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DIAMETERS IN LOCALES: HOW BAD THEY CAN BE

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Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: General diameters in locales are shown to be close to diameters with very good properties. They can, however, still behave badly in some respects. A few notes on diameters of sublocales are added.

Key words: Diameter, frame, locale, metric.

Classification: 06D99, 18B35, 54E35, (54J05)

Extending the metric structure to locales one can use a notion of diameter to replace that of metric ([3],[4],[5]). When formalizing the intuition one has of a diameter on a frame, a few conditions seem to be natural: it should be a monotone real function (with possible values $+\infty$) with $d(0)=0$ and with the conditioned subadditivity

$$a\wedge b \neq 0 \Rightarrow d(a\vee b)\leq d(a)+d(b).$$

The diameters induced by distance functions in metric spaces satisfy, moreover, special conditions which cannot be deduced from these demands, most notably the following:

$$\forall a \quad \forall \varepsilon > 0 \quad \exists\, b,c \leq a \text{ such that } d(b),d(c) < \varepsilon \quad \text{and } d(a) < d(b\vee c)+\varepsilon .$$

This assumption, which we call metricity (see 1.2 below), is very handy and seems to imply all one can wish of a reasonable diameter function. The question naturally arises as to how badly a general diameter can differ from a metric one. In this article we show, on the one hand, that if a diameter d agrees with the underlying frame structure (see Section 2) then it is necessarily very close to a metric one. More specifically, there is a metric diameter $\tilde{d}$ producing the same ε-neighbourhoods, and the difference $d-\tilde{d}$ is in a sense small. On the other hand, we show that even so it can be rather badly behaved.

The paper is divided into five sections. In the first one we discuss the basic notions. Section 2 is devoted to the relation of a diameter to the underlying frame structure. In Section 3 we deal with diameters inducing identical ε-neighbourhoods and show that each equivalence class contains exactly one metric diameter. Some consequences and counterexamples are presented in Section 4. Section 5 contains a few remarks on diameters of sublocales.

To make the reading easy we have included, in three instances, explicit proofs of facts proved (in essence) elsewhere. Statements 1.4 and 3.4 are proved in [3] under more special conditions (1.4, moreover, rather clumsily). Showing we do not really need them would take some space anyway. Proposition 3.3, which is very substantial here, is in an author's paper which has not yet appeared. Leaving these proofs out would hardly save more space than one page.

1. Prediameters in lattices

1.1. The set of reals augmented by $+\infty$ and $-\infty$ will be denoted by $\mathbf{R}$; we put $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$. The bottom resp. top of a lattice L - if it exists - will be usually denoted by 0 resp. 1.

A prediameter on a lattice L is a mapping $d: L \to \mathbf{R}_+$ satisfying

(i) $d(0)=0$,

(ii) $a \leq b \Rightarrow d(a) \leq d(b)$,

(iii) $a \wedge b \neq 0 \Rightarrow d(a \vee b) \leq d(a)+d(b)$.

Examples: 1. (Bad) A measure on L.

2. (Good) Let (X, ρ) be a metric space, L a sublattice of exp X. Consider the usual diameter function.

1.2. We will consider some additional properties. The most important of them is the following: A prediameter is said to be metric if

(M) $\forall a \in L \ \forall \varepsilon > 0 \ \exists u,v \leq a$ s.t. $d(u),d(v) < \varepsilon$ and

$$d(a) \leq d(u \vee v) + \varepsilon .$$

Notes: 1. Sometimes the following reformulation of (M) is more handy:

$\forall a \in L,\ a \neq 0,\ \forall \varepsilon > 0 \ \exists u,v$ s.t. $u \wedge a \neq 0 \neq v \wedge a$,

$$d(u),d(v) < \varepsilon \text{ and } d(u) < d(u \vee v) + \varepsilon .$$

2. If the L in Example 2 above has the property that $\forall u \subseteq X,\ u \neq 0,\ \exists u'$ s.t. $0 \neq u' \subsetneq u$, then the diameter induced by ρ is metric.

