Pavel Drábek; Petr Tomiczek

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Remark on the structure of the range of second order nonlinear elliptic operator

PAVEL DRÁBEK, PETR TOMICZEK

Dedicated to the memory of Svatopluk Fučík

Abstract. In this paper we study the solvability of the boundary value problem for semilinear second order elliptic partial differential equation at resonance. We consider nonlinearities g satisfying the sign condition and investigate the set of right hand sides for which the problem has a solution.

Keywords: Nonlinear second order elliptic equation, semilinear problems, nonlinearities with linear growth

Classification: 35J65, 35J60, 35J25

1. Introduction.

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with the smooth boundary $\partial\Omega$, we suppose $\partial\Omega$ is at least of a class $C^{1,\mu}$, $0 < \mu < 1$, and let

$$Lu := \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) - a_0(x)u$$

be a second order symmetric uniformly elliptic operator with smooth coefficients. More precisely, we suppose

$$a_{ij}(x) = a_{ji}(x),\ 1 \leq i,j \leq N,\ a_0(x) \geq 0 \quad \text{on } \Omega,$$
$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j > 0,$$
$$\text{for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^N \setminus \{0\},\ a_{ij} \in C^1(\overline{\Omega}),\ 1 \leq i,j \leq N, a_0 \in L^\infty(\Omega).$$

We shall discus the solvability of the selfadjoint boundary value problem

$$\begin{aligned} Lu + \lambda_1 u + g(x,u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of -L, $f \in L^p(\Omega)$ with $p > N$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory's function which grows at most linearly, i.e. $g(\cdot, u)$ is measurable function for any $u \in \mathbb{R}$, $g(x,\cdot)$ is continuous for a.e. $x \in \Omega$, and there exist a constant $c_1 > 0$ and a function $c_2 \in L^p(\Omega)$, $p > N$ such that

$$|g(x,u)| \le c_1|u| + c_2(x) \tag{1.2}$$

for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$.

When this is the case, the first eigenvalue $\lambda_1 > 0$ of $-L$ is simple and the corresponding eigenspace is generated by a smooth function φ. It is $\varphi > 0$ in Ω and $\frac{\partial\varphi}{\partial n} < 0$ on $\partial\Omega$, where $\frac{\partial}{\partial n}$ is the outer normal derivative. These facts follow from the Bony's maximum principle and the abstract Krein–Rutman theorem (see e.g. Bers, John, Schechter [2]).

In what follows we shall denote by P the orthogonal $L^2(\Omega)$–projection onto the eigenspace generated by $\varphi, \|\varphi\|_{L^2} = 1$ and by $Q = I - P$ the complementary projection.

2. Preliminaries.

The following lemma is proved in Iannacci, Nkashama, Ward [7].

Lemma 2.1. *Let $\Gamma_- \in L^p(\Omega)$, $p > N$. Then there exists a constant $d = d(\Gamma_-) > 0$ such that for all $p_+, p_- \in L^p(\Omega)$ satisfying*

$$\begin{aligned} &0 \le p_+(x) \le d, \\ &0 \le p_-(x) \le \Gamma_-(x) \end{aligned} \tag{2.1}$$

for a.e. $x \in \Omega$, and all $u \in W^{2,p}(\Omega)$, $p > N$, for which

$$\begin{aligned} Lu + \lambda_1 u + p_+(x)u^+ - p_-(x)u^- &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

one of the following assertions hold:

(i) *$u = 0$ on $\overline{\Omega}$;*
(ii) *$u(x) > 0$ for all $x \in \Omega$ and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$;*
(iii) *$u(x) < 0$ for all $x \in \Omega$ and $\frac{\partial u}{\partial n} > 0$ on $\partial\Omega$.*

Remark 2.1. Similarly it is possible to prove a "dual version" of Lemma 2.1. with an arbitrary $\Gamma_+ \in L^p(\Omega)$, $p > N$, a constant $d = d(\Gamma_+) > 0$ and functions $p_\pm(x)$ satisfying $0 \le p_+(x) \le \Gamma_+(x)$, $0 \le p_-(x) \le d$.

