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Autohomeomorphism groups of spaces with unique non–isolated point

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Abstract. Let X be a set, $J \subset 2^X$ an ideal of subsets of X, S_X the group of all permutations of X, $S_X(J) = \{f \in S_X : f(J) = J\}$ the stabilizer of J, and let $H_J = \{f \in S_X : \operatorname{supp} f \in J\}$, where $\operatorname{supp} f = \{x \in X : f(x) \neq x\}$. There exists a one–to–one correspondence between the pairs (X, J) and the topological spaces $X \cup \{*\}$ with unique non–isolated point $*$, the stabilizers $S_X(J)$ being associated with the autohomeomorphism groups of these spaces. It is shown that the autohomeomorphism group of a strongly non–homogeneous T_1–space with unique non–isolated point is complete, i.e. has the trivial center and nò outer automorphisms. Specifically, the stabilizer $S_X(J)$ of every maximal ideal $J \subset 2^X$ is complete. Furthermore, it is established under CH that the stabilizer $S_R(J)$ of the σ–ideal J of Lebesque measure zero sets or of meager sets on R is a complete group and the quotient group $S_R(J)/H_J$ is not simple.

Keywords: autohomeomorphism group, stabilizer

Classification: Primary 54H05, 04A20, 03E5; Secondary 28A05

For a set X we denote by 2^X the Boolean algebra of subsets X and, respectively, by $[X]^{<\omega}$ and $[X]^{<\omega_1}$ the ideals in 2^X of finite and at most countable subsets of X. If $J \subset 2^X$ is an ideal of subsets of X and J^c its dual filter, then the family $\{\emptyset\} \cup J^c$ is obviously a topology on X. However, one can associate with J another topological space by adjoining a new point $*$ to X and viewing $\{(X \setminus a) \cup \{*\} : A \in J\} \cup 2^X$ as the topology of this space. We thus have a topological space with unique non–isolated point. Conversaly, if $Y = X \cup \{*\}$ is such a space, then by setting $J = \{A \subset X : * \notin \operatorname{cl} A\}$, we obtain an ideal in 2^X and the topology constructed from J on Y coincides with that of the original space Y. We observe that $J \supset [X]^{<\omega}$ if and only if the topology constructed on $X \cup \{*\}$ form J is T_1. As usual, we denote by S_X the general symmetric group of X and by $S_X(J) = \{f \in S_X : f(J) = J\}$ the stabilizer of an ideal $J \subset 2^X$, where $f(J) = \{f(A) : A \in J\}$. For each $f \in S_X$ we denote by $\operatorname{supp} f = \{x \in X : f(x) \neq x\}$ the support of f and set $H_J = \{f \in S_X : \operatorname{supp} f \in J\}$. It is easily seen that H_J is a normal subgroup in $S_X(J)$. If X is a space with unique non–isolated point and J is the ideal in 2^X related to it, then the autohomeomorphism group of $X \cup \{*\}$ is obviously isomorphic to $S_X(J)$. Thus, the study of the pairs (X, J), where $J \subset 2^X$ is a set ideal, is equivalent to the study of topological spaces $(X, \{J^c \cup \{\emptyset\}\})$ or the study of topological spaces $X \cup \{*\}$ with unique non–isolated point.

Let us consider, for an arbitrary set ideal $J \subset 2^X$ the quotient algebra $2^X/J$ and the representation $\Psi : S_X(J) \to \operatorname{Aut}(2^X/J)$ of the group $S_X(J)$ by automorphisms of the Boolean algebra $2^X/J$ defined as follows: if $\pi : 2^X \to 2^X/J$ is a canonical epimorphism, then $\Psi_J(f)(\pi(A)) := \pi(f(A))$ for all $f \in S_X(J)$ and $A \subset X$.