1.3. Other additional assumptions: (Cf. [3].) A prediameter is said to be strong if

$$\forall S \subseteq L,\ d(\bigvee S) \leq \sup\{\inf \{\sum_{i=1}^{n} (d(a_i) \mid a_i \in S,\ a_1 = a,\ a_n = b,$$
$$a_i \wedge a_{i+1} \neq 0\} \mid a,b \in S\}.$$

A prediameter is said to be star-additive if

$$(a,b \in S \Rightarrow a \wedge b \neq 0) \Rightarrow d(\bigvee S) \leq \sup\{d(a)+d(b) \mid a,b \in S, a \neq b\}.$$

We speak about a star-prediameter (here the terminology of [3] has been radically changed; cf. [5]) if

$$\forall a \in L \quad \forall S \subseteq L \text{ such that } b \in S \Rightarrow b \wedge a \neq 0,$$
$$d(a \vee \bigvee S) \leq d(a) + \sup\{d(b)+d(c) \mid b,c \in S, b \neq c\}.$$

A prediameter is said to be continuous if

for each up-directed $S \subseteq L$, $d(\bigvee S) = \sup\{d(a) \mid a \in S\}$.

Observation: Each strong diameter is a star-additive star-prediameter.

1.4. Proposition: Let the lattice L satisfy the implication $(a \wedge \bigvee S \neq 0 \Rightarrow \exists\ b \in S,\ a \wedge b \neq 0)$. Then each metric prediameter on L is continuous and strong.

Proof: In both cases we can assume that $\bigvee S \neq 0$. Then, for each $\varepsilon > 0$, we have u, v such that $u \wedge \bigvee S \neq 0 \neq v \wedge \bigvee S$, $d(u), d(v) < \varepsilon$ and $d(\bigvee S) < d(u \vee v) + \varepsilon$. Choose $a,b \in S$ such that $a \wedge u \neq 0 \neq b \wedge v$.

1. Let S be up-directed. Take $c \in S$ such that $a,b \leq c$. Then $d(\bigvee S) < d(u \vee c \vee v) + \varepsilon < d(c) + 3\varepsilon$.

2. Let $a = a_1, a_2, \dots, a_n = b$ be such that $a_i \wedge a_{i+1} \neq 0$. Then $d(u \vee v) \leq d(u \vee \bigvee a_i \vee v) < \sum d(a_i) + 2\varepsilon$

and hence

$$d(\bigvee S) \leq \inf\{\sum d(a_i) \mid a = a_1, b = a_n, a_i \wedge a_{i+1} \neq 0\} + 3\varepsilon. \quad \square$$

1.5. Proposition: Let L be as in 1.4. Let d be a metric prediameter and let $a \leq \bigvee S$ where $\forall b \in S$, $d(b) < \varepsilon$. Then there are $b,c \in S$ such that $d(a) < d(b \vee c) + \varepsilon$.

Proof: Let $a \neq 0$. There are u, v such that $u \wedge a \neq 0 \neq v \wedge a$, $d(u), (d(v) < \frac{1}{3}\varepsilon$ and $d(a) < d(u \vee v) + \frac{1}{3}\varepsilon$. Choose $b,c \in S$ such that $u \wedge b \neq 0 \neq v \wedge c$. Then

$$d(a) < d(u \vee b \vee c) + \frac{1}{3}\varepsilon \leq d(b \vee c) + d(u) + d(v) + \frac{1}{3}\varepsilon < d(b \vee c) + \varepsilon. \quad \square$$

2. Diameters on frames

2.1. As usual (see e.g. [2]), a frame is a complete lattice satisfying the distributivity law $(\bigvee a_i)\wedge b=\bigvee(a_i\wedge b)$ (thus, in particular, a frame satisfies the implication from 1.4 and 1.5). A frame homomorphism $f:A\longrightarrow B$ is a mapping preserving general joins and finite meets. The resulting category is denoted by Frm, its dual, the category of locales is denoted by Loc.