If $0 \le \Gamma_-(x) \le \lambda_2 - \lambda_1$ for a.e. $x \in \Omega$, then the assertion of Lemma 2.1. holds with any $d < \lambda_2 - \lambda_1$ (λ_2 is the second eigenvalue of $-L$). This follows immediately from Lemma 1 in [7].

Remark 2.2. Using the shooting argument in one–dimensional case ($N = 1$) we can find the explicit relationship between Γ_- and d (see Drábek [4]). That is why in the case $N = 1$ the results of this paper can be proved with more accurately formulated assumptions.

Remark 2.3. Let us consider the operator $A : W^{2,p}(\Omega) \cap H_0^1(\Omega) \to L^p(\Omega)$, $p \geq 2$, defined by

$$A: \quad y \mapsto Lu + \lambda_1 u.$$

Then it is well known that

$$L^2(\Omega) = N(A) \oplus R(A)$$

(the orthogonal decomposition of $L^2(\Omega)$), where $N(A)$ is the kernel and $R(A)$ is the range of A. Moreover, $K = A^{-1}$ is a well-defined operator from $R(A)$ onto $D(A) \cap R(A)$ ($D(A) \subset L^2(\Omega)$ is the domain of A), $K(R(A) \cap L^P(\Omega)) \subset W^{2,p}(\Omega) \cap H_0^1(\Omega)$, $p \geq 2$, and

$$\|Kf\|_{W^{2,p}} \leq c_p \|f\|_{L^p},$$

for any $f \in R(A) \cap L^p(\Omega)$. The operator K is called *the right inverse* of A.

Any function $f \in L^p(\Omega)$, $p \geq 2$, can be written in the form

$$f = s\varphi + h = Pf + Qf,$$

where $s \in \mathbf{R}$, $h \in L^p(\Omega) \cap R(A)$.

In what follows G will be the Němyckij's operator generated by $g = g(x, u)$, i.e.

$$G(u)(x) = g(x, u(x)).$$

Due to (1.2) G is a continuous operator from $L^p(\Omega)$ into itself, $p > N$.

Due to our notation the boundary value problem can be written in the equivalent form

$$Au + G(u) = f. \tag{2.3}$$

3. Main result.

Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory's function satisfying the growth condition (1.2). Then we can assume, without loss of generality, that for the functions $\Gamma_\pm$ defined by

$$\limsup_{u \to +\infty} \frac{g(x,u)}{u} = \Gamma_+(x), \qquad \limsup_{u \to -\infty} \frac{g(x,u)}{u} = \Gamma_-(x), \tag{3.1}$$

for a.e. $x \in \Omega$, we have $\Gamma_\pm \in L^p(\Omega)$, $p > N$.

Let us suppose that g satisfies the following sign condition

$$g(x, u)u \geq 0 \tag{3.2}$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$.

Theorem 3.1. *Let us suppose that $\Gamma_- \in L^p(\Omega)$, $p > N$, is the function defined in (3.1) and let $d = d(\Gamma_-)$ be the constant associated with Γ_- by Lemma 2.1. Suppose that the function Γ_+ from (3.1) is such that*

$$0 \le \Gamma_+(x) \le d$$

for a.e. $x \in \Omega$. Moreover, assume the validity of (3.2). Then the boundary value problem (1.1) has at least one solution $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, $p > N$, for any $f \in L^p(\Omega)$ satisfying the orthogonality condition

$$\int_\Omega f(x)\varphi(x)\,dx = 0. \tag{3.3}$$

The proof of Theorem 3.1 can be found in Iannacci, Nkashama and Ward [7].

In addition to (3.2) we shall assume

(g) let at least one of the following conditions be fulfilled:

(i) there are open sets of positive measure $\Omega_\pm \subset \Omega$, $\partial\Omega_\pm \cap \partial\Omega \neq \emptyset$ and real numbers $u_+ > 0, u_- < 0$ such that $g(x, u_+) > 0$, for a.e. $x \in \Omega_+$, $g(x, u_-) < 0$, for a.e. $x \in \Omega_-$;

(ii) there are real numbers $u_+ > 0, u_- < 0$ such that $g(x, u) > 0$ for a.e. $x \in \Omega_+(u)$ and all $u \ge u_+$, $g(x, v) < 0$ for a.e. $x \in \Omega_-(v)$ and all $v \le u_-$, respectively. Here $\Omega_+(u), \Omega_-(v)$ are subsets of Ω of positive measure.

