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Landesman–Lazer conditions for strongly nonlinear boundary value problems

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Dedicated to the memory of Svatopluk Fučík

Abstract. The solvability of the equations of the type

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{p-2}u + f(x,u) = g$$

with Dirichlet or Neumann boundary conditions is proved for right–hand sides satisfying Landesman–Lazer type conditions. Here λ_1 is the smallest eigenvalue of the corresponding boundary value problem for the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{p-2}u = 0,$$

$p > 1$, the growth of f is not greater then $|u|^{p-1}$. Further, the solvability of the boundary value problem for the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda_1|u|^{p-2}u - |u|^{q-2}u + f(x,u) = g$$

for any right–hand side is proved under the assumption $q > p$. In both cases, a generalization to the equation with $(p-1)$–quasihomogeneous term with the second derivative is given.

Keywords: Solvability of strongly nonlinear boundary value problems, p–Laplacian, Landesman–Lazer condition, degree of the mapping

Classification: 35J65, 35J25, 35D05

Introduction

We shall consider the equation

$$\operatorname{div}\left(|\nabla u(x)|^{p-2}\nabla u(x)\right) + \lambda_1|u(x)|^{p-2}u(x) + f(x,u(x)) = g(x) \quad \text{in} \quad \Omega, \tag{0.1}$$

with the boundary condition

$$|\nabla u|^{p-2}\nabla u\cdot\vec{n} = 0 \qquad \text{on } \partial\Omega \tag{0.2}$$

or

$$u = 0 \qquad \text{on } \partial\Omega, \tag{0.3}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$, $\nabla u = \text{grad } u$, $p > 1$, $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory's function, λ_1 is the smallest eigenvalue of the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0 \tag{0.4}$$

with the boundary conditions (0.2) or (0.3), respectively, $\vec{n}$ is the outer normal. Under certain general assumptions laid on f (see (1.1), (1.2) and (2.1) or (2.5)) we shall show that our problem is solvable for right-hand sides $g \in L_{p'}(\Omega)$ satisfying conditions of the Landesman–Lazer type (see Theorems 2.1, 2.3, 2.5). We give also a generalization to the case when the main term is only asymptotically close to $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (see Theorem 2.2, 2.4, 2.6). The term $\lambda_1 + \dfrac{f(x,u)}{|u|^{p-2}u}$ can meet the eigenvalue λ_1. That means we can speak about the problem at resonance. Note that also unbounded nonlinearities f satisfy our assumptions (see Remark 2.1). These results represent a modification of those received in a semilinear case ($p = 2$) by Landesman, Lazer [7], Ahmad [2], Drábek [5] and others. They are also related to our previous paper [4] where the corresponding nonresonance case was considered. Moreover, we shall consider the equations of the type

$$\begin{aligned} \operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + \lambda_1|u(x)|^{p-2}u(x) - |u(x)|^{q-2}u(x) + \\ + f(x, u(x)) = g(x) \end{aligned} \tag{0.5}$$

with $q > p$ and f having the growth not stronger then the $(p-1)$-th power. The existence of a solution for any right-hand side will be proved (see Theorems 2.7, 2.8).

Note that the solutions of our problems are considered in the weak sense. The regularity of solutions of the equation of type considered is discussed e.g. in Tolksdorf [11].

The equations with the principal part $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ arrise in the theory of quasiregular and quasiconformal mappings or in physics (see e.g. [9], [10], [11]).

1. Notation, general remarks

Throughout the whole paper, we suppose that Ω is a bounded domain in $\mathbf{R}^n$ with its boundary $\partial\Omega$ of the class $C^{2,\alpha}$ (with some $\alpha \in (0,1)$) and that $1 < p < +\infty$. We denote $X = W_p^1(\Omega)$ or $X = \overset{\circ}{W}{}_p^1(\Omega)$ if the problem (0.1), (0.2) or (0.1), (0.3) is considered, respectively. $W_p^1(\Omega)$ and $\overset{\circ}{W}{}_p^1(\Omega)$ is the usual Sobolev space and its subspace of functions having zero traces on $\partial\Omega$, with the norm

$$\|u\| = (\|u\|_{1,p}^p + \|u\|_p^p)^{1/p}$$

where $\|u\|_{1,p} = (\int_\Omega |\nabla u|^p \, dx)^{1/p}$, $\|u\|_p = (\int_\Omega |u|^p \, dx)^{1/p}$. In the case of the equation (0.5), the space $X = W_p^1(\Omega) \cap L_q(\Omega)$ or $X = \overset{\circ}{W}{}_p^1(\Omega) \cap L_q(\Omega)$ with the norm

$\|u\| = (\|u\|_{1,p}^p + \|u\|_q^p)^{1/p}$ will be considered. Further, we denote by X^*, $\|\cdot\|_*$ and $(\cdot,\cdot)$ the dual space to X, the norm in X^* and the pairing between X and X^*. The symbols $B_R(0)$ and $\deg[T, B_R(0), 0]$ are used for the ball in X with the radius R centered at the origin and for the Leray–Schauder degree of the mapping $T : X \to X$ at 0 with respect to $B_R(0)$. (For the basic properties of this degree see e.g. [6]). The strong and the weak convergence will be denoted by $\to$ and $\rightharpoonup$, respectively.

It will be always supposed that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a function satisfying the Caratheodory condition

$$(1.1) \qquad \begin{array}{ll} f(x,\cdot) & \text{is continuous on } \mathbf{R} \text{ for a.a. } x \in \Omega, \\ f(\cdot,s) & \text{is measurable for all } s \in \mathbf{R} \end{array}$$

and the growth condition

$$(1.2) \qquad |f(x,s)| \le m(x) + c|s|^{p-1} \quad \text{with some } m \in L_{p'}(\Omega), \quad c \ge 0,$$

where $p'^{-1} + p^{-1} = 1$.

Let us introduce the operators $J, S, F : X \to X^*$ and an element $g^* \in X^*$ associated with a given $g \in L_{p'}(\Omega)$ by

$$(1.3) \qquad (J(u), v) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx$$

$$(1.4) \qquad (S(u), v) = \int_\Omega |u|^{p-2} uv \, dx$$

$$(1.5) \qquad (F(u), v) = \int_\Omega f(x, u(x)) v(x) \, dx$$

$$(1.6) \qquad (g^*, v) = \int_\Omega g(x) v(x) \, dx$$

for all $u, v \in X$. Further, denote $J_d = J + dS$ for any $d \in \mathbf{R}$.

