufficient to choose one point $(\lambda, \mu) \in \kappa$, to define the corresponding hyperplane $\rho(T_{\lambda,\mu}) \subset R^n$ and to use a modification of the procedure, which was developed in [8]. If either $deg(T_{\lambda,\mu}) \neq 0$ or κ is not completely contained in one of the regions (66) and (67), then $k(T_{\lambda,\mu}) = |deg(T_{\lambda,\mu})|$ and $k(T_{\lambda,\mu})$ is constant in κ. If $deg(T_{\lambda,\mu}) = 0$ and κ is completely contained in one of the regions (66) and (67), $k(T_{\lambda,\mu})$ can be calculated from the intersection properties of $\rho(T_{\lambda,\mu})$ and C^n. Once more , $k(T_{\lambda,\mu})$ is constant in κ.

This reasoning gives the values of $deg(T_{\lambda,\mu})$,which are written in the Fig. 3,4,5. In any component of $R^2 - \sigma(T)$ we have

$$k(T_{\lambda,\mu}) = |deg(T_{\lambda,\mu})|,$$

only in the component, which is denoted by the sign $*$,

$$deg(T_{\lambda,\mu}) = 0 \quad \text{and} \quad k(T_{\lambda,\mu}) = 2.$$

(Of course, we have always

$$deg(T_{\lambda,\mu}) = deg(T_{\mu,\lambda}),$$
$$k(T_{\lambda,\mu}) = k(T_{\mu,\lambda}),$$

hence if the value of the degree of a component can be found in none of the Fig. 3,4,5, one has to look onto the corresponding anticomponent.)

In the case of S (see(2)), the reasoning is completely analogous and gives the values of the degree, which can be found in Fig. 1,2. In all components of $R^2 - \sigma(S)$ we have (68), only in the component, which is denoted by $*$ (and in its anticomponent), we have (69).

Section 6. Concluding remarks

The calculation of $\sigma(S)$ and $\sigma(T)$ has been very complicated. Especially the calculation of the polynomials $W_{\xi,\zeta}$ and their roots is very lengthy. Nevertheless, this calculation seems to be indispensable. E.g., I would never suspect, that the curves $C_{\{1,2\}}$ and $C_{\{1,3\}}$ (in the case $T_{\lambda,\mu}$) intersect themselves in the point T, if I had not found the λ -coordinate of T as one of the roots of $W_{\{1,2\},\{1,3\}}$. Of course, this can well be my fault, but without calculating, e.g., all the roots of $W_{\{1,2\},\{1,3\}}$ we can hardly be sure, that there is not another intersection point of $C_{\{1,2\}}$ and $C_{\{1,3\}}$ in a small neighbourhood of the point J. The existence of such a point would imply the existence of another component of $R^2 - \sigma(T)$, which could be invisible in Fig.5, but could be seen in another scaling. We have met a similar situation in the case of the component, which is bounded by $C_{\{1,2\}}$ and $C_{\{1,3\}}$ between the points J and T. It is invisible in the Fig.4, but can be easily seen in the scaling of the Fig.5.

From the results of [8],[9] follows the observation, that the most interesting spectra seem to have those matrices S, which "almost have" some multiple linear eigenvalues (in the sense that some simple linear eigenvalues of S are contained in a small interval of R). Unfortunately, the "almost-multiplicity" of the eigenvalues of S considerably complicates the calculation of $\sigma(S)$. Hence, in the most interesting cases in the dimension $n \geq 5$ the calculation of $\sigma(S)$ is in principle very similar to the calculations, which have been described in this article, but especially the calculation of the roots of $W_{\xi,\zeta}$ must be expected to be extremely lengthy and difficult because of the following reasons:

(i) In the case of T we had to calculate 49 polynomials $W_{\xi,\zeta}$. In the n-dimensional case an analogous list of $W_{\xi,\zeta}$ contains

$$(2^{n-1} - 1)^2$$

polynomials. (In the case of T we have

$$W_{\xi,\zeta} = W_{\zeta-\xi,\zeta},$$

whenever $card\,\zeta = 2$, $card\,\xi = 1$, $\xi \subset \zeta$. Hence, there are only 43 different polynomials on the list of $W_{\xi,\zeta}$ and we can expect, that in the n-dimensional case some of the polynomials $W_{\xi,\zeta}$ can coincide as well. Thus the list of different $W_{\xi,\zeta}$ may contain less than $(2^{n-1} - 1)^2$ polynomials.
On the other hand, the number of different $W_{\xi,\zeta}$ can be hardly <u>substantially</u> smaller than $(2_{n-1} - 1)^2$).