Let X be a topological space. The open sets constitute a frame which will be denoted by $\Omega(X)$. If $f:X\longrightarrow Y$ is a continuous mapping, we obtain a frame homomorphism $\Omega(f):\Omega(Y)\longrightarrow\Omega(X)$ by putting $\Omega(f)(u)=f^{-1}(u)$. Thus one obtains a functor Ω:Top $\longrightarrow$ Loc.

Let U be a subset of a frame A, $a\in A$. Put

$$Ua=\bigvee\{u\mid u\in U,\ u\wedge a\neq 0\}.$$

Obviously, $U(\bigvee a_i)=\bigvee Ua_i$. Thus, there is a Galois correspondence

$$Ua\leq b \text{ iff } a\leq\alpha_U(b).$$

The symbol α_U (later in modifications) will be used in this sense throughout the paper.

A cover of A is a subset $U\subseteq A$ such that $\bigvee U=1$.

2.2. **Proposition:** Let (X,τ) be a topological space, ρ a metric on the set X, d the resulting diameter, τ_ρ the induced topology. Then

(1) $\tau_\rho\subseteq\tau$ iff

$(*)$ $\forall\varepsilon>0$, $U^d_\varepsilon=\{a\mid a\in\tau,\ d(a)<\varepsilon\}$ is a cover.

(2) $\tau_\rho=\tau$ iff $(*)$ and

$(**)$ $\forall a\in\tau$, $a=\bigcup\{b\mid\exists\varepsilon>0,\ U^d_\varepsilon b\subseteq a\}$.

Proof: (1) If $\tau_\rho\subseteq\tau$ then $(*)$ obviously holds. On the other hand, let $(*)$ hold and let $a\in\tau_\rho$. Take an $x\in a$ and choose an $\varepsilon>0$ such that $\Omega(x,\varepsilon)=\{y\mid\rho(x,y)<\varepsilon\}\subseteq a$. By $(*)$, $a=\bigcup\{b\cap a\mid b\in\tau,\ d(b)<\varepsilon\}$ and hence there is a $b\in\tau$ such that $x\in b$ and $d(b)<\varepsilon$ so that $b\subseteq\Omega(x,\varepsilon)$. Thus, $a=\bigcup\{b\in\tau\mid b\subseteq a\}\in\tau$.

(2) Let $\tau=\tau_\rho$. If $x\in a\in\tau$, we can find an $\varepsilon>0$ such that $\Omega(x,2\varepsilon)\subseteq a$. Thus, $x\in U^d_\varepsilon\Omega(x,\varepsilon)\subseteq\Omega(x,2\varepsilon)\subseteq a$. On the other hand let $(*)$ and $(**)$ hold and let a be in τ. Choose an $x\in a$. By $(**)$ there is a $b\in\tau$ and $\varepsilon>0$ such that $x\in b$ and $U^d_\varepsilon b\subseteq a$. Let $\rho(x,y)<\varepsilon$. Choose $\eta>0$ so that $\rho(x,y)+2\eta<\varepsilon$ and, by $(*)$, $u,v\in U^d_\eta$ such that $x\in u$ and $y\in v$. Then $d(u\vee v)<\varepsilon$ and hence $y\in U^d_\varepsilon b\subseteq a$. Thus, $a\in\tau_\rho$. □

2.3. A diameter on a frame A is a prediameter d on A satisfying, moreover,

(iv) $\forall \varepsilon > 0$, $U_\varepsilon^d = \{a \mid d(a) < \varepsilon\}$ is a cover.