A function $u \in X$ is said to be a weak solution of (0.1), (0.2) or (0.1), (0.3) if $X = W_p^1(\Omega)$ or $X = \overset{\circ}{W}{}_p^1(\Omega)$, respectively, and

$$(1.7) \qquad J(u) - \lambda_1 S(u) - F(u) + g^* = 0.$$

Remark 1.1. The operators J, S are $(p-1)$–homogeneous, i.e. $J(tu) = t^{p-1} J(u)$ for any $t > 0, u \in X$ (and analogously for S). The operators S, F are compact with respect to the compact imbedding $W_p^1(\Omega) \subset L_p(\Omega)$. Further, J_d for any $d > 0$ is a homeomorphism of X onto X^*. Let us prove it. It follows from (1.3), (1.4) and Hölder inequality that

$$(1.8) \qquad (J_d(u), u) \ge C\|u\|^p \quad \text{for all } u \in X (\text{with some } C > 0),$$

$$(1.9) \qquad \begin{aligned} (J_d(u) - J_d(v), u - v \ge (\|u\|_{1,p}^{p-1} - \|v\|_{1,p}^{p-1})(\|u\|_{1,p} - \|v\|_{1,p}) + \\ + d(\|u\|_p^{p-1} - \|v\|_p^{p-1})(\|u\|_p - \|v\|_p) > 0 \quad \text{for all } u, v \in X, u \ne v. \end{aligned}$$

Hence, J_d is monotone and coercive and it follows from the theory of monotone operators that J_d maps X onto X^* (see e.g. [8]). The last inequality ensures also the existence of J_d^{-1}. Suppose that J_d^{-1} is not continuous, i.e. there are $u_n, u_0 \in X$ such that $J_d(u_n) \to J_d(u_0), \|u_n - u_0\| \geq \delta > 0$. Then $\{u_n\}$ is bounded by (1.8) and we can suppose $u_n \rightharpoonup \widetilde{u}$ in X for some $\widetilde{u} \in X$. Hence,

$$(J_d(u_n) - J_d(\widetilde{u}), u_n - \widetilde{u}) =$$
$$= (J_d(u_n) - J_d(u_0), u_n - \widetilde{u}) + (J_d(u_0) - J_d(\widetilde{u}), u_n - \widetilde{u}) \to 0.$$

It follows from (1.9) that $\|u_n\| \to \|\widetilde{u}\|$ and therefore $u_n \to \widetilde{u}$. Further, $J_d(u_n) \to J_d(\widetilde{u}) = J_d(u_0)$. That means $\widetilde{u} = u_0$ which contradicts the assumption.

Remark 1.2. There exists the smallest eigenvalue λ_1 of the problem

$$J(u) - \lambda S(u) = 0, \tag{1.10}$$

i.e. the smallest real λ such that (1.10) has a nontrivial solution. This eigenvalue is isolated and simple (i.e. there is $\varphi \in X$ such that (1.10) with $\lambda = \lambda_1$ holds if and only if $u = \xi\varphi, \xi \in \mathsf{R}$). Moreover, $\varphi \in C^{1,\beta}(\overline{\Omega})$ with $\beta \in (0,1)$ and φ does not change its sign in Ω, i.e. we can suppose $\varphi > 0$ on Ω. It follows that λ_1 is simultaneously the smallest eigenvalue of (0.4), (0.2) or (0.4), (0.3) if $W_p^1(\Omega)$ or $X = \overset{\circ}{W}{}_p^1(\Omega)$, respectively. Further,

$$\lambda_1 = \min_{\substack{u \in X \\ u \neq 0}} \frac{(J(u), u)}{(S(u), u)}$$

and this minimum is attained only in the points $\xi\varphi, \xi \in \mathsf{R}, \xi \neq 0$. Particularly,

$$(J(u), u) - \lambda_1(S(u), u) \geq 0 \quad \text{for all } u \in X,$$

and the equality holds only for $u = \xi\varphi$, $\xi \in \mathsf{R}$. All these assertions for the case of the boundary conditions (0.3) can by found in [3]. It is easy to see that they are true also for the boundary conditions (0.2) because then clearly $\lambda_1 = 0$, $\varphi \equiv 1$. Let us show that λ_1 is isolated in this case. This is, perhaps, the only property which is not clear at the first sight in the case considered. Suppose by contradiction that there are eigenvalues λ_n of (1.10) with the corresponding normed eigenfunctions $u_n, \lambda_n \neq 0, \lambda_n \to 0$. We can suppose $u_n \rightharpoonup u$ in X. Then (1.10) (which can be written as $J_d(u_n) - (\lambda_n + d)S(u_n) = 0$), the compactness of S and the fact that J_d is a homeomorphism (see Remark 1.1) yield $u_n \to u$, $J(u) = 0$. That means either $u \equiv \varphi$ or $u \equiv -\varphi, \varphi \equiv 1$. Simultaneously, it should be

$$0 = (J(u_n), \varphi) - \lambda_n(S(u_n), \varphi) = -\lambda_n \int_\Omega |u_n|^{p-2} u_n \, dx$$

because $(J(w), \varphi) = 0$ for $\varphi \equiv 1$ and any $w \in X$. This is the contradiction.

2. Main results

A. Landesman–Lazer conditions: nonlinearities of a "decreasing type"

Let us denote

$$f_{-\infty}(x) = \lim_{s\to-\infty} \inf f(x,s),\ f^{+\infty}(x) = \lim_{s\to+\infty} \sup f(x,s)$$

and assume that

$$(2.1)\qquad \begin{cases} \text{there exist } r>0 \text{ and functions } h_{-\infty}, h^{+\infty} \in L_{p'}(\Omega) \text{ such that} \\ f(x,s) \ge h_{-\infty}(x) \text{ for } s<-r, \text{ a.a. } x\in\Omega, \\ f(x,s) \le h^{+\infty}(x) \text{ for } s>r, \text{ a.a. } x\in\Omega. \end{cases}$$

Theorem 2.1. *Suppose (1.1), (1.2) and (2.1). Then the problem (0.1), (0.2) and (0.1), (0.3) has at least one weak solution for any $g \in L_{p'}(\Omega)$ satisfying the condition*

$$(2.2)\qquad \int_\Omega f^{+\infty}(x)\varphi(x)\,dx < \int_\Omega g(x)\varphi(x)\,dx < \int_\Omega f_{-\infty}(x)\varphi(x)\,dx$$

where φ is the positive eigenfunction corresponding to the smallest eigenvalue of the problem (0.4), (0.2) and (0.4), (0.3), respectively.

Definition 2.1. Let $A_0 : X \to X^*$ be an a–homogeneous operator, i.e. $A_0(tu) = t^a A_0(u)$ for any $t>0, u\in X$. Then $A : X \to X^*$ is said to be a–quasihomogeneous with respect to A_0 if

$$t_n \to 0, t_n > 0, u_n \rightharpoonup u, t_n^a A(\frac{u_n}{t_n}) \to g^* \text{ in } X^* \Rightarrow g^* = A_0(u).$$

Further, we shall replace J by an operator A which is $(p-1)$–quasihomogeneous with respect to J, i.e. we shall consider the equation

$$(2.3)\qquad A(u) - \lambda_1 S(u) - F(u) + g^* = 0.$$

Of course, we could also replace S by an operator which is only asymptotically homogeneous in some sense, but this nonhomogeneity can be contained in F.