Note that in the case (ii) it is possible $\partial\Omega_+(u) \cap \partial\Omega = \emptyset, \partial\Omega_-(v) \cap \partial\Omega = \emptyset$.

If $g = g(u)$ does not depend on $x \in \Omega$, the previous condition (g) has this more simple form:

($\widetilde{g}$) there are $u_1 > 0$ and $u_2 < 0$ such that $g(u_1) > 0$ and $g(u_2) < 0$.

Theorem 3.2. *In addition to the hypothesis of Theorem 3.1 suppose that (g) is fulfilled. Then for any fixed $h \in R(A) \cap L^p(\Omega)$, $p > n$, there exist $T_1 = T_1(h) < 0 < T_2(h) = T_2$ (where possibly $T_1 = -\infty$ or $T_2 = +\infty$) such that the boundary value problem*

$$\begin{aligned} Lu + \lambda_1 u + g(x, u) &= s\varphi + h \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.4}$$

has at least one solution $u \in W^{2,p}(\Omega) \cap H_0^1\Omega$ provided that

$$s \in (T_1, T_2).$$

Remark 3.1. If $g(x, u) \equiv 0$, the Fredholm alternative implies that the problem

$$\begin{aligned} Lu + \lambda_1 u &= f \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has a solution for f satisfying (3.3). Theorem 3.2 asserts that if nonlinearity g is in some sense "nontrivial" (see condition (g)) then right hand sides f satisfying the orthogonality condition (3.3) form a proper subset of the set of all right hand sides for which (1.1) has a solution. Moreover, the orthogonal decomposition of f gives more precise information about the structure of the range of the operator defined by the left hand side of (3.4) (see the definition of $T_1 = T_1(h)$, $T_2 = T_2(h)$ in section 4).

Remark 3.2. The proof of Theorem 3.2 uses essentially the assertion of Lemma 2.1. Taking into the account the Remark 2.1 then also "dual version" of Theorems 3.1 and 3.2 hold: the function $\Gamma_+ \in L^p(\Omega)$, $p > N$ given in (3.1) may be arbitrary and Γ_- (given also in (3.1)) must be such that

$$o \leq \Gamma_-(x) \leq d$$

for a.e. $x \in \Omega$, where $d = d(\Gamma_+)$ is the constant associated to Γ_+ by a "dual version" of Lemma 2.1.

Remark 3.3. Theorem 3.2 completes Theorems 1 and 2 in Iannacci, Nkashama and Ward [7]. Our result is also a generalization of the result of de Figueiredo, Ni[5], Gupta [6] and Drábek [3].

4. Proof of the main result.

Let $f = s\varphi + h$, $s \in \mathbf{R}, h \in R(A) \cap L^p(\Omega), p > N$, be arbitrary but fixed. We shall suppose that g fulfills both (3.2) and (g). Assume, at first, that $g = g(x, u)$ is bounded in the following sense: there exists $b \in L^p(\Omega)$ such that

$$|g(x,u)| \leq b(x)$$

for a.e. $x \in \Omega$ and for all $u \in \mathbf{R}$.

Step 1. (Liapunov – Schmidt reduction). Using the usual decomposition of (2.3) we obtain an equivalent bifurcation system

$$v + KQG(t\varphi + v) - KQf = 0, \tag{4.1}$$

$$PG(t\varphi + v) - Pf \quad = 0, \tag{4.2}$$

$u = t\varphi + v$, $v \in R(A)$, $t \in \mathbf{R}$.

Step 2. (solvability of (4.1)). Let $v \in R(A) \cap L^p(\Omega)$ be an eventual solution of (4.1) for arbitrary but fixed $t \in \mathbf{R}$. It follows that $v \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$, $p > N$, and moreover

$$\|v\|_{W^{2,p}} \leq c_p\left[\|b\|_{L^p} + \|h\|_{L^p}\right], \; p > N \tag{4.3}$$

(for c_p see Remark 2.3). Applying the Schauder fixed point theorem and using (4.3) we can prove that for any fixed $t \in \mathbf{R}$ there is at least one $v \in R(A)$ satisfying (4.1).