(ii) In the case of T the maximal degree of $W_{\xi,\zeta}$ is equal to 6 and some of the sixth degree polynomials $W_{\xi,\zeta}$ have all the roots different from the values (61), thus they can't be easily decomposed. In the n-dimensional case the maximal degree of $W_{\xi,\zeta}$ is equal to

$$(n-1)(n-2).$$

(iii) In the n-dimensional case the formulae (57), (58), (59), (60) are not sufficient for the calculation of all the $W_{\xi,\zeta}$ and we should derive some other formulae, which turn out to be substantially more complicated than (57), (58), (59), (60).

(iv) In the case of T the major part of the roots of $W_{\xi,\zeta}$ is contained in a small neighbourhood of the value -0.25. Hence $W_{\xi,\zeta}$ have "almost multiple" roots. The same situation should be expected in the most interesting examples in higher dimensions. But the "almost - multiplicity" of the roots of $W_{\xi,\zeta}$ causes serious numerical problems, if we want to calculate them. E.g., the Newton's method gives only few digits of the roots, because it is sensitive to rounding errors in such cases.

(v) Nevertheless, we need rather good approximations of the roots of $W_{\xi,\zeta}$ if we want at least to distinguish one from another. Namely, from (i), (ii) and (iv) follows, that many roots of $W_{\xi,\zeta}$ are contained in a small neighbourhood of one value, hence some pairs of the roots must necessarily almost coincide. In fact, we can show, that, e.g., in the case of T one root of $W_{\{2,3\},\{1,4\}}$ is

$$-0.2292226,$$

one root of $W_{\{3\},\{1,4\}}$ is

$$-0.2292256.$$

(vi) In the case of T, all the coefficients of $W_{\xi,\zeta}$ are rational, but they have many digits. E.g.,

$$\begin{aligned} W_{\{4\},\{1,2,3\}} = & \; 14876.533224\lambda^6 + 23235.594228\lambda^5 + \\ & + 14987.115504\lambda^4 + 5112.162\lambda^3 + \\ & + 973.1313\lambda^2 + 98.07\lambda + 4.09 \end{aligned}$$

It is hard to calculate with numbers, which have eleven digits. But because of (iv) the roots of $W_{\xi,\zeta}$ must be expected to be very sensitive to the errors in the coefficients, hence according to (v) we have to work with very good approximations of the coefficients of $W_{\xi,\zeta}$ (if not with their exact values). The situation must be expected to be even more unpleasant in the higher dimensions.

Remark. In the case of S (see(2)) the list of $W_{\xi,\zeta}$ reduces to 9 polynomials and

$$9 = (2^{n-1} - 1)^2$$

for $n = 3$. All these polynomials can be easily decomposed, because they contain factors $(4.5\lambda + 1)$ and $(3.5 + 1)$. After dividing them by these factors we obtain polynomials, which are at most quadratic. But

$$2 = (n-1)(n-2)$$

for $n = 3$. Hence S , which has a double linear eigenvalue, exhibits some "3-dimensional features".

The most interesting points in $\sigma(T)$ seem to be the points E, F, G, H, I, J and their antipoints. All these points are intersection points of four straight lines and their antilines and to each of these points correspond very special eigenvectors (every such eigenvector has two coordinates equal to zero). These properties of the points seem to be a consequence of the special form of T, which is a matrix of the type (65).

Let us look at the Fig.7. The curve $\bar{C}_{\{1,4\}}$ contains the points A, B, C, D, because for every $\omega \subset \bar{4}$ the curve $\bar{C}_\omega$ contains them. But $\bar{C}_{\{1,4\}}$ contains also the points $\widetilde{E}, G, \widetilde{H}$ and I.