Let d be a diameter on A and d´ one on B. A homomorphism $f:A \to B$ is said to ve contractive (w.r.t. d and d´) if

$$\forall \varepsilon > 0 \quad \forall b \in B \quad d'(b) < \varepsilon \Rightarrow \exists a \in A,\ d(a) < \varepsilon \text{ and } f(a) \geq b.$$

Note: Let (X,ρ), (Y,σ) be metric spaces, $f:X \to Y$ a mapping. It is easy to see that f is a contraction (i.e., $\sigma(f(x),f(y)) \leq \rho(x,y)$) iff $\Omega(f)$ is contractive w.r.t. the resulting diameters on $\Omega(X),\Omega(Y)$.

2.4. We will write

α_ε^d (or simply α_ε) instead of $\alpha_{U_\varepsilon^d}$.

Observation: Let A be a frame and d a diameter on A. Then we have

$(**)$ $\forall a \in A,\ a = \bigvee\{b \mid \exists \varepsilon > 0,\ U_\varepsilon^d b \leq a\}$

iff

(v) $\forall a \in A,\ a = \bigvee_{\varepsilon > 0} \alpha_\varepsilon^d a.$

(Indeed, since $U_\varepsilon \alpha_\varepsilon a \leq a$, (v) $\Rightarrow$ $(**)$. On the other hand, if $(**)$ holds, we have $a = \bigvee\{b \mid \exists \varepsilon > 0,\ b \leq \alpha_\varepsilon a\} = \bigvee_{\varepsilon > 0} \alpha_\varepsilon a$.)

2.5. (Cf. 2.2.) A diameter on a frame A is said to be compatible if there holds (v) from 2.4. A diametric frame is a couple (A,d) where d is a compatible diameter on A. If d is a metric diameter, we speak of a metric frame.

The category of diametric resp. metric frames and contractive homomorphisms will be denoted by

DFrm resp. MFrm.

IN the context of the dual categories (denoted DLoc, MLoc) we speak of (dia)metric locales.

Remark: Note that a frame A is metrizable in the sense of Isbell ([1]) iff it can be made to a metric frame (A,d) ([3],[5]).

3. Metric diameters associated with general ones

3.1. Diameters d_1, d_2 on a frame A are said to be similar if

$\forall a \quad \forall \varepsilon > 0 \quad U_\varepsilon^{d_1} a = U_\varepsilon^{d_2} a$ (which, of course, is equivalent to $\forall \varepsilon > 0$,

$\alpha_\varepsilon^{d_1}=\alpha_\varepsilon^{d_2}$. In such a case we write

$$d_1 \approx d_2 .$$

3.2. A mapping $\mu: A \longrightarrow \mathbf{R}$ is said to be thin if
$\forall \varepsilon > 0 \quad \exists$ cover U s.t. $x \in U,\ y \in U \implies |\mu(x \vee y)| < \varepsilon$.

Proposition: Let d_1, d_2 be diameters on A, let d_1-d_2 be thin. Then $d_1 \approx d_2$.

Proof: Since the conditions are symmetric, it suffices to prove that $U_\varepsilon^{d_2}a \le U_\varepsilon^{d_1}a$. Put $\mu=d_1-d_2$. Let $u \wedge a \neq 0$ and $d_2(u) < \varepsilon$. Choose an $\eta > 0$ such that $d_2(u)+\eta < \varepsilon$. Choose a cover U satisfying the formula in the definition for the η . We have $u=\bigvee\{x \wedge u \mid x \in U\}$. Choose an $x_0 \in U$ such that $x_0 \wedge u \wedge a \neq 0$. Now let $x \in U$ be general. We have $|\mu((x_0 \wedge u) \vee (x \wedge u))| < \eta$ and hence

$$d_1((x_0 \wedge u) \vee (x \wedge u)) \le d_2(u)+|\mu((x_0 \wedge u) \vee (x \wedge u))| < \varepsilon$$

so that $x \wedge u \le U_\varepsilon^{d_1}a$. Thus, $u \le U_\varepsilon^{d_1}a$. $\square$

3.3. **Proposition:** Let d be a metric diameter on A, d' a metric diameter on B. Let $f: A \longrightarrow B$ be a homomorphism. Then the following statements are equivalent:

(1) f is contractive,

(2) $\forall \varepsilon > 0 \quad \forall a \in A,\ U_\varepsilon^{d'} f(a) \le f(U_\varepsilon^{d} a)$,

(3) $\forall \varepsilon > 0 \quad \forall a \in A,\ f(\alpha_\varepsilon^{d}(a)) \le \alpha_\varepsilon^{d'}(f(a))$.