We observe that $\operatorname{Ker}\Psi_J = H_J$. Indeed, if Δ is the operation of symmetric difference on 2^X, then for any $f \in H_J$ and $A \subset X$ we have that $f(A)\Delta A \in J$ and, hence $\pi(f(A)) = \pi(A)$, i.e. $\Psi_J(f) = \mathrm{id}$ or $H_J \subset \operatorname{Ker}\Psi_J$. Conversaly, if $f \in \mathcal{S}_X(J) \setminus H_J$, then $\operatorname{supp} f \notin J$ and by Zorn's lemma there exists the maximal $E \subset \operatorname{supp} f$ with $E \cap f(E) = \emptyset$. It is easily seen that $E \notin J$, for otherwise $f(E) \in J$, and hence $C := \operatorname{supp} f \setminus (E \cup f(E)) \notin J$. Since E is maximal and f is injective, $f(C) \subset E \in J$, hence $f(C) \in J$ and this contradicts $f \in \mathcal{S}_X(J)$. Thus $E \notin J$, and hence $f(E) \notin J$ and $f(E)\Delta E = E \cup f(E) \notin J$, that is $\pi(E) \neq \pi(f(E)) = \Psi_J(f)(\pi(E))$. We thus have that $f \notin \operatorname{Ker}\Psi_J$ and so $\operatorname{Ker}\Psi_J \subset H_J$. From this we deduce that $\Psi_J = \{id\}$ if and only if $J = \{\emptyset\}$, i.e. the representation Ψ_J is faithful only for $J = \{\emptyset\}$. Since the isotropy group $\mathcal{S}_X(J)_{\pi(A)} \supset H_J$ for all $A \subset X$, the representation Ψ_J is effective (i.e. $\mathcal{S}_X(J)_{\pi(A)}$ does not contain non-trivial normal subgroup of $\mathcal{S}_X(J)$) only for $J = \{\emptyset\}$ and only in this case Ψ_J is regular, i.e. all the stabilizers $\mathcal{S}_X(J)_{\pi(A)}, A \subset X$, are trivial. Thus, for a non-trivial ideal $J \subset 2^X$, the representation Ψ_J is neither faithful, nor effective, and nor regular.

We observe that for a maximal ideal $J \subset 2^X$, one has $2^X/J = \{0,1\}$, so $\operatorname{Aut}(2^X/J) = \{id\}$, and hence Ψ_J is trivial and $H_J = \mathcal{S}_X(J)$. It follows from the equality $\operatorname{Ker}\Psi_J = H_J$ that the diagram

$$\begin{array}{ccc} \mathcal{S}_X(J) & \xrightarrow{\Psi_J} & \operatorname{Aut}(2^X/J) \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{S}_X(J)/H_J & & \end{array}$$

can be completed up to the commutative one by the homomorphism $\widetilde{\Psi_J}$, that is , the quotient group $\mathcal{S}_X(J)/H_J$ operates naturally by automorphisms on $2^X/J$.

We remark also that the triviality of the representation Ψ_J is equivalent to the condition that for any $f \in \mathcal{S}_X(J)$ and $A \in 2^X \setminus J$, $f(A) \cap A \neq \emptyset$.

The representation Ψ_J can be trivial for non-maximal ideals J as well. Indeed, let us consider two disjoint subsets X' and X'', with $|X'| = |X''|$, and let $J_1 \subset 2^{X'}$ and $J_2 \subset 2^{X''}$ be non-equivalent maximal ideals (for example one of them is principal and the other is not). Then the ideal $J_1 + J_2$ on $X = X' \cup X''$ generated by $J_1 \cup J_2$ is not maximal, because $2^X/(J_1 + J_2) = \{0, 1, \pi(X'), \pi(X'')\}$ and, since J_1 is not equivalent to J_2, no automorphism of the quotient algebra transposes $\pi(X')$ and $\pi(X'')$. Thus, $\operatorname{Aut}(2^X/(J_1 + J_2)) = \{id\}$ and $\Psi_{J_1+J_2}$ is trivial. If J_1 and J_2 are not equivalent, then $\Psi_{J_1+J_2}$ is not trivial and surjective.