Because $card\{1,4\} = 2$, we know a priori, that $\bar{C}_{\{1,4\}}$ must be quadratic in both λ and μ.Thus the equation of $\bar{C}_{\{1,4\}}$ contains nine coefficients.Now the coefficients (hence the equation of $\bar{C}_{\{1,4\}}$) can be calculated from the eight conditions, which assert that $\bar{C}_{\{1,4\}}$ contains the eight points A, B, C, D , $\widetilde{E}, G, \widetilde{H}, I$. This way we can obtain the equation of $\bar{C}_{\{1,4\}}$ up to a scalar multiple. Similarly we can calculate the equation of any other curve $\bar{C}_\omega, \omega \subset \bar{4}$.

As we have just seen the points E, F, G, H, I, J together with their antipoints and the points A, B, C, D (which correspond to the linear spectrum of T) define all the curves $\bar{C}_\omega$, $\omega \subset \bar{4}$. Hence, we can ask, whether in the case of a general (or generic) matrix S there exists an analogous small subset of $\sigma(S)$, which together with the linear spectrum of S uniquely determines all the curves $\bar{C}_\omega$, $\omega \subset \bar{4}$. If the answer were affirmative, it could eventually provide a more simple and more general way, how to calculate $\sigma(S)$.

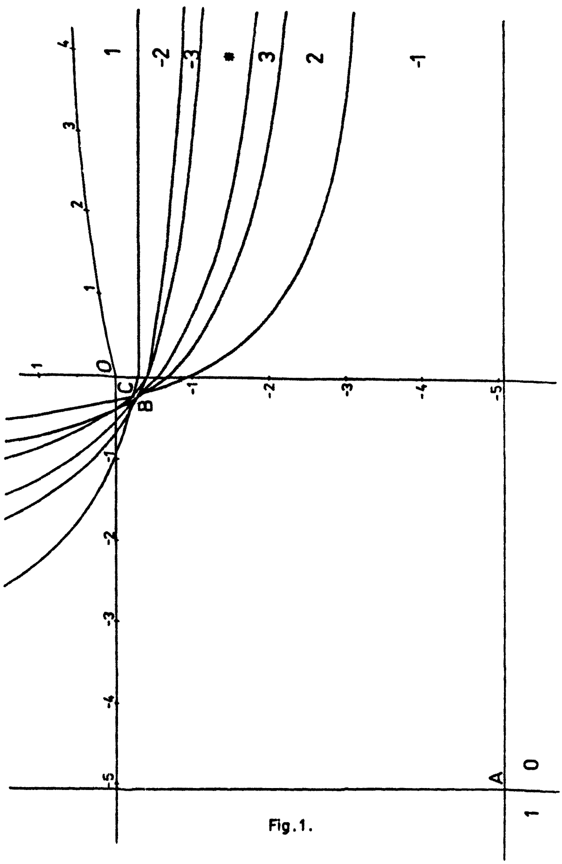

Fig.1.