Theorem 2.2. *Suppose (1.1), (1.2) and (2.1). Let $A : X \to X^*$ be an odd mapping which is $(p-1)$–quasihomogeneous with respect to J, such that $A_d = A + dS$ is a homeomorphism for any $d>0$ and $(A(u),u) \ge (J(u),u)$ for all $u\in X$. Then (2.3) has at least one solution for any $g^* \in X^*$ defined by (1.6) with $g\in L_{p'}(\Omega)$ satisfying (2.2).*

Example 2.1. Consider the boundary value problem

$$\operatorname{div}[(a(x) + |\nabla u(x)|^{p-2})\nabla u(x)] + f(x,u(x)) = g(x) \quad \text{in } \Omega,$$
$$(a + |\nabla u|^{p-2})\nabla u \cdot \vec{n} = 0 \quad \text{on } \partial\Omega,$$

where a is a smooth function on Ω, $0 \le a(x) \le C$. The weak solution of this problem is a solution of (2.3) with A defined by

$$(A(u), v) = \int_\Omega (a + |\nabla u|^{p-2})\nabla u \nabla v \, dx \quad \text{for all } u, v \in X = W^1_p(\Omega).$$

It is easy to see that for $p \ge 2$, the operator A is $(p-1)$–quasihomogeneous with respect to J. Further, A_d is a homeomorphism of X onto X^*. This can be proved by the same considerations as in Remark 1.1 observing that $(A(u) - A(v), u - v) \ge (J(u) - J(v), u - v)$. Hence, if f fulfils (1.1), (1.2) and (2.1) then there is a weak solution for any $g \in L_{p'}(\Omega)$ satisfying (2.2) with $\varphi \equiv 1$ (see Remark 1.2). Analogously we can consider the boundary conditions (0.3). (In that case $X = \overset{\circ}{W}{}^1_p(\Omega)$, φ is not constant and $\lambda_1 > 0$, that means the term $\lambda_1 |u|^{p-2}u$ must be added.)

Remark 2.1. Note that we can deal also with nonlinearities f for which $f^{+\infty}(x) \equiv -\infty$ or $f_{-\infty}(x) \equiv +\infty$. Consider, for instance, the equation

$$\text{(2.4)} \qquad \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda_1 |u|^{p-2}u - |u|^{q-2}u = g \quad \text{in } \Omega$$

with the Dirichlet boundary conditions (0.3), $1 < q \le p$. Then $f(x, u) = -|u|^{q-2}u$ fulfils (1.1), (1.2) and (2.1). Moreover, $f^{+\infty}(x) \equiv -\infty$, $f_{-\infty} \equiv +\infty$. Hence, the problem (2.4), (0.3) has at least one weak solution for any $g \in L_{p'}(\Omega)$. If the nonlinearity in (0.1) has the form

$$f(x, u) = \begin{cases} -|u|^{q-2}u & \text{for } x \in \Omega, u \ge 0, \\ 0 & \text{for } x \in \Omega, u < 0, \end{cases}$$

$1 < q \le p$, then (0.1), (0.3) has at least one weak solution for any $g \in L_{p'}(\Omega)$ satisfying

$$\int_\Omega g(x)\varphi(x) \, dx < 0.$$

B. Landesman–Lazer conditions: nonlinearities of an "increasing type" for Neumann boundary conditions

Further, denote

$$f^{-\infty}(x) = \lim_{s \to -\infty} \sup f(x, s), \; f_{+\infty}(x) = \lim_{s \to +\infty} \inf f(x, s)$$

and suppose that

$$\text{(2.5)} \qquad \begin{cases} \text{there exist } r > 0 \text{ and functions } h^{-\infty}, h_{+\infty} \in L_{p'}(\Omega) \text{ such that} \\ f(x, s) \le h^{-\infty}(x) \text{ for } s < -r, \text{ a.a. } x \in \Omega, \\ f(x, s) \ge h_{+\infty}(x) \text{ for } s > r, \text{ a.a. } x \in \Omega. \end{cases}$$

Moreover, assume that

$$\text{(2.6)} \qquad \lim_{|s| \to \infty} \frac{f(x, s)}{|s|^{p-1}} = 0 \quad \text{for a.a. } x \in \Omega.$$

Now, we can formulate in a certain sense dual version of Theorems 2.1, 2.2 for the case of Neumann boundary conditions.

Theorem 2.3. *Let us suppose (1.1), (1.2) and (2.5), (2.6), $p > n$. Then the problem (0.1), (0.2) has at least one weak solution for any $g \in L_{p'}(\Omega)$ satisfying the condition*

(2.2')
$$\int_\Omega f^{-\infty}(x)\,dx < \int_\Omega g(x)\,dx < \int_\Omega f_{+\infty}(x)\,dx$$

Theorem 2.4. *Suppose (1.1), (1.2), (2.5), (2.6), $p > n, X = W_p^1(\Omega)$. Let $A : X \to X^*$ be an odd mapping which is $(p-1)$-quasihomogeneous with respect to J, such that $A_d = A + dS$ is a homeomorphism for any $d > 0$ and $A(u), u) \geq (J(u), u), (A(u), \varphi) = 0$ for all $u \in X$. Then (2.3) has at least one solution for any $g^* \in X^*$ defined by (1.6) with $g \in L_{p'}(\Omega)$ satisfying (2.2').*

Note that $\varphi \equiv 1$ in the situation of Theorems 2.3, 2.4, 2.5, 2.6 (see Remark 1.2). Hence, the assumption $(Au, \varphi) = 0$ for all $u \in X$ is fulfilled for $A = J$ from (1.3) as well as for A from Example 2.1. Of course, this is not true in the case of Dirichlet boundary conditions when $\varphi \not\equiv 1$. Unfortunately, we cannot solve the case (2.2') without the assumption $(Au, \varphi) = 0$ and that is why we consider only Neumann boundary conditions in Theorems 2.3 - 2.6.

Let us remark that Theorem 2.3 is a consequence of Theorem 2.4 according to Remark 1.1.

The assumption $p > n$ in Theorems 2.3, 2.4 ensures the compact imbedding $W_p^1(\Omega) \subset C(\overline{\Omega})$. In the case $1 < p \leq n$ we need some additional assumptions on f. Precisely, suppose that

(2.7) $\quad |f(x,s)| \leq h(x)$ for all $s \in \mathbb{R}, x \in \Omega$ with some $h \in L_{p'}(\Omega)$.