Proof: Obviously (2)$\iff$(3) and the proof of (1)$\implies$(2) is straightforward (also note that all this holds without the assumption (M)).

Now let $d'(b) < \varepsilon$. Choose $\eta > 0$ so that $d'(b) < \varepsilon - 3\eta$. Let x, y be such that

$$f(x) \wedge b \neq 0 \neq f(y) \wedge b \text{ and } d(x), d(y) < \eta .$$

Let (1) hold. Then we have $b \le U_{\varepsilon-3\eta} f(x) \le f(U_{\varepsilon-3\eta} x)$ and since $f(y) \wedge b \neq 0$ there is a u such that

$$d(u) < \varepsilon - 3\eta \ ,\ u \wedge x \neq 0 \text{ and } f(u) \wedge f(y) \wedge b \neq 0.$$

Since U_η is a cover, we have a $z \neq 0$ with $d(z) < \eta$ such that $z \le u$ and $f(z \wedge y) \wedge b = f(z) \wedge f(y) \wedge b \neq 0$. Thus, $d(x \vee z) \le d(x \vee u) \le d(x)+d(u) < \varepsilon - 2\eta$ and hence further $d(x \vee y) \le d(x \vee z \vee y) \le d(x \vee z)+d(y) < \varepsilon - \eta$. Put $a=\bigvee\{x \mid d(x) < \eta,\ f(x) \wedge b \neq 0\}$. By 1.5 and the inequality just proved, $d(a) < \varepsilon$. Finally, for $c=\bigvee\{f(x)|d(x)<\eta, f(x)\wedge b=0\}$ we have $f(a)\vee c=f(\bigvee U_\eta)=1$ and hence $b=$ $=(f(a)\vee c)\wedge b=f(a)\wedge b$ so that $b\leq f(a)$. □

Having considered the identity mapping $(A,d_1)\longrightarrow(A,d_2)$, we immediately obtain

Corollary: For metric diameters d_1, d_2,

$$d_1 \approx d_2 \text{ iff } d_1=d_2.$$

3.4. Let d be a diameter on A. Put

$$\tilde{d}(a)=\inf_{\varepsilon>0}\sup\{d(u\vee v)|u\wedge a\neq 0\neq v\wedge a,\ d(u),d(v)<\varepsilon\}=$$

$$=\inf_{\varepsilon>0}\sup\{d(u\vee v)|u,v\leq a,\ d(u),d(v)<\varepsilon\}.$$

The equality of the two formulas is obvious. Also obviously

$$\tilde{d}\leq d.$$

Lemma: $\tilde{d}(a\vee b)\geq d(a\vee b)-d(a)-d(b)$.

Proof: The statement obviously holds for a=0 or b=0. Thus, we can assume $a\neq 0\neq b$. Let $u\wedge a\neq 0\neq v\wedge b$, $d(u),d(v)<\varepsilon$. Then

$$d(a\vee b)\leq d(a\vee u\vee v\vee b)\leq d(u\vee v)+d(a)+d(b)$$

and hence

$$d(a\vee b)-d(a)-d(b)\leq d(u\vee v)\leq\sup\{d(x\vee y|x\wedge a\neq 0\neq y\wedge b,\ d(x),d(y)<\varepsilon\}.\ \square$$

Proposition: $\tilde{d}$ is a metric diameter.

Proof: The properties (i), (ii) and (iv) are obvious.