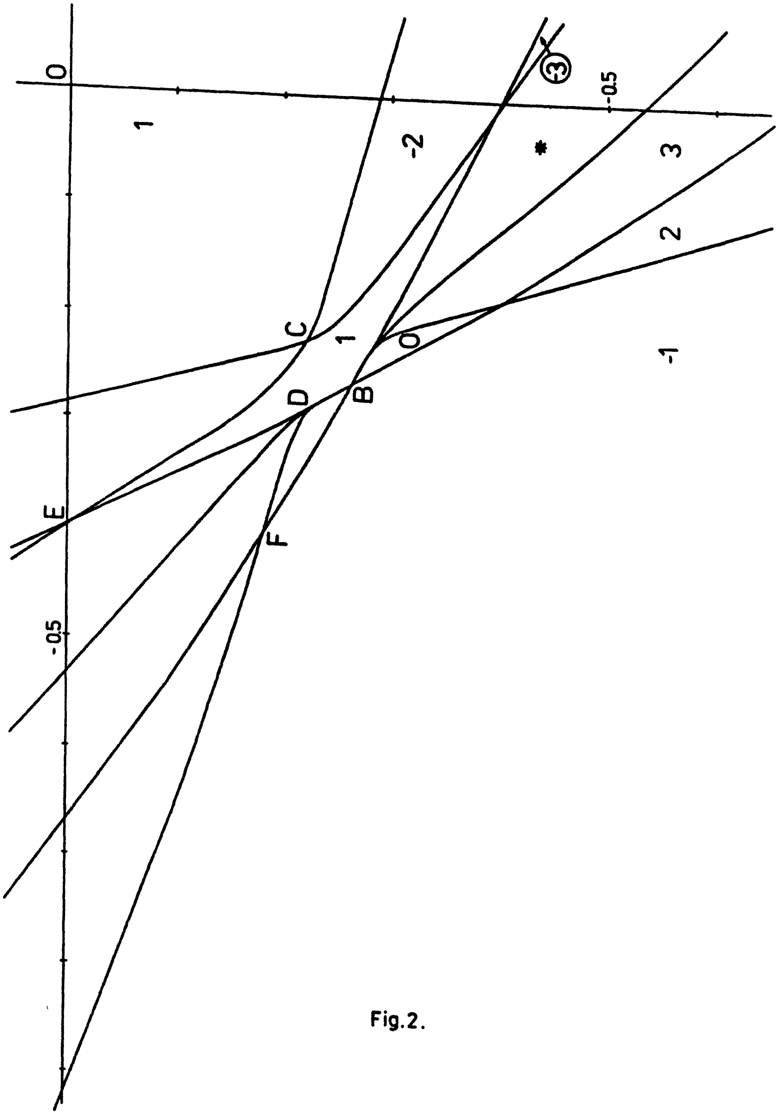

Fig.2.