Theorem 2.5. *Suppose (1.1) and (2.7). Then the problem (0.1), (0.2) has at least one weak solution for any $g \in L_{p'}(\Omega)$ satisfying (2.2').*

Theorem 2.6. *Suppose (1.1), (2.7), $X = W_p^1(\Omega)$. Let $A : X \to X^*$ be an odd mapping which is $(p-1)$-quasihomogeneous with respect to J such that $A_d = A+dS$ is a homeomorphism of X onto X^* for any $d > 0$ and $(A(u), u) \geq (J(u), u), (A(u), \varphi) = 0$ for all $u \in X$. Then (2.3) has at least one solution for any $g^* \in X^*$ defined by (1.6) with $g \in L_{p'}(\Omega)$ satisfying (2.2').*

Analogously as in the case of Theorems 2.1, 2.2 and 2.3, 2.4, Theorem 2.6 is a generalization of Theorem 2.5.

C. Equations with a higher order term: solvability for all right–hand sides

As we have mentioned in Remark 2.1, the equation (2.4) is included in our theory provided $1 < q \leq p$ (and in fact also if $1 < q < p^*$ - se Remark 2.3 below). In case of a general $q > p$, it is necessary to work in the spaces

(2.8)
$$X = W_p^1(\Omega) \cap L_q(\Omega) \quad \text{or } X = \overset{\circ}{W}{}_p^1(\Omega) \cap L_q(\Omega)$$

if (0.2) or (0.3) is considered, respectively. This is a locally uniformly convex Banach space with the norm

$$\|u\| = (\|u\|_{1,p}^p + \|u\|_q^p)^{1/p}. \tag{2.9}$$

We shall consider equations of the more general type (0.5) for which the term of the order $q (q > p)$ plays a substantial role.

The mappings $J, S, F : X \to X^*$ and an element $g^* \in X^*$ are again well-defined by (1.3), (1.4), (1.5), (1.6) (for all $u, v \in X$ from (2.8), $g \in L_{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$) under the assumptions (1.1), (1.2). Moreover, introduce the mappings $T, J_T : X \to X^*$ by

$$\begin{gathered}(T(u), v) = \int_\Omega |u|^{q-2} uv \, dx \quad \text{for all } u, v \in X, \\ J_T = J + T.\end{gathered} \tag{2.10}$$

The weak solution of (0.5), (0.2) or (0.5), (0.3) is defined as $u \in X$ satisfying

$$J(u) - \lambda_1 S(u) + T(u) - F(u) + g^* = 0. \tag{2.11}$$

Remark 2.2. The mapping J_T is a homeomorphism of X onto X^*. This can be shown analogously as for J_d in Remark 1.1 but by using the estimates

$$\begin{gathered}\frac{(J_T(u), u)}{\|u\|} = \frac{\|u\|_{1,p}^p + \|u\|_q^q}{(\|u\|_{1,p}^p + \|u\|_q^p)^{1/p}} \geq \frac{\max(\|u\|_{1,p}^p, \|u\|_q^q)}{[2\max(\|u\|_{1,p}^p, \|u\|_q^p)]^{1/p}} \to +\infty \\ (\text{for} \quad \|u\| \to +\infty), \\ (J_T(u) - J_T(v), u - v) \geq (\|u\|_{1,p}^{p-1} - \|v\|_{1,p}^{p-1})(\|u\|_{1,p} - \|v\|_{1,p}) + \\ + (\|u\|_q^{q-1} - \|v\|_q^{q-1})(\|u\|_q - \|v\|_q) > 0 \quad \text{for all } u, v \in X, u \neq v\end{gathered}$$

instead of (1.8), (1.9). Note that the mappings S, F are compact again under the assumptions (1.1), (1.2), but T is not compact for q so large that $W_p^1(\Omega)$ is not compactly imbedded into $L_q(\Omega)$.

Theorem 2.7. *Suppose (1.1), (1.2), $q > p$. Then each of the problems (0.5), (0.2) and (0.5), (0.3) has at least one weak solution for any $g \in L_{q'}(\Omega)$.*

Further, we shall replace the terms $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ and $|u|^{q-2} u$ by a quasihomogeneous operator and by function $f_q : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying (1.1) and

$$f_q(x, s) \leq m_q(x) + C|s|^{q-1} \quad \text{with some } m_q \in L_{q'}(\Omega), C \geq 0, \tag{1.2$_q$}$$

$$f_q(x, s) \cdot s \geq C_0 |s|^q \quad \text{with some } C_0 > 0. \tag{2.12}$$

More precisely, introduce the operator $B : X \to X^*$ by

$$(B(u), v) = \int_\Omega f_q(x, u(x)) v(x) \, dx \quad \text{for all } u, v \in X \tag{2.13}$$

and consider the equation

$$A(u) - \lambda_1 S(u) + B(u) - F(u) + g^* = 0 \tag{2.14}$$

Theorem 2.8. *Suppose (1.1), (1.2), $q > p$, (2.12), (1.2$_q$). Let $A : X \to X^*$ be $(p-1)$-quasihomogeneous with respect to J and such that $A_B = A + B$ is a homeomorphism of X onto X^* and $(A(u), u) \geq (J(u), u)$ for all $u \in X$. Then (2.14) has at least one solution for any g^* defined by (1.6) with $g \in L_{q'}(\Omega)$.*

Note that Theorem 2.7 is a special case of Theorem 2.8 according to Remark 2.2.

Example 2.2. Consider the boundary value problem

$$\begin{aligned}\operatorname{div}[(a(x) + |\nabla u(x)|^{p-2})\nabla u(x)] + \lambda_1 |u(x)|^{p-2}u(x) - \\ - (b(x) + |u(x)|^{q-2})u(x) + f(x, u(x)) = g(x) \quad \text{on } \Omega\end{aligned}$$

with the boundary conditions (0.3), where a, b are smooth functions on $\Omega, 0 \leq a(x) \leq C, 0 \leq b(x) \leq C, q > p$, f satisfies (1.1), (1.2). The weak solution of this problem is a solution of (2.14) with A, B defined by

$$(A(u), v) = \int_\Omega (a + |\nabla u|^{p-2})\nabla u \nabla v \, dx \quad \text{for all } u, v \in X,$$

$$(B(u), v) = \int_\Omega (b + |u|^{q-2})uv \, dx \quad \text{for all } u, v \in X,$$

$X = \overset{\circ}{W}{}^1_p(\Omega) \cap L_q(\Omega)$. Suppose that $p \geq 2$. Then it is easy to see that A is $(p-1)$–quasihomogeneous with respect to J. Further,

$$(A_B(u) - A_B(v), u - v) \geq (J_T(u) - J_T(v), u - v)$$

and therefore we can show by considerations analogous to those from Remark 1.1 that A_B is a homeomorphism of X onto X^* (cf. Example 1.1, Remark 2.2). Hence, our problem has for any $g \in L_{q'}(\Omega)$ at least one weak solution by Theorem 2.8. Analogous considerations can be made for the boundary conditions from Example 2.1. (Then $\lambda_1 = 0, X = W^1_p(\Omega) \cap L_q(\Omega)$.)