We call two ideals $J_1 \subset 2^{X'}$ and $J_2 \subset 2^{X''}$ weakly equivalent if there exist $A_1 \in 2^{X'} \setminus J_1$ and $A_2 \in 2^{X''} \setminus J_2$ such that the ideals $J_1 |_{A_1}$ and $J_2 |_{A_2}$ are equivalent, i.e. there exists a bijection $f : A_1 \to A_2$ such that $f(J_1 |_{A_1}) = J_2 |_{A_2}$. We remark that if two ideals J and $J' \subset 2^X$ are equivalent, then their stabilizers $\mathcal{S}_X(J)$ and $\mathcal{S}_X(J')$ are conjugated in $\mathcal{S}_X$, because for any $f \in S_x, \mathcal{S}_X(f(J)) = f\mathcal{S}_X(J)f^{-1}$.

All the ideals $J \subset 2^X$ for which Ψ_J is trivial can be described as follows: for every partition of X into two subsets $X', X'' \notin J$, the ideals $J |_{X'}$ and $J |_{X''}$ are not weakly equivalent. One can call the ideals of such a kind weakly indecomposable (it is impossible to decompose them into weakly equivalent ones) or strongly non-homogeneous (for any $A_1, A_2 \in 2^X \setminus J, A_1 \cap A_2 = \emptyset$, we have that $J|_{A_1}$ and $J|_{A_2}$ are not equivalent). In topological terms this means that in the space $X \cup \{*\}$ with unique non-isolated point related to J no two disjoint non-closed subsets are homeomorphic.

We recall that a group G with trivial center and no outer automorphisms is called complete [$\mathbf{S}_1$].

Theorem 1. *If an ideal $J \subset 2^X$ is weakly indecomposable and $J \subset [X]^{<\omega}$, then the group $\mathcal{S}_X(J)$ is complete.*

PROOF: Since $J \supset [X]^{<\omega}$, the stabilizer $\mathcal{S}_X(J)$ contains the alternating group A_X consisting of compositions of even many transpositions of X. Therefore, $A_X \subset \mathcal{S}_X(J) \subset S_X$ and by a theorem of Wielandt [W] every automorphism of the group $\mathcal{S}_X(J)$ is induced by an inner automorphism of S_X, i.e. for any $\varphi \in \operatorname{Aut}(\mathcal{S}_X(J))$ there exits $f \in S_X$ such that $\varphi(g) = fgf^{-1}$ for all $g \in \mathcal{S}_X(J)$. Then f belongs to the normalizer $N_{S_X}(\mathcal{S}_X(J))$. Let us verify that $\mathcal{S}_X(J) = N_{S_X}(\mathcal{S}_X(J))$. Let $f \in N_{S_X}(\mathcal{S}_X(J))$, that is $f\mathcal{S}_X(J)f^{-1} = \mathcal{S}_X(J)$. As we have observed above, $f\mathcal{S}_X(J)f^{-1} = \mathcal{S}_X(f(J))$. Since J is weakly indecomposable, $\mathcal{S}_X(J) = H_J$, and hence $\mathcal{S}_X(f(J)) = H_J$. Since the ideal $f(J)$ is weakly indecomposable as well, $\mathcal{S}_X(f(J)) = H_{f(J)}$. Therefore we have that $H_J = H_{f(J)}$, hence $\{\operatorname{supp} h : h \in H_J\} = \{\operatorname{supp} h : h \in H_{f(J)}\}$. It is not hard to show that for each $A \in J$ there exists an involution $h \in S_X$ with $\operatorname{supp} h = A$. Thus $J \subset \{\operatorname{supp} h : h \in H_J\}$, and hence $\{\operatorname{supp} h : h \in H_{f(J)}\} = f(J)$ and $\{\operatorname{supp} h : h \in H_J\} = J$, so $f(J) = J$. We thus have that $f \in \mathcal{S}_X(J)$. Since $J \supset [X]^{<\omega}$, we have that the center of $\mathcal{S}_X(J)$ is trivial and hence $\mathcal{S}_X(J)$ is complete. ■

Corollary 1. *For every maximal ideal $J \subset 2^X$ the group $\mathcal{S}_X(J)$ is complete.*

Indeed, if J is principal, i.e. $J = 2^{X \setminus \{x_0\}}$, then $\mathcal{S}_X(J)$ is isomorphic to $S_{X \setminus \{x_0\}}$ and by the Schreier–Ulam theorem [S–U] $\mathcal{S}_X(J)$ is complete. For non-principal J we have $J \supset [X]^{<\omega}$ and since any maximal ideal is weakly indecomposable one can apply Theorem 1.