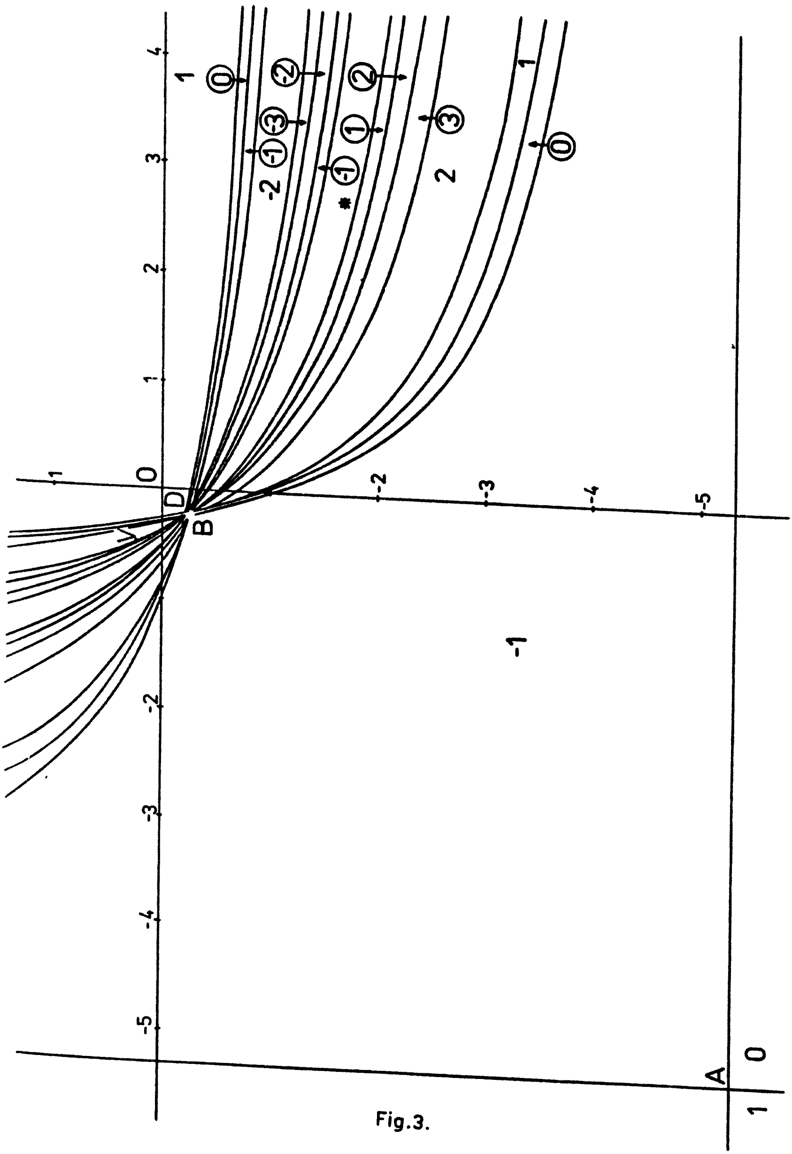

Fig.3.

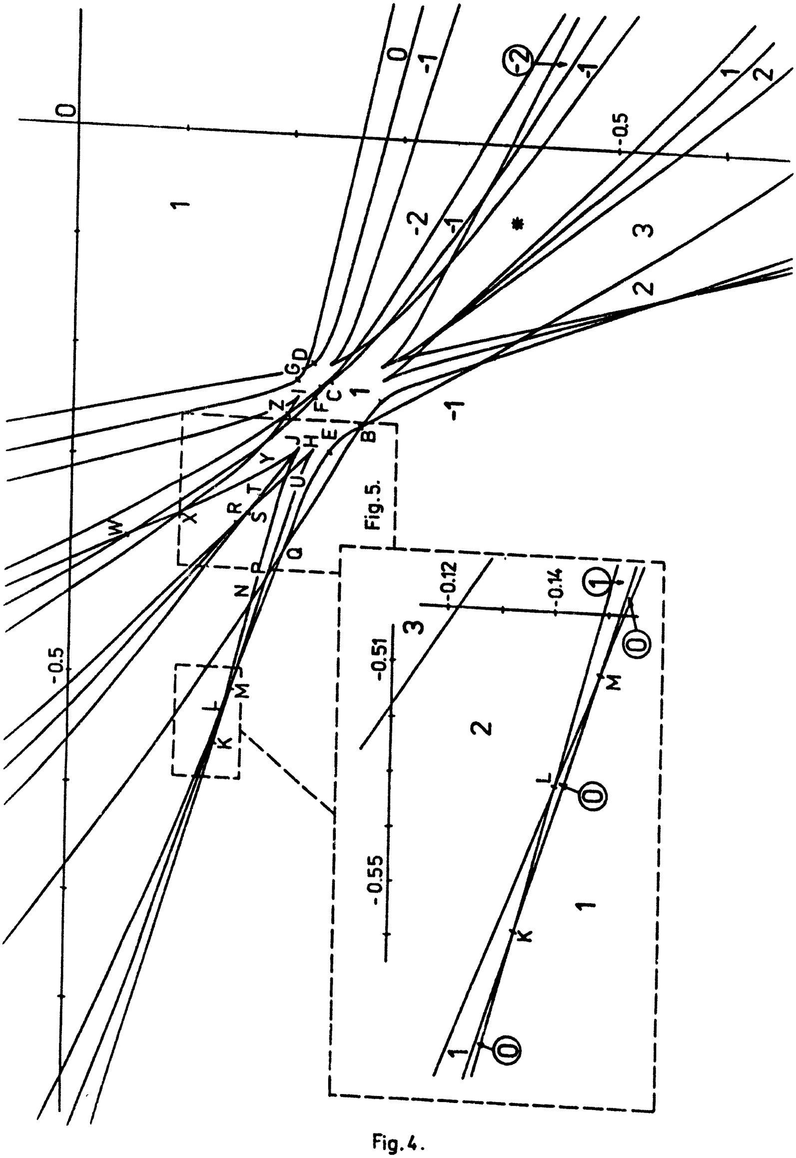

Fig.4.