(iii): Let $a\wedge b\neq 0$ and let $\varepsilon>0$ be given. Choose $\eta>0$ such that

$$\alpha=\sup\{d(u\vee v)|u\wedge a\neq 0\neq v\wedge a,\ d(u),d(v)<\eta\}<\tilde{d}(a)+\varepsilon,$$

$$\beta=\sup\{d(u\vee v)|u\wedge b\neq 0\neq v\wedge b,\ d(u),d(v)<\eta\}<\tilde{d}(b)+\varepsilon.$$

Take a $c\in U^d_\eta$, $0\neq c\leq a\wedge b$. Let $x\wedge(a\vee b)\neq 0\neq y\wedge(a\vee b)$. If $x\wedge a\neq 0\neq$ $\neq y\wedge a$ or $x\wedge b\neq 0\neq y\wedge b$ we have $d(x\vee y)\leq\alpha$ resp. β, otherwise $d(x\vee y)\leq$ $\leq d(x\vee c)+d(y\vee c)\leq\alpha+\beta$. Thus, always $d(x\vee y)\leq\alpha+\beta$ and we obtain $\tilde{d}(a\vee b)\leq\alpha+\beta<\tilde{d}(a)+\tilde{d}(b)+2\varepsilon$.

(M): By Lemma, we have

$$\tilde{d}(a)\leq\inf_{\varepsilon>0}\sup\{\tilde{d}(u\vee v)+2\varepsilon\ |u,v\leq a,\ d(u),d(v)<\varepsilon\}\leq$$

$$\inf_{\varepsilon>0}(\sup\{\tilde{d}(u\vee v)|u,v\leq a,\ \tilde{d}(u),\tilde{d}(v)<\varepsilon\}+2\varepsilon).\ \square$$

3.5. Theorem: For each diameter d on a frame A there is a metric diameter $\tilde{d}$ such that d-$\tilde{d}$ is non-negative and thin.

Proof: Use the results and notation from 3.4. Thus it suffices to show that d-$\tilde{d}$ is thin. By Lemma in 3.4,

$$d(a \vee b)-\tilde{d}(a \vee b) \leq d(a)+d(b)$$

so that one can put $U=U^d_{\varepsilon/2}$ to satisfy the condition for thinness. □

3.6. Theorem: $d_1 \approx d_2$ iff d_1-d_2 is thin.

Proof: By 3.2 it suffices to show that if $d_1 \approx d_2$ then d_1-d_2 is thin. If $d_1 \approx d_2$ then by 3.5, 3.2 and Corollary in 3.3, $\tilde{d}_1=\tilde{d}_2=d$ and we have $d_I=d+$ $+\mu_i$ with μ_i thin. Using common refinements of the covers in question we see that $d_1-d_2=\mu_1-\mu_2$ is thin. □

3.7. Remarks: 1. Summarizing 3.2, 3.5 and 3.3, we also see that each similarity class contains a minimum (in the natural order of real functions). This minimum diameter is the unique metric diameter in the class.

2. If metric diameters d_1 and d_2 are distinct, their difference is not thin. More explicitly, there is an $\varepsilon_0 > 0$ such that for each cover U there are $u,v \in U$ such that

$$|d_1(u \vee v)-d_2(u \vee v)| \geq \varepsilon_0.$$

3. The correspondence $d \mapsto \tilde{d}$ constitutes a mono-coreflection of DFrm into MFrm. Indeed, let (A,d) be an object of DFrm; since $\tilde{d} \leq d$, id:$(A,\tilde{d}) \longrightarrow$ $\longrightarrow (A,d)$ is contractive. Now let $f:(B,d') \longrightarrow (A,d)$ be a contractive homomorphism. This implies that $U^d_\varepsilon f(b) \leq f(U^{d'}_\varepsilon b)$ and, since $U^{\tilde{d}}_\varepsilon a=U^d_\varepsilon a$, also $U^{\tilde{d}}_\varepsilon f(b) \leq$ $\leq f(U^{d'}_\varepsilon b)$. Thus, by 3.3, f is also a contractive homomorphism $(B,d') \longrightarrow$ $\longrightarrow (A,\tilde{d})$. □

4. Compatible diameters

4.1. A non-zero element a of a frame A is said to be an <u>isolated point</u> if $b < a \Rightarrow b=0$ (this term will be explained in Section 5).