Step 3. (solvability of (4.2)). The Sobolev imbedding theorem and (4.3) yield that $v \in C^{1,\mu}(\overline{\Omega})$ and

$$\|v\|_{C^{1,\mu}} \leq \text{const}. \tag{4.4}$$

for any solution of (4.1) with the constant independent on $t \in \mathbf{R}$. Set

$$S = \{(t,v) \in \mathbf{R} \times [R(A) \cap L^p(\Omega)];\; v + KQG(t\varphi + v) = KQf\}$$

and define a real function $\psi : S \to \mathbf{R}$ by

$$\psi(t,v) = \int_\Omega g(x, t\varphi(x) + v(x))\varphi(x)\,dx,$$

$(t,v) \in S$. Then the solution of (2.3) is exactly $u = t\varphi + v$ such that $(t,v) \in S$ and $\psi(t,v) = \int_\Omega f(x)\varphi(x)\,dx = s$.

From (3.2), (g) and (4.4) follows that there exists $t_1 > 0$ such that

$$\psi(t_1, v) > 0 \quad \text{and} \quad \psi(-t_1, w) < 0, \tag{4.5}$$

for all $(t_1, v) \in S$ and $(-t_1, w) \in S$. according to Lemma 1.2 from Amann, Ambrosetti, Mancini [1] there exists a connected subset $S_{t_1} \subset S$ such that $[-t_1, t_1] \subset \operatorname{proj}_R S_{t_1}$. Since the function $\psi = \psi(t,v)$ is continuous on connected set S_{t_1}, there are due to (4.5) at least one $t \in (-t_1, t_1)$ and $v \in R(A)$ such that $(t,v) \in S$ and

$$\psi(t,v) = 0,$$

i.e. $u = t\varphi + v$ is a solution of (1.1) with the right hand side f satisfying the orthogonality condition (3.3). For fixed $h \in R(A)$ set

$$T_1 = \inf_{t_1} \inf_{(t,v)\in S_{t_1}} \psi(t,v),\ T_2 = \sup_{t_1} \sup_{(t,w)\in S_{t_1}} \psi(t,w),$$

where the first "inf" and "sup" are taken over all t_1 satisfying (4.5). Note that $T_1 < 0 < T_2$. Then for any $s \in (T_1, T_2)$ we can find $t \in \mathbf{R}$ and $v \in R(A)$ such that $(t,v) \in S$ and

$$\psi(t,v) = s,$$

i.e. $u = t\varphi + v$ is a solution of (1.1) with the right hand side $f = s\varphi + h$. This completes the proof of Theorem 3.2 for a bounded g.

Further, let us suppose, that g is not bounded in the sense mentioned above. For fixed $n \in \mathbf{N}$ we shall define a new function g_n in the following way

$$g_n(x,u) = \begin{cases} g(x,u)\ , & x \in \Omega, \quad |u| < n, \\ g(x,n), & x \in \Omega, \quad u \geq n, \\ g(x,-n), & x \in \Omega, \quad u \leq -n. \end{cases}$$

Then, with respect to (1.2), for any $n \in \mathbf{N}$ there exists $b_n \in L^p(\Omega)$, $p > N$, such that

$$|g_n(x,u)| \leq b_n(x)$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$, i.e. each g_n is bounded.

Step 4. (an apriori estimate). Let us suppose that $f \in L^p(\Omega)$ satisfies (3.3). We shall prove that there exists $n_0 \in \mathbb{N}$ such that $\|u\|_{C^1} < n_0$ for any solution of

$$\text{(4.6)} \qquad \begin{aligned} Lu + \lambda_1 u + g_{n_0}(x, u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Suppose the contrary, i.e. there is a sequence of $u_n \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ with $\|u_n\|_{C^1} \geq n$ such that

$$\text{(4.7)} \qquad Lu_n + \lambda_1 u_n + g_n(x, u_n) = f \qquad \text{in } \Omega.$$