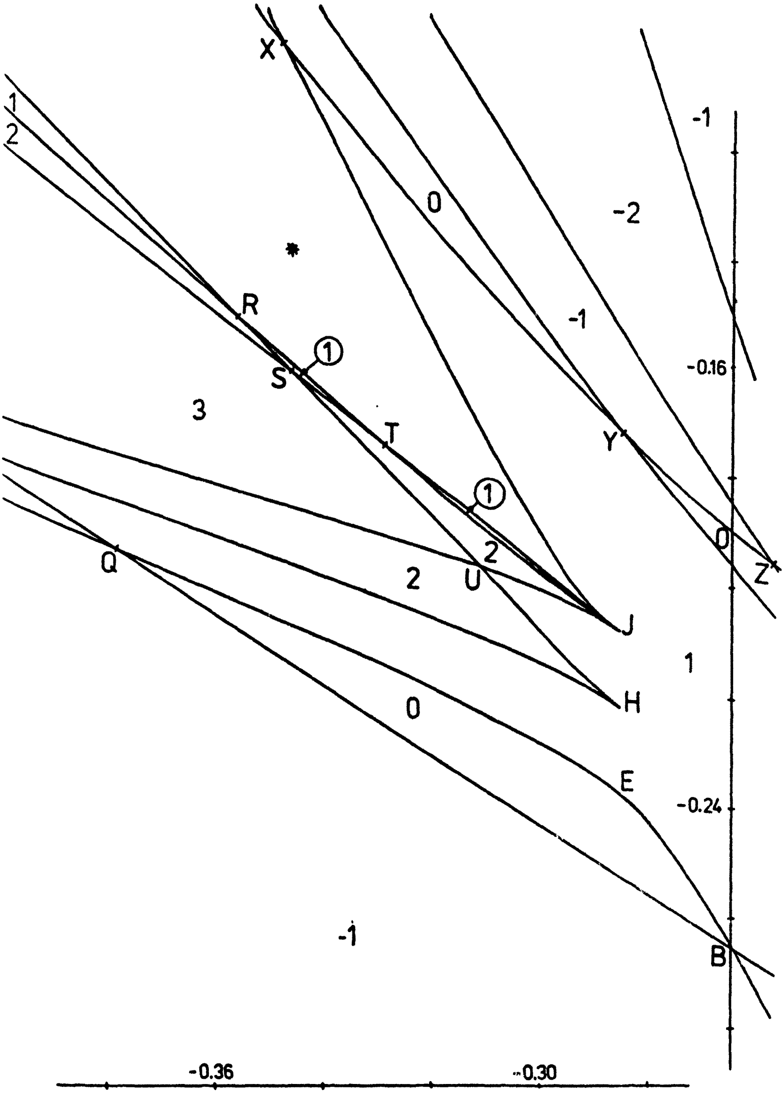

Fig. 5.