Remark 2.3. Let us note that the assumption (1.2) could be replaced by a slightly weaker growth restriction

$$|f(x, s)| \leq m(x) + C|s|^{\alpha - 1}$$

with an arbitrary fixed $\alpha < p^*, m \in L_{\alpha'}(\Omega)$, where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$. The space $W^1_p(\Omega)$ is compactly imbedded into $L_\alpha(\Omega)$ for such α and the proofs of our results can be easily modified using this fact.

3. Proof of main results

PROOF of Theorem 2.2: Let us choose a fixed $d > 0$ and define the mapping $H : \langle 0,1\rangle \times X \to X$ by

$$H(\tau, u) = u - A_d^{-1}((\lambda_1 + \tau d)S(u) + \tau(F(u) - g^*)).$$

The equation $H(1,u) = 0$ is equivalent to (2.3). Suppose that

$$H(\tau, u) \neq 0 \quad \text{for all } \tau \in \langle 0,1\rangle, u \in X, \|u\| = R \tag{3.1}$$

with some $R > 0$. The mapping $A_d^{-1}((\lambda_1 + \tau d)S + \tau(F - g^*))$ is compact (see Remark 1.1). The operator $H(0,u) = u - A_d^{-1}(\lambda_1 S(u))$ is odd. Hence,

$$\deg[H(0,\cdot), B_R(0), 0] \neq 0$$

by the Borsuk theorem. Further, (3.1) ensures

$$\deg[H(1,\cdot), B_R(0), 0] = \deg[H(0,\cdot), B_R(0), 0] \neq 0$$

by the homotopy invariance property of the degree. It follows that there exists a solution $u \in B_R(0)$ of $H(1,u) = 0$, i.e. of (2.3). (For the properties of the degree see e.g. [6].) Hence, it is sufficient to show that (3.1) holds for $R > 0$ large enough. Suppose by contradiction that there exist $\tau_n \in \langle 0,1\rangle, u_n \in X (n = 1,2,\dots)$ such that $\|u_n\| \to +\infty$ and $H(\tau_n, u_n) = 0$, i.e.

$$A(u_n) - \lambda_1 S(u_n) + (1-\tau_n)dS(u_n) - \tau_n(F(u_n) - g^*) = 0. \tag{3.2}$$

We can suppose that $\tau_n \to \tau \in \langle 0,1\rangle, v_n = \dfrac{u_n}{\|u_n\|} \rightharpoonup v$ in X, $v_n \to v$ in $L_p(\Omega)$ (according to the compactness of the imbedding $X \subset L_p(\Omega)$) and $v_n(x) \to v(x)$ a.e. on Ω. Dividing (3.2) by $\|u_n\|^{p-1}$ we receive

$$\|u_n\|^{1-p}A(u_n) - \lambda_1 S(v_n) + (1-\tau_n)dS(v_n) - \tau_n\|u_n\|^{1-p}(F(u_n) - g^*) = 0. \tag{3.3}$$

It follows from (1.2) that the sequence $\{\|u_n\|^{1-p} f(\cdot, u_n(\cdot))\}$ is bounded in $L_{p'}(\Omega)$, i.e. we can suppose that

$$\|u_n\|^{1-p} f(\cdot, u_n(\cdot)) \rightharpoonup \tilde{f} \text{ in } L_{p'}(\Omega) \tag{3.4}$$

for some $\tilde{f} \in L_{p'}(\Omega)$. The compactness of the imbedding $L_{p'}(\Omega) \subset X^*$ implies

$$\|u_n\|^{1-p}F(u_n) \to f^* \text{ in } X^*, (f^*, w) = \int_\Omega \tilde{f} w \, dx \text{ for all } w \in X. \tag{3.5}$$

This together with the compactness of S and the equation (3.3) ensures that $\{\|u_n\|^{1-p}A(u_n)\}$ is convergent, that means

$$\|u_n\|^{1-p}A(u_n) \to J(v) \quad \text{in } X^* \tag{3.6}$$

because A is $(p-1)$–quasihomogeneous with respect to J. Simultaneously $\|u_n\|^{1-p}$ $\|A_d(u_n)\| \geq C > 0$ according to (1.8) and the assumption $(A(u),u) \geq (J(u),u)$. Therefore

$$\|u_n\|^{1-p} \quad A_d(u_n) \to J_d(v) \neq 0$$

because of the compactness of S. That means

$$v \neq 0. \tag{3.7}$$

It follows easily from (1.2), (3.4) and the fact $v_n \to v$ in $L_p(\Omega)$ that $\widetilde{f} = 0$ a.e. on $M_0 = \{x \in \Omega; v(x) = 0\}$. Hence, we can write

$$\widetilde{f}(x) = \chi(x)|v(x)|^{p-2}v(x). \tag{3.8}$$

The function χ is uniquely defined a.e. on $\Omega \setminus M_0$ and we set $\chi \equiv 0$ on M_0. Further, the conditions (1.2), (2.1) yield

$$\chi(x) \leq 0 \quad \text{a.e. on } \Omega \tag{3.9}$$

(precisely see Remark 3.1 below). Define the mapping $\widetilde{S} : X \to X^*$ by

$$(\widetilde{S}(u), w) = \int_\Omega \chi |u|^{p-2} u w \, dx \quad \text{for all } u, w \in X. \tag{3.10}$$

Now, (3.3) together with (3.5), (3.6), (3.8), (3.10) imply

$$J(v) - \lambda_1 S(v) + (1-\tau) dS(v) - \tau \widetilde{S}(v) = 0.$$

Particularly

$$(J(v) - \lambda_1 S(v), v) = -(1-\tau) d(S(v), v) + \tau (\widetilde{S}(v), v). \tag{3.11}$$

The left–hand side is nonnegative and it can equal zero only for $v = \xi\varphi$ (see Remark 1.2), the right–hand side is nonpositive by (1.4), (3.9), (3.10). Hence, both sides in (3.11) must equal zero, that means $\tau = 1, v = \xi\varphi$ with some $\xi \neq 0$ because of (3.7). It follows from (3.2) that

$$(A(u_n) - \lambda_1 S(u_n), v_n) + (1-\tau_n) d(S(u_n), v_n) =$$
$$= \tau_n \int_\Omega [f(x, u_n(x)) - g(x)] v_n(x) \, dx.$$