Setting $v_n = u_n/\|u_n\|_{C^1}$, we have from (4.7)

$$\text{(4.8)} \qquad \begin{aligned} Lv_n &= \frac{f}{\|u_n\|_{C^1}} - \frac{g_n(x, u_n)}{\|u_n\|_{C^1}} - \lambda_1 v_n \qquad && \text{in } \Omega, \\ v_n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

From the growth condition (1.2) it follows that $\frac{g_n(x,u_n)}{\|u_n\|_{C^1}}$ is bounded in $L^p(\Omega)$. Hence the right side of (4.8) is bounded in $L^p(\Omega)$. Using a standard L^p–estimate and the compact imbedding of $W^{2,p}(\Omega)$ into $C^1(\overline{\Omega})$ (for $p > N$), we deduce from (4.8) that there exists $v \in C^1(\overline{\Omega})$ such that

$$\text{(4.9)} \qquad \begin{aligned} v_n &\to v \qquad \text{in } C^1(\overline{\Omega}), \quad \|v\|_{C^1} = 1, \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

(we pass to a subsequence if necessary).

Since $\|Lv_n\|_{L^p} \leq \text{const}$, $L^p(\Omega)$ is reflexive Banach space and L is weakly closed operator, we get that $v \in W^{2,p}(\Omega) \cap H_0^1(\Omega), Lv_n \rightharpoonup Lv$ in $L^p(\Omega)$. Hence we can pass to the limit in (4.8) and obtain that v solves the problem

$$\text{(4.10)} \qquad \begin{aligned} Lv &= -P(x) - \lambda_1 v \qquad && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The function $P \in L^p(\Omega)$ is the weak limit in $L^p(\Omega)$ of the sequence

$$\left\{ \frac{g_n(x, u_n)}{\|u_n\|_{C^1}} \right\}_{n=1}^{\infty}.$$

Let us define function $p = p(x)$ by $p(x) = \frac{P(x)}{v(x)}$ if $v(x) \neq 0$, $p(x) = 0$ if $v(x) = 0$ and set

$$\begin{aligned} p_+(x) &= p(x) \qquad \text{for } x \in \{x \in \Omega;\ v(x) > 0\}, \\ p_-(x) &= p(x) \qquad \text{for } x \in \{x \in \Omega;\ v(x) < 0\}. \end{aligned}$$

Then clearly

$$0 \le p_+(x) \le \Gamma_+(x),$$
$$0 \le p_-(x) \le \Gamma_-(x),$$

for a.e. $x \in \Omega$ (see (3.1)) and the equation (4.10) can be written in an equivalent form

$$\begin{aligned} Lv + \lambda_1 v + p_+(x)v^+ - p_-(x)v^- &= 0 \qquad \text{in } \Omega, \\ v &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

It follows from Lemma 2.1 that either

$$\begin{aligned} &v > 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} < 0 \qquad \text{on } \partial\Omega, \quad \text{or} \\ &v < 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial n} > 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Let us assume that $v > 0$ (the case $v < 0$ can be treated similarly). Since by (4.9) $v_n \to v$ in $C^1(\overline{\Omega})$ with $v > 0$ in Ω and $\frac{\partial v}{\partial n} < 0$ on $\partial\Omega$, we have $u_n(x) \to \infty$ uniformly on each compact subset of Ω and

$$u_n(x) > 0$$

for all $x \in \Omega$ and n sufficiently large. Multiplying the equation (4.7) by the eigenfunction φ and integrating over Ω, we get

$$\int_\Omega g_n(x, u_n(x)) \quad \varphi(x)\, dx = 0.$$

But our hypotheses (3.2) and (g) imply

$$\int_\Omega g_n(x, u_n(x)) \quad \varphi(x)\, dx > 0,$$

for n large enough, which is a contradiction.

The apriori estimate just proved yields that any solution of the problem (4.6) is simultaneously the solution of (1.1).

Step 5. Take $f \in L^p(\Omega)$, $p > N$, satisfying (3.3). Define g_{n_0} with n_0 so large that g_{n_0} satisfies (g) and any solution of (4.6) satisfies the apriori estimate $\|u\|_{C^1} < n_0$. Since g_{n_0} is bounded, (4.6) has at least one solution by Step 3. It is the solution of (1.1) too (see Step 4).