Now we describe the automorphism group of the kernel of Ψ_J.

Theorem 2. *For every ideal $J \subset 2^X$ we have $\mathcal{S}_X(J) \cong \mathcal{S}_{\cup J}(J) \times S_{X \setminus \cup J}$ and* $\operatorname{Aut}(H_J) \cong \mathcal{S}_{\cup J}(J)$.

PROOF: The correspondence $f \to (f|_{\cup J}, f|_{X \setminus \cup J}), f \in \mathcal{S}_X(J)$, is obviously an isomorphism of the group $\mathcal{S}_X(J)$ onto the group $\mathcal{S}_{\cup J}(J) \times S_{X \setminus \cup J}$. Let $H'_J = H_J|_{\cup J} = \{f|_{\cup J} : f \in H_J\}$. Since for each $f \in H_J$ we have $\operatorname{supp} f \subset \cup J$, $f|_{X \setminus \cup J} = id$, and hence the image of H_J under this isomorphism is $H'_J \times \{id\}$. That is H_J is isomorphic to $H'_J \subset \mathcal{S}_{\cup J}(J)$, where $J \supset [\cup J]^{<\omega}$, and we have realized the reduction to the case that J contains the ideal of finite subsets of X. Thus, we may assume that $J \supset [X]^{<\omega}$. We will show that $\operatorname{Aut}(H_J) \cong \mathcal{S}_X(J)$. Since $J \supset [X]^{<\omega}$, $A_X \subset H_J \subset S_X$. By Wielandt's theorem [W] every automorphism $\alpha \in \operatorname{Aut}(H_J)$ is induced by an inner automorphism $\beta \in \operatorname{Inn}(S_X)$, i.e. there exists $g \in S_X$ such that $\alpha(f) = gfg^{-1}$ for all $f \in H_J$. We then have that $g \in N_{S_X}(H_J)$. Let us verify that

$N_{\mathcal{S}_X}(H_J) = \mathcal{S}_X(J)$. Suppose the contrary, then there exists $g \in N_{\mathcal{S}_X}(H_J) \setminus \mathcal{S}_X(J)$, i.e. for some infinite $A \in J$ either $g(A) \notin J$ or $g^{-1}(A) \notin J$. Let us consider an involution $h \in S_X$ with $\operatorname{supp} h = A$. Clearly $h \in H_J$ but either $\operatorname{supp} ghg^{-1} = g(\operatorname{supp} h) = g(A) \notin J$ or $\operatorname{supp} g^{-1}hg = g^{-1}(\operatorname{supp} h) = g^{-1}(A) \notin J$. Contradiction. Thus, $N_{\mathcal{S}_X}(H_J) \subset \mathcal{S}_X(J)$. On the other hand, if $g \in \mathcal{S}_X(J)$, then $gH_Jg^{-1} = H_J$, because H_J is a normal subgroup of $\mathcal{S}_X(J)$. Hence $\mathcal{S}_X(J) \subset N_{\mathcal{S}_X}(H_J)$ and we have that $N_{\mathcal{S}_X}(H_J) = \mathcal{S}_X(J)$. Thus, every automorphism $\alpha \in \operatorname{Aut} H_J$ is induced by an inner automorphism $\beta \in \operatorname{Inn}(\mathcal{S}_X(J))$. If $g \in C_{\mathcal{S}_X(J)}(H_J)$, then $ghg^{-1} = h$ for all $h \in H_J$. But H_J contains all the transpositions of S_X and an element of S_X commuting with any transposition is id. Thus, $C_{\mathcal{S}_X(J)}(H_J) = \{id\}$ and since $C(\mathcal{S}_X(J)) = \{id\}$ and $\operatorname{Inn}(\mathcal{S}_X(J)) \cong \mathcal{S}_X(J)/C(\mathcal{S}_X(J)) \cong \mathcal{S}_X(J)$ we have that $\operatorname{Aut}(H_J) \cong \operatorname{Inn}(\mathcal{S}_X(J))/C_{\mathcal{S}_X(J)}(H_J) \cong \mathcal{S}_X(J)$. Therefore $\operatorname{Aut}(H_J) \cong \operatorname{Aut}(H'_J) \cong \mathcal{S}_{\cup J}(J)$ and the proof is complete. ∎