The left–hand side is nonnegative (see Remark 1.2 and the assumption $(A(u), u) \geq (J(u), u)$) and therefore

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n)(x)) v_n(x) \, dx \geq \int_\Omega g(x) v(x) \, dx.$$

Now, it is sufficient to use Lemma 3.1 below and we receive

$$\int_\Omega f^{+\infty}(x)\varphi(x) \, dx \geq \int_\Omega g(x)\varphi(x) \, dx \quad \text{or} \quad \int_\Omega f_{-\infty}(x)\varphi(x) \, dx \leq \int_\Omega g(x)\varphi(x) \, dx$$

if $\xi > 0$ or $\xi < 0$, respectively, which contradicts the assumption (2.2). ∎

Lemma 3.1. *Let (1.1), (1.2), (2.1) be fulfilled and let $u_n \in X$ be such that $\|u_n\| \to +\infty, v_n = \frac{u_n}{\|u_n\|} \to \xi\varphi$ both in $L_p(\Omega)$ and a.e. on $\Omega, \xi \neq 0$. Then*

$$\liminf_{n\to\infty} \int_\omega f(x, u_n(x))v_n(x)\,dx \le \int_\Omega f^{+\infty}(x)\xi\varphi(x)\,dx \tag{3.12}$$

or

$$\liminf_{n\to\infty} \int_\omega f(x, u_n(x))v_n(x)\,dx \le \int_\Omega f_{-\infty}(x)\xi\varphi(x)\,dx \tag{3.13}$$

if $\xi > 0$ or $\xi < 0$, respectively.

PROOF : Introduce functions v_n by

$$\begin{aligned}
\widetilde{v}_n(x) &= v_n(x) && \text{for } x \in \Omega, v_n(x) \in \langle -|\xi|\varphi(x), |\xi|\varphi(x)\rangle,\\
\widetilde{v}_n(x) &= |\xi|\varphi(x) && \text{for } x \in \Omega, v_n(x) > |\xi|\varphi(x),\\
\widetilde{v}_n(x) &= -|\xi|\varphi(x) && \text{for } x \in \Omega, v_n(x) < -|\xi|\varphi(x).
\end{aligned}$$

Clearly $\widetilde{v}_n \to \xi\varphi$ in $L_p(\Omega)$ and a.e. on Ω. It follows from (1.2) and (2.1) (where we can suppose $h^{+\infty} \ge 0, h_{-\infty} \le 0$ without loss of generality) that

$$\int_\Omega f(x, u_n(x))(v_n(x) - \widetilde{v}_n(x))\,dx \le \int_{M_n} (m(x) + cr^{p-1})|v_n(x) - \widetilde{v}_n(x)|\,dx +$$
$$\int_{M_n^+} h^{+\infty}(x)(v_n(x) - \widetilde{v}_n(x))\,dx + \int_{M_n^-} -h_{-\infty}(x)|v_n(x) - \widetilde{v}_n(x)|\,dx \le$$
$$\int_\Omega (m(x) + cr^{p-1} + h^{+\infty}(x) - h_{-\infty}(x))|v_n(x) - \widetilde{v}_n(x)|\,dx \to 0,$$

where $M_n = \{x \in \Omega; |u_n(x)| \le r\}, M_n^+ = \{x \in \Omega; u_n(x) > r\}, M_n^- = \{x \in \Omega; u_n(x) < -r\}$. Hence,

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x))v_n(x)\,dx \le \limsup_{n\to\infty} \int_\Omega f(x, u_n(x))\widetilde{v}_n(x)\,dx.$$

If we knew that

$$f(x, u_n(x))\widetilde{v}_n(x) \le h(x) \quad \text{a.e. on } \Omega \quad \text{for some } h \in L_1(\Omega) \tag{3.14}$$

then Fatou's lemma would imply that

$$\limsup_{n\to\infty} \int_\Omega f(x, u_n(x))\widetilde{v}_n(x)\,dx \le \int_\Omega \limsup_{n\to\infty} f(x, u_n(x))\widetilde{v}_n(x)\,dx.$$

The assertion would follow because $u_n(x) \to +\infty$ or $u_n(x) \to -\infty$ for a.a. $x \in \Omega$ and therefore the last integral equals the right–hand side in (3.12) or in (3.13) if $\xi > 0$ or $\xi < 0$, respectively. Hence, it is sufficient to show (3.14). The definition of $\widetilde{v}_n$ and (1.2), (2.1) imply

$$\begin{aligned}
f(x, u_n)(x))\widetilde{v}_n(x) &\le h^{+\infty}(x)|\xi|\varphi(x) && \text{on } M_n^+,\\
&\le -h_{-\infty}(x)|\xi|\varphi(x) && \text{on } M_n^-,\\
&\le (m(x) + cr^{p-1})|\xi|\varphi(x) && \text{on } M_n
\end{aligned}$$

and (3.14) follows. ■

Remark 3.1. For the completness, let us show precisely that (3.9) holds. Suppose by contradiction that there is $M \subset \Omega \setminus M_0$ such that meas $M > 0, \chi > 0$ on M. We can suppose $v > 0$ on M. (The case $v < 0$ can be treated analogously.) If χ_M is the characteristic function of M then (3.4), (3.8) imply

$$\text{(3.15)} \qquad \lim_{n\to\infty} \int_\Omega \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} \chi_M(x)\,dx = \int_\Omega \chi(x)|v(x)|^{p-2}v(x)\chi_M(x)\,dx > 0.$$

It follows from (1.2) and (2.1) that

$$\text{if } |u_n(x)| \le \|u_n\| \text{ then } \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} \le \frac{m(x) + C\|u_n\|^{p-1}}{\|u_n\|^{p-1}},$$

$$\text{if } u_n(x) > \|u_n\| \text{ then } \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} \le \frac{h^{+\infty}(x)}{\|u_n\|^{p-1}},$$

$$\text{if } u_n(x) < -\|u_n\| \text{ then } \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} = \frac{f(x,u_n(x))}{\|u_n\|^{p-2}u_n(x)}(v_n(x) - v(x)) +$$

$$+ \frac{f(x,u_n(x))}{\|u_n\|^{p-2}u_n(x)} v(x) \le \frac{m(x) + C|u_n(x)|^{p-1}}{\|u_n\|^{p-1}} |v_n(x) - v(x)| - \frac{h_{-\infty}(x)}{\|u_n\|^{p-1}} v(x)$$

for n large such that $\|u_n\| > r$. Hence,

$$\frac{f(x,u_n(x))}{\|u_n\|^{p-1}} \chi_M(x) \le \frac{m(x) + C|u_n(x)|^{p-1}}{\|u_n\|^{p-1}} |v_n(x) - v(x)|\chi_M(x) + h(x)$$

with $h = \frac{1}{r^{p-1}}(h^{+\infty} - h_{-\infty}v + m) + C \in L_1(\Omega)$. It follows from Fatou's lemma and (2.1) that

$$\limsup_{n\to\infty} \int_\Omega \left[\frac{f(x,u_n(x))}{\|u_n\|^{p-1}} - \frac{m(x) + C|u_n(x)|^{p-1}}{\|u_n\|^{p-1}} |v_n(x) - v(x)| \right] \chi_M(x)\,dx \le$$
$$\int_\Omega \limsup_{n\to\infty} \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} \chi_M(x)\,dx = 0.$$