Step 6. (proof of Theorem 3.2). Let $h \in R(A) \cap L^p(\Omega), p > N$, be fixed. Let us consider the boundary value problem

$$(4.11) \qquad \begin{aligned} Lu + \lambda_1 u + g_{n_0}(x,u) &= s\varphi + h \qquad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where g_{n_0} was defined in Step 5 for $f \equiv h$. Then any solution u_0 of (4.11) with $s = 0$ satisfies

$$(4.12) \qquad \|u_0\|_{C^1} < n_0.$$

It can be also written in the form $u_0 = t_0\varphi + v_0$, where

$$\psi_{n_0}(t_0, v_0) = \int_\Omega g_{n_0}(x, t_0\varphi(x) + v_0(x))\varphi(x)\,dx = 0,$$

$(t_0, v_0) \in S^{n_0}$ (see Step 3). Moreover, there exists $t_1 > 0$ (which only depends on h and not on s) and a connected set $S^{n_0}_{t_1} \subset S^{n_0}$ such that $[-t_1, t_1] \subset \operatorname{proj}_R S^{n_0}_{t_1}$,

$$(4.13) \qquad \psi_{n_0}(t_1, v) > 0, \quad \psi_{n_0}(-t_1, w) < 0$$

for any $(t_1, v) \in S^{n_0}_{t_1}$, $(-t_1, w) \in S^{n_0}_{t_1}$. Since ψ_{n_0} is continuous on $S^{n_0}_{t_1}$, its Darboux property together with (4.12) and (4.13) imply that for any $s \in (T_1(h), T_2(h))$ with $T_1(h) < 0 < T_2(h)$, $|T_1(h)|, T_2(h)$ sufficiently small, there exists at least one solution u of (4.11) such that

$$\|u\|_{C^1} < n_0.$$

This completes the proof of Theorem 3.2.

Remark 4.1. Let

$$g^{-\infty}(x) = \limsup_{u \to -\infty} g(x,u) \quad \text{and} \quad g_{+\infty}(x) = \liminf_{u \to +\infty} g(x,u)$$

be well defined functions bounded from above and from below respectively, and instead of (3.2) assume that

$$\int_\Omega g^{-\infty}(x)\varphi(x)\,dx < \int_\Omega g_{+\infty}(x)\varphi(x)\,dx.$$

Then the boundary value problem (1.1) has at least one solution $u \in W^{2,p}(\Omega) \cap H^1_0(\Omega)$ for any $f \in L^p(\Omega)$, $p > N$, satisfying

$$(4.14) \qquad \int_\Omega g^{-\infty}(x)\varphi(x)\,dx < \int_\Omega f(x)\varphi(x)\,dx < \int_\Omega g_{+\infty}(x)\varphi(x)\,dx.$$

Let us give a sketch of the proof. We can make an apriori estimate similarly to the Step 4 for any right hand side f satisfying condition (4.14). Then using a truncation of g outside of a sufficiently large interval we prove the solvability of (1.1). We proceed by the same way as in Steps 1–3.

Remark 4.2. The same result as mentioned in the previous remark can be proved using the degree–theoretical approach (see e.g. [7] and [4]).

Remark 4.3. The same result as our Theorems 3.1 and 3.2 holds also for Neumann boundary value problem

$$\begin{aligned} Lu + g(x,u) &= f && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

To prove it we have to use the corresponding modification of Lemma 2.1 and a "stronger version" of condition (g):

(g_N) there are $\Omega_\pm(u) \subset \Omega, \operatorname{meas}\Omega_\pm(u) > 0$ such that $g(x,u) > 0$ for all u large enough and a.e. $x \in \Omega_+(u)$, $g(x,u) < 0$ for all $-u$ large enough and a.e. $x \in \Omega_-(u)$.

If $g = g(u)$ is independent on $x \in \Omega$, then the condition (g_N) is of the form:

($\tilde{g}_N$) $g(u) > 0$ for u large enough and $g(u) < 0$ for $-u$ large enough.

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Technical University of Plzeň, Nejedlého sady 14, 306 14 Plzeň, Czechoslovakia

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