Proposition: Let d be a compatible diameter on A. Then d(a)=0 iff a=0 or a is an isolated point.

Proof: Let d(a) > 0. Since by (i) and (iv) $0 \neq a=\bigvee\{b \mid b \leq a,\ d(b)< d(a)\}$, there has to be a $b \neq 0$ with $b < a$.

Let d(a)=0, let $0 \neq x \leq a$. By (v) there is an $\varepsilon > 0$ such that $\alpha_\varepsilon(x) \neq 0$.

Since $d(a)=0$ and $a \wedge \alpha_\varepsilon(b) \neq 0$, we have $a \leq U_\varepsilon \alpha_\varepsilon(b) \leq b$. □

Corollary: 1. $d \equiv 0$ is compatible only on the one-point and the two-point frame (representing respectively the void and the one point space).

2. More generally, by 3.5, no thin d (in particular, no measure) is compatible with a non-trivial frame.

4.2. Since, trivially, a diameter similar to a compatible one is compatible itself, we obtain by 4.1 also

Observation: Let d be compatible. Then $d(a) \neq 0$ iff $\tilde{d}(a) \neq 0$.

4.3. **Proposition:** Let d be a compatible diameter on A, let $d(a)>0$. Then for each K there are $u,v \leq a$ such that $d(u \vee v)>0$ and $d(u \vee v) \geq K(d(u)+ +d(v))$.

Proof: Take an $\varepsilon > 0$. We have x, y such that $x \wedge a, y \wedge a \neq 0$ and $\tilde{d}(a) < < \tilde{d}(x \vee y)+ \varepsilon \leq d(x \vee y)+ \varepsilon$, $\tilde{d}(x), \tilde{d}(y) < \varepsilon$. Choose $0 \neq u \leq x \wedge a$, $0 \neq v \leq y \wedge a$ such that $d(u), d(v) < \varepsilon$. We have $d(x \vee y) \leq d(x \vee u \vee v \vee y) \leq d(u \vee v)+2\varepsilon$. Thus

$$d(u \vee v) > \tilde{d}(a)-3\varepsilon \ , \ d(u)+d(v) < 2\varepsilon \ .$$

Since $\tilde{d}(a)>0$ and ε has been arbitrary, the statement follows. □

4.4. Thus we have seen that a compatible diameter is very much unlike a measure. On the other hand it can still behave rather badly as we shall see in the following examples.

Examples: 1. Let I be the unit interval, d the diameter induced on $\Omega(I)$ by the usual metric ($\rho(x,y)=|x-y|$), μ the Lebesgue measure. Put

$$\bar{d}(u)=d(u)+ \mu(u).$$

Obviously μ is thin so that $\bar{d} \approx d$ and hence, in particular, $\bar{d}$ is compatible. It is a well known fact that μ is continuous in the sense of 1.3. The diameter d, being metric, is also continuous (see 1.4) and hence $\bar{d}$ is continuous. Obviously, $\bar{d}$ is not a star-diameter (not even star-additive).

For some purposes, being a star-diameter is an unnecessarily strong property. What one often needs is just that

$(*)$ $\forall a>0 \ \ \forall \varepsilon > 0 \ \ \exists \delta > 0$ such that $d(U_\delta a) < d(a)+ \varepsilon$.

For $\bar{d}$ even this fails to hold. Consider a dense $a \subseteq I$ such that $\mu(a) < \frac{1}{2}$. Then $\bar{d}(a) < 1+ \frac{1}{2}$ while, for each $\delta > 0$, $U_\delta a=I$ and hence $\bar{d}(U_\delta a)=2$.