(We use the fact that the second term on the left is nonpositive and that $u_n(x) \to +\infty$ a.e. on M.) Thus,

$$\liminf_{n\to\infty} \int_\Omega \frac{f(x,u_n(x))}{\|u_n\|^{p-1}} \chi_M(x)\,dx \le \limsup_{n\to\infty} \int_\Omega \frac{m(x) + C|u_n(x)|^{p-1}}{\|u_n\|^{p-1}} \cdot$$
$$|v_n(x) - v(x)|\chi_M(x)\,dx.$$

But the integral on the right-hand side tends to zero because $v_n \to v$ in $L_p(\Omega)$ and this contradicts (3.15).

Remark 3.2. Let us observe that the proof of Theorem 2.2. above becomes very simple and short for the special case of Theorem 2.1 (i.e. if we replace A by J) and a bounded nonlinearity f. Indeed, we get $\widetilde{f} \equiv 0$ and it follows directly from (3.3) by using the compactness argument and the properties of J that $v_n \to v$ in X and that (3.11) holds with $\widetilde{S} = 0$. (The last part of the proof remains without changes.)

Remark 3.3. Consider the problem (0.1), (0.3) with f bounded (cf. Remark 3.2). Let us explain very briefly on this particular situation an other approach which can be used for the proof of results of the type of Theorem 2.1. Consider the equation

$$(0.1_\varepsilon) \qquad \operatorname{div}(|\nabla u_\varepsilon(x)|^{p-2}\nabla u_\varepsilon(x)) + \lambda_1 \frac{|u_\varepsilon(x)|^{p-2}u_\varepsilon(x)}{1+\varepsilon^{p-1}|u_\varepsilon(x)|^{p-1}} + f(x,u(x)) = g(x)$$

with the boundary conditions (0.3). It follows from the theory of monotone operators (see e.g. [8]) that for any $\varepsilon > 0$ there exists a weak solution of this problem, i.e. $u_\varepsilon \in X = \overset{\circ}{W}{}^1_p(\Omega)$ satisfying

$$(2.3_\varepsilon) \qquad J(u_\varepsilon) - \lambda_1 S_\varepsilon(u_\varepsilon) - F(u_\varepsilon) + g^* = 0,$$

where $S_\varepsilon : X \to X^*$ is defined by

$$(S_\varepsilon(u), v) = \int_\Omega \frac{|u|^{p-2}uv}{1+\varepsilon^{p-1}|u|^{p-1}}\, dx.$$

Multiplying (2.3_ε) by $\varepsilon^p u_\varepsilon$, we can derive that $\|\varepsilon u_\varepsilon\|$ is bounded. Hence, we can suppose $\varepsilon_n u_{\varepsilon_n} \rightharpoonup v$ in X for some $v \in X$ and some sequence $\{\varepsilon_n\}, \varepsilon_n \to 0$. Multiplying (2.3_ε) by ε_n^{p-1} and using the compactness argument, we receive $\varepsilon_n u_{\varepsilon_n} \to v$,

$$J(v) - \lambda_1 S_1(v) = 0,$$

i.e. v is a weak solution of

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \lambda_1 \frac{|v|^{p-2}v}{1+|v|^{p-1}} = 0$$

with (0.3). This implies $v = 0$ because λ_1 is the smallest eigenvalue. Hence, $\varepsilon_n u_{\varepsilon_n} \to 0$. If we were able to exclude the case $\|u_{\varepsilon_n}\| \to +\infty$ then we would obtain $u_{\varepsilon_n} \to u$ for some $u \in X$ from (2.3_ε) by using the compactness argument again, and u would be a solution of our problem. But if $\|u_{\varepsilon_n}\| \to +\infty$ then we can suppose $v_n = \dfrac{u_{\varepsilon_n}}{\|u_{\varepsilon_n}\|} \rightharpoonup v$ for some v. Dividing (2.3_ε) by $\|u_{\varepsilon_n}\|^{p-1}$, we get $v_n \to v$,

$$J(v) - \lambda_1 S(v) = 0,$$

i.e. $v = \pm\varphi$. This leads to the contradiction with (2.2) similarly as in the last part of the proof of Theorem 2.2.

Proof of Theorem 2.4.: First, choose $d > 0$ such that there is no eigenvalue of (0.4), (0.2) (i.e. of (1.10)) in the interval $(0, d)$ (see Remark 1.2). Note that $\lambda_1 = 0$ in the case under consideration. Define the homotopy $H : \langle 0,1\rangle \times X \to X$ by

$$H(\tau, u) = u - A_d^{-1}((2-\tau)dS(u) + \tau(F(u) - g^*)).$$

We can follow the proof of Theorem 2.2 until

$$A(u_n) - (1-\tau_n)dS(u_n) - \tau_n(F(u_n) - g^*) = 0, \tag{3.2'}$$

$$\|u_n\|^{1-p}A(u_n) - (1-\tau_n)dS(v_n) - \tau_n\|u_n\|^{1-p}(F(u_n) - g^*) = 0, \tag{3.3'}$$

where $v_n = \|u_n\|^{-1}\, u_n \rightharpoonup v, \tau_n \to \tau$. It follows again from (1.2) that we can suppose

$$\|u_n\|^{1-p}f(\cdot, u_n(,)) \rightharpoonup \widetilde{f} \quad \text{in } L_{p'}(\Omega).$$

But (2.5), (2.6) ensure that $\widetilde{f} = 0$, i.e. $\chi \equiv 0$ in the procedure of the proof of Theorem 2.2. (Precisely, considerations analogous to those from Remark 3.1 can be made.) Then using the facts that A_d is a homeomorphism, A is $(p-1)$–quasihomogeneous with respect to J, (1.8) and the assumption $(A(u),u) \geq (J(u),u)$, we respect to J, (1.8) and the assumption $(A(u),u) \geq (J(u),u)$, we derive $v \neq 0$ and

$$J(v) - (1-\tau)dS(v) = 0$$

by the same way as in the proof of Theorem 2.2. But $(1-\tau)d$ is not an eigenvalue for $\tau \in \langle 0,1)$ according to the choice of d. Therefore $\tau = 1, v = \xi\varphi = \xi$ for some $\xi \neq 0$ (because $\varphi \equiv 1$, see Remark 1.2). Suppose $\xi > 0$ (the case $\xi < 0$ can be treated similarly). It follows from (3.2') that

$$(A(u_n),\xi) - (1-\tau_n)d(S(u_n),\xi) = \tau_n \int_\Omega [f(x,u_n(x)) - g(x)]\xi\, dx. \tag{3.16}$$