Theorem 3. *(CH). If J is the σ-ideal of null-sets or of meager sets on the real line R, then the group $\mathcal{S}_R(J)$ is complete and the quotient group $\mathcal{S}_R(J)/H_J$ is not simple.*

PROOF: Since $J \supset [R]^{<\omega}$, we have $C(\mathcal{S}_R(J)) = \{id\}$ and $A_R \subset \mathcal{S}_R(J)$. By Wielandt's theorem [W] for each $\varphi \in \operatorname{Aut}(\mathcal{S}_R(J))$ there exists $h \in S_R$ such that $\varphi(g) = hgh^{-1}$ for all $g \in \mathcal{S}_R(J)$, that is $h \in N_{S_R}(\mathcal{S}_R(J))$. We will show that $N_{S_R}(\mathcal{S}_R(J)) = \mathcal{S}_R(J)$. Let $h \in N_{S_R}(\mathcal{S}_R(J))$. Obviously, $h([R]^{<\omega_1}) = [R]^{<\omega_1} \subset J$. Therefore, using CH, it suffices to show that for any infinite uncountable $A \in J$, $h(A) \in J$. Let us consider the σ-ideal $\mathcal{G}$(resp. $\mathcal{Z}$) of Sierpinski (resp. Lusin) sets on R, distinct from $[R]^{<\omega_1}$ under CH, the is of those $S \subset R$ every uncountable subset of which having positive outer measure (resp. being not a meager set in R) [S$_2$]. We observe that $\mathcal{S}_R(J) = \mathcal{S}_R(\mathcal{G})$, if J is the σ-ideal of null-sets and $\mathcal{S}_R(J) = \mathcal{S}_R(\mathcal{Z})$, if J is the σ-ideal of meager sets on R. Indeed, if $f \in \mathcal{S}_R(\mathcal{G})$(resp. $\mathcal{S}_R(\mathcal{Z})$) and $f(A) \notin J$ for some uncountable $A \in J$, then there exists an uncountable $S \in \mathcal{G}$(resp. $S \in \mathcal{Z}$) such that $S \subset f(A)$ [S$_2$]. Then $f^{-1}(S) \in \mathcal{G}$ (resp. $\mathcal{Z}$), but $f^{-1}(S) \subset A \in J$, and hence $f^{-1}(S) \in J$. This contradicts the equality $\mathcal{G} \cap J$ (resp. $\mathcal{Z} \cap J$)= $[R]^{<\omega_1}$. Thus, $\mathcal{S}_R(\mathcal{G}) \subset \mathcal{S}_R(J)$ (resp. $\mathcal{S}_R(\mathcal{Z}) \subset \mathcal{S}_R(J)$). Conversaly, if $g \in \mathcal{S}_R(J)$ and for some uncountable $S \in \mathcal{G}$ (resp. $\mathcal{Z}$) $g(S) \notin \mathcal{G}$ (resp. $\mathcal{Z}$), then there exists an uncountable $A \subset g(S)$ with $A \in J$. Hence $g^{-1}(A) \in J$ and $g^{-1}(A) \subset S$, i.e. $g^{-1}(A) \in J \cap \mathcal{G}$ (resp. $J \cap \mathcal{Z}$), a contradiction. Thus, $\mathcal{S}_R(J) \subset \mathcal{S}_R(\mathcal{G})$ (resp. $\mathcal{S}_R(\mathcal{Z})$). We further observe that for each $f \in S_R$ there exists either an uncountable $B \in J$ with $f(B) \in J$ or an uncountable $C \in \mathcal{G}$ (resp. $\mathcal{Z}$) with $f(C) \in \mathcal{G}$ (resp. $\mathcal{Z}$). Indeed, suppose that $f(J) \cap J = [R]^{<\omega_1}$ and $f(\mathcal{G}) \cap \mathcal{G} = [R]^{<\omega_1}$. Then $f(J) = \mathcal{G}$ (resp. $\mathcal{Z}$). If on the contrary $f(B) \notin \mathcal{G}$ (resp. $\mathcal{Z}$) for some uncountable $B \in J$, then there exists an uncountable $C \subset f(B)$ of