2. Consider the set $X=I \times \omega / \{0\} \times \omega$ (i.e., $I \times \omega$ in which all the points $(0,n)$ are glued into one) endowed with the metric

$$\rho((x,m),(y,n)) = \begin{cases} |x-y| & \text{if } m=n \\ x+y & \text{if } m \neq n. \end{cases}$$

Let d be the resulting diameter. Put $M=\{(1,n) \mid n=0,1,\ldots\}$. On $\Omega(X)$ consider the diameters

$$d_1(u) = \begin{cases} d(u)+1 & \text{if } u \cap M \text{ is infinite} \\ d(u) & \text{otherwise,} \end{cases}$$

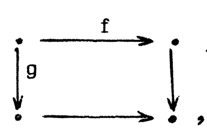
$$d_2(u) = \begin{cases} d(u)+1 & \text{if } \forall \varepsilon > 0 \ \exists \text{ infinitely many } (x,n) \in u \text{ with } x > 1-\varepsilon \\ d(u) & \text{otherwise.} \end{cases}$$

None of the d_i is continuous (consider $u_n = I \times n / \{0\} \times n$) and d_1 also fails to satisfy (*) (consider $a = \langle 0,1) \times \omega / \{0\} \times \omega$). On the other hand, $d_2(U_\varepsilon a) \leq$ $\leq d_2(a) + 2\varepsilon$.

5. Remarks on diameters of sublocales

5.1. (For more details on sublocales see e.g. [1],[2].) A sublocale of A is a surjective frame homomorphism $f:A \longrightarrow A'$. If A' is the two-element frame, we speak of a point, if A' is the one-element frame we say that f is void and write f=0. With the elements $a \in A$ we associate the sublocales $(a):A \longrightarrow$ $\longrightarrow [a] = \{x \mid x \leq a\}$ defined by $(a)(x) = a \wedge x$. (Note that a is an isolated point in the sense of 4.1 iff (a) is a point.)

We write $f \sqsubseteq g$ if there is a homomorphism h such that $f = h \circ g$. If f, g are sublocales, $f \sqcap g$ is defined as the diagonal in the pushout



$f \sqcup g$ as the natural projection $A \longrightarrow A/\sim$ where $\sim$ is the congruence $a \sim b$ iff f(a)=f(b) and g(a)=g(b). Note that $f \sqcap g$ is the infimum and $f \sqcup g$ the supremum of f, g in the preorder $\sqsubseteq$, and that $(a) \sqcap (b) = (a \wedge b)$ and $(a) \sqcup (b) =$ $= (a \vee b)$. Also note that $f \sqsubseteq (a)$ iff f(a)=1.

5.2. A sublocale $f:A \longrightarrow A'$ is said to be closed if there is a $c \in A$ such that f(a)=f(b) iff $a \vee c = b \vee c$. For a general sublocale f put $c_f = \bigvee \{x \mid f(x)=0\}$ and define cl(f) as the natural projection $A \longrightarrow A/\sim$ where $a \sim b$ iff $a \vee c_f =$ $= b \vee c_f$. cl(f), the closure of f, is the $\sqsubseteq$ -minimal closed sublocale g such that $f \sqsubseteq g$.

5.3. Let d be a diameter on A. For sublocales $f:A \longrightarrow A'$ put

$$d(f) = \inf \{ d(a) \mid f(a)=1 \}.$$

Observations: 1. $d((a))=d(a)$.

2. $d(f)=\inf\{d(a) \mid f \sqsubseteq (a)\}$.

3. $d(0)=0$ and $f \sqsubseteq g \Rightarrow d(f) \leq d(g)$.

5.4. **Proposition:** Let $f \sqcap g \neq 0$. Then $d(f \sqcup g) \leq d(f)+d(g)$.

Proof: Since $f \sqcap g \neq 0$, we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ A'' & \xrightarrow{\ f'\ } & B \end{array}$$

such that $1_B \neq 0_B$. Let a, b be such that

$d(a) < d(f)+\varepsilon$, $f(a)=1$; $d(b) < d(g)+\varepsilon$, $