Since $(A(u_n),\xi) = 0$ by the assumption and $d > 0$, the left–hand side of (3.16) is nonpositive for n large enough. Hence,

$$\liminf_{n\to\infty} \int_\Omega f(x,u_n(x))\, dx \leq \int_\Omega g(x)\, dx. \tag{3.17}$$

With respect to the compact imbedding $W^1_p(\Omega)$ into $C(\overline{\Omega})$ it is $u_n(x) > r$ for all $x \in \Omega$ if n is large enough. Then (3.17), (2.5) and Fatou's lemma imply

$$\int_\Omega f_{+\infty}(x)\, dx \leq \int_\Omega g(x)\, dx$$

which is the contradiction with (2.2'). ■

PROOF of Theorem 2.6: is a simple modification of that of Theorem 2.4. ■

PROOF of Theorem 2.8: Let us define the mapping $H : \langle 0,1\rangle \times X \to X$ by

$$H(\tau,u) = u - A_B^{-1}(\tau(\lambda_1 S(u) + F(u) - g^*)).$$

Similarly as in the proof of Theorem 2.2, it is sufficient to show that (3.1) holds with R large enough. (Note that $H(1,u) = 0$ is equivalent to (2.14) and deg$[H(0,\cdot),$

$B_R(0),0] = \deg[I, B_R(0),0] = 1$.) Suppose by contradiction that there exist $\tau_n \in \langle 0,1\rangle, u_n \in X (n = 1,2,\dots)$ such that $\|u_n\| \to +\infty$ and

$$A(u_n) + B(u_n) - \tau_n(\lambda_1 S(u_n) + F(u_n) - g^*) = 0. \tag{3.18}$$

Particularly,

$$(B(u_n), u_n) = -(A(u_n) - \tau_n\lambda_1 S(u_n), u_n) - \tau_n(F(u_n) - g^*, u_n).$$

The first term on the right–hand side is nonpositive by the assumption $(A(u), u) \geq (J(u), u)$ and Remark 1.2. Hence, (1.2) and the imbedding $L_q(\Omega) \subset L_p(\Omega)$ imply that

$$C_0\|u_n\|_q^q \leq \|m\|_{p'}\|u_n\|_q + C\|u_n\|_q^p + \|g\|_{q'} \cdot \|u_n\|_q.$$

This yields that

$$\{\|u_n\|_q\} \quad \text{is bounded.} \tag{3.19}$$

Hence, (1.2_q) gives

$$\left(\int_\Omega \left|\frac{f_q(x, u_n(x))}{\|u_n\|^{p-1}}\right|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \leq \left[\int_\Omega \left(\frac{m_q(x) + C|u_n(x)|^{q-1}}{\|u_n\|^{p-1}}\right)^{\frac{q}{q-1}} dx\right]^{\frac{q-1}{q}} \leq$$
$$\leq \|u_n\|^{1-p}(\|m_q\|_{q'} + \|u_n\|_q^{q-1}) \to 0,$$

that means

$$\frac{B(u_n)}{\|u_n\|^{p-1}} \to 0 \quad \text{in } X^* \tag{3.20}$$

according to the imbedding $L_{q'}(\Omega) \subset X^*$. Analogously we receive (by using (1.2) and the imbedding $L_q(\Omega) \subset L_p(\Omega)$)

$$\frac{F(u_n)}{\|u_n\|^{p-1}} \to 0 \quad \text{in } X^*. \tag{3.21}$$

We can suppose $\tau_n \to \tau$, $v_n = \dfrac{u_n}{\|u_n\|} \rightharpoonup v$ in X. Dividing (3.18) by $\|u_n\|^{p-1}$ and using (3.20), (3.21) and the compactness of S we see that $\{\dfrac{A(u_n)}{\|u_n\|^{p-1}}\}$ is convergent. Hence, $\dfrac{A(u_n)}{\|u_n\|^{p-1}} \to J(v)$ by the assumption that A is $(p-1)$–quasihomogeneous with respect to J. The limiting process in (3.18) yields

$$J(v) - \tau\lambda_1 S(v) = 0. \tag{3.22}$$

Further,

$$(J(v), v) = \lim_{n\to\infty}\frac{(A(u_n), u_n)}{\|u_n\|^p} \geq \lim_{n\to\infty}\frac{(J(u_n), u_n)}{\|u_n\|^p} = \lim_{n\to\infty}\frac{\|u_n\|_{1,p}^p}{\|u_n\|_{1,p}^p + \|u_n\|_q^p} = 1.$$

Hence, $v \neq 0$ and it follows from (3.22) that $\tau = 1, v = \xi\varphi$ with $\xi \neq 0$. The compactness of the imbedding $X \subset L_p(\Omega)$ ensures $\|v_n\|_p \to \xi\|\varphi\|_p$ and we have $\|u_n\| \to \infty$. Simultaneously,

$$\|u_n\| \cdot \|v_n\|_p = \|u_n\|_p \leq C_1\|u_n\|_q$$

which contradicts (3.19). ∎

Remark 3.4. Analogously as in the case of Theorem 2.1, also a different approach to the proof of Theorem 2.7 can be used (cf. Remark 3.3). We can consider the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda_1\frac{|u|^{p-2}u}{1+\varepsilon^{p-1}|u|^{p-1}}-|u|^{q-2}u+f(x,u)=g$$

with (0.2) or (0.3). It follows from the theory of monotone operators that for any $\varepsilon > 0$ there exists a weak solution. It is possible to show (by using suitable test functions again) that $\varepsilon_n u_{\varepsilon_n} \to 0$ and that the assumption $\|u_{\varepsilon_n}\| \to +\infty$ implies $\frac{u_{\varepsilon_n}}{\|u_{\varepsilon_n}\|} \to \pm\varphi$, which leads to the contradiction similarly as in the last part of the proof of Theorem 2.8 above. It follows $u_{\varepsilon_n} \to u$, where u is a solution of our problem. This approach does not use the degree theory but the precise proof is technically more complicated than the procedure from the proof of Theorem 2.8 above.

Additional remark. When the manuscript of this paper was finished, the authors received a preprint by A.Anane, J.P. Gossez [12] where results similar to our Theorem 2.1 are proved by using a variational method.

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