measure 0 (resp. of first category) and hence $f^{-1}(C) \in J$, i.e. $C \in f(J) \cap J$, a contradiction. Thus, $f(J) \subset \mathcal{G}$ (resp. $\mathcal{Z}$). On the other hand, if $f^{-1}(D) \notin J$ for some uncountable $D \in \mathcal{G}$ (resp. $\mathcal{Z}$), then by choosing an uncountable $E \subset f^{-1}(D)$ with $E \in \mathcal{G}$ (resp. $\mathcal{Z}$), we have $f(E) \subset D$, that is $f(E) \in \mathcal{G}$ (resp. $\mathcal{Z}$) and hence $f(E) \in \mathcal{G} \cap f(\mathcal{G})$ (resp. $\mathcal{Z} \cap f(\mathcal{Z})$), a contradiction again. Therefore, $f^{-1}(\mathcal{G}) \subset J$ (resp. $f^{-1}(\mathcal{Z}) \subset J$) or, in other words, $\mathcal{G} \subset f(J)$ (resp. $\mathcal{Z} \subset f(J)$). Thus, $f(J) = \mathcal{G}$ (resp. $\mathcal{Z}$). However, we will show that this equality does not hold for any $f \in S_R$. Since every null-set (resp. meager set) is contained in a G_δ-null-set (resp. meager F_σ-set), we obtain a family $J' \subset J$ of cardinality c such that for any $A \in J$ there exists $A' \in J'$ with $A' \supset A$. If for some $f \in S_X$, $f(J) = \mathcal{G}$ (resp. $\mathcal{Z}$), then $f(J')$ is a subfamily of $\mathcal{G}$ (resp. $\mathcal{Z}$) with the same property as J' in J. We will show that $\mathcal{G}$ (resp. $\mathcal{Z}$) does not contain a subfamily of such a kind of cardinality c. Suppose the contrary, i.e. there exists a family $\{C_\alpha : \alpha < \omega_1\} \subset \mathcal{G}$ (resp. $\mathcal{Z}$) such that every $C \in \mathcal{G}$ (resp. $\mathcal{Z}$) is contained in some C_α. Let $\{A_\alpha : \alpha < \omega_1\}$ be the family of all G_δ-null-sets (resp. meager F_σ-sets). For each $\alpha < \omega_1$, $R \setminus (\bigcup_{\beta<\alpha} C_\beta \cup \bigcup_{\beta<\alpha} A_\beta) \neq \emptyset$, because $\bigcup_{\beta<\alpha} C_\beta \in \mathcal{G}$ (resp. $\mathcal{Z}$) and $\bigcup_{\beta<\alpha} \in J$ and the complement to any null-set (resp. meager set) contains a null-set (resp. meager set) of cardinality 2^ω [0, Theorem 19.1]. Let $x_\alpha \in R \setminus (\bigcup_{\beta<\alpha} (C_\beta \cup A_\beta))$ and $C = \{x_\alpha : \alpha < \omega_1\}$. If $A \subset C$ is a null-set (resp. meager set), then there exists $\alpha_0 < \omega_1$ such that $A \subset A_{\alpha_0}$, and hence A does not contain x_α with $\alpha > \alpha_0$, i.e. $|A| \leq \omega$. Thus, $C \in \mathcal{G}$ (resp. $\mathcal{Z}$) and from the diagonal construction it follows that $C \not\subset C_\alpha$ for all $\alpha < \omega_1$. A contradiction. We thus have that either there is an uncountable $B \in J$ with $h(B) \in J$ or there exists an uncountable $C \in \mathcal{G}$ (resp. $\mathcal{Z}$) such that $h(C) \in \mathcal{G}$ (resp. $\mathcal{Z}$). Suppose there is an uncountable $B \in J$ such that $h(B) \in J$. If for an uncountable $A \in