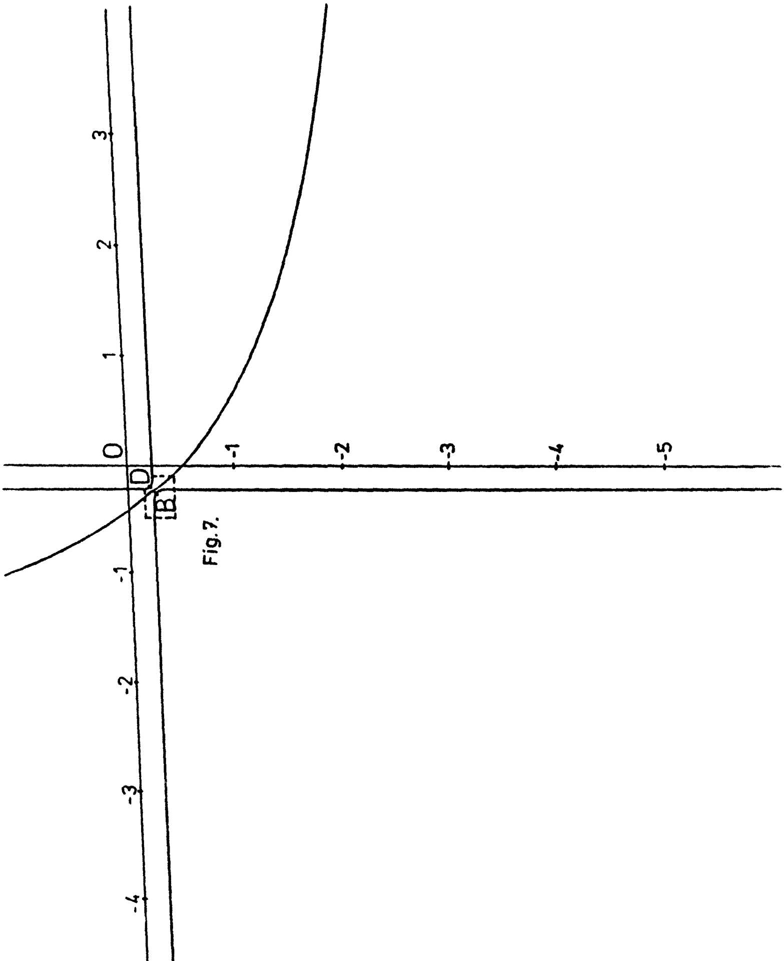

Fig. 7.

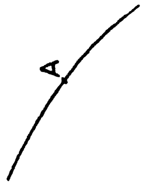

Fig. 6.

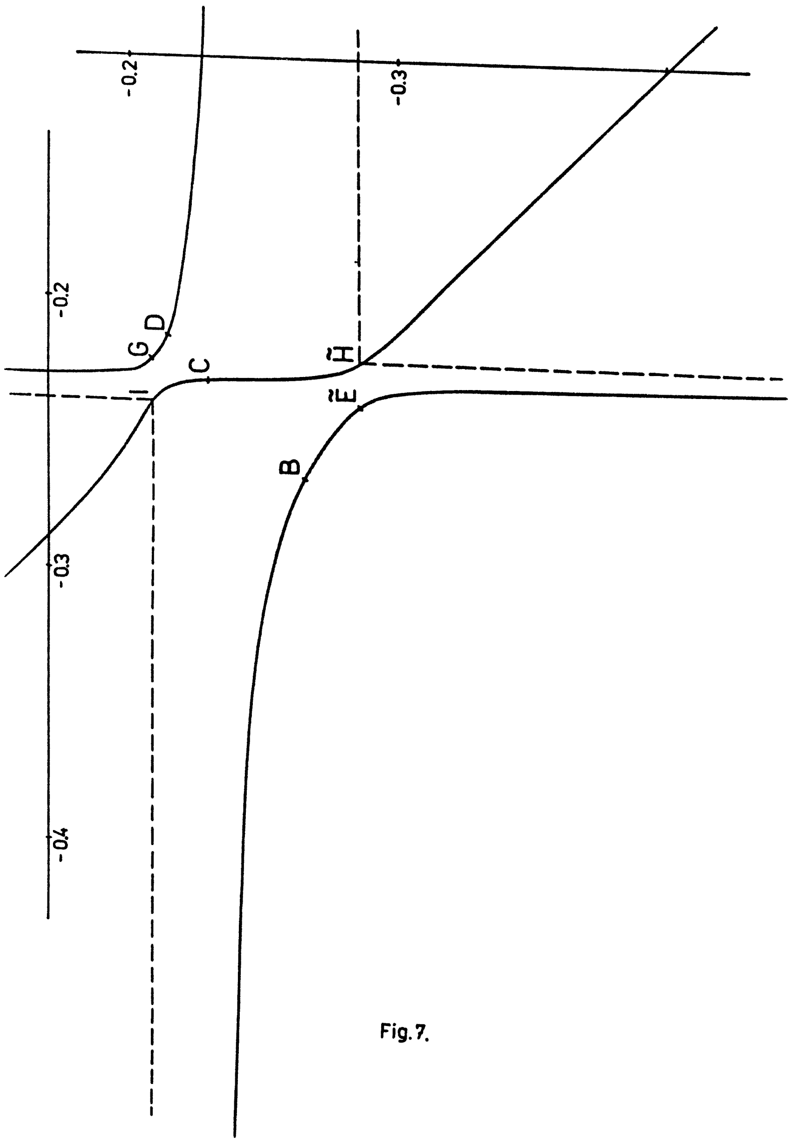

Fig. 7.

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