J$ we have that $h(A \setminus B) \in J$, then $h(A) = h(A \setminus B) \cup h(A \cap B) \in J$, i.e. we may assume that $A \cap B = \emptyset$. Obviously, any involution $g \in S_R$ with $\operatorname{supp} g = A \cup B$ and $g(B) = A$ belongs to $\mathcal{S}_R(J)$ (in other words, the group $\mathcal{S}_R(J)$ acts transitively on disjoint elements of J of common cardinality). We then have, since $hgh^{-1} \in \mathcal{S}_R(J)$ and $h(B) \in J$, that $hgh^{-1}(h(B)) = h(A) \in J$, i.e. $h(J) \subset J$. The same argument for h^{-1} shows that $h^{-1}(J) \subset J$, i.e. $J \subset h(J)$, and hence $h(J) = J$, i.e. $h \in \mathcal{S}_R(J)$. Suppose there is an uncountable $C \in \mathcal{G}$ (resp. $\mathcal{Z}$) such that $h(C) \in \mathcal{G}$ (resp. $\mathcal{Z}$). Let us verify that $h(D) \in \mathcal{G}$ (resp. $\mathcal{Z}$) for all $D \in \mathcal{G}$ (resp. $\mathcal{Z}$). We may assume as above that D and C are disjoint. If $g \in S_R$ is an involution with $\operatorname{supp} g = C \cup D$ and $g(C) = D$, then $g \in \mathcal{S}_R(\mathcal{G})$ (resp. $\mathcal{S}_R(\mathcal{Z})$), because $\mathcal{G}$ and $\mathcal{Z}$ are set ideals. We then have that $hgh^{-1} \in h\mathcal{S}_R(\mathcal{G})h^{-1}$ (resp. $h\mathcal{S}_R(\mathcal{Z})h^{-1}$)$= h\mathcal{S}_R(J)h^{-1} = \mathcal{S}_R(J) = \mathcal{S}_R(\mathcal{G})$ (resp. $\mathcal{S}_R(\mathcal{Z})$), because $h(C) \in \mathcal{G}$ (resp. $\mathcal{Z}$) implies that $hgh^{-1}(h(C)) = h(D) \in \mathcal{G}$ (resp. $\mathcal{Z}$). Thus, $h \in \mathcal{S}_R(\mathcal{G})$ (resp. $\mathcal{S}_R(\mathcal{Z})$)$= \mathcal{S}_R(J)$, and hence $N_{S_R}(\mathcal{S}_R(J)) \subset \mathcal{S}_R(J)$. Since H_J and $H_\mathcal{G}$ (resp. $H_\mathcal{Z}$) are distinct normal subgroups in $\mathcal{S}_R(J)$. the subgroup $H_J H_\mathcal{G}$ (resp. $H_J H_\mathcal{Z}$) generated by $H_J \cup H_\mathcal{G}$ (resp. $H_J \cup H_\mathcal{Z}$) is a proper normal subgroup in $\mathcal{S}_R(J)$ distinct from H_J and $H_\mathcal{G}$ (resp. $H_\mathcal{Z}$). Indeed, $H_J \not\subset H_\mathcal{G}$ (resp. $H_\mathcal{Z}$) and $H_\mathcal{G}$ (resp. $H_\mathcal{Z}$)$\not\subset H_J$ and $H_J H_\mathcal{G}$ (resp. $H_J H_\mathcal{Z}$)$\neq \mathcal{S}_R(J)$, because $H_J H_\mathcal{G}$ (resp. $H_J H_\mathcal{Z}$)$\subset H_{J+\mathcal{G}}$ (resp. $H_{J+\mathcal{Z}}$) and for instance, the reflection f of $R : x \to -x, x \in R$, belongs obviously to $\mathcal{S}_R(J)$ and does not belong to $H_{J+\mathcal{G}}$ (resp. $H_{J+\mathcal{Z}}$), since $\operatorname{supp} f = R \setminus \{0\} \notin J + \mathcal{G}$ (resp. $J + \mathcal{Z}$). Thus, the quotient group $\mathcal{S}_R(J)/H_J$ is not simple, as well as $\mathcal{S}_R(J)/H_\mathcal{G}$ (resp. $\mathcal{S}_R(J)/H_\mathcal{Z}$), and the proof is complete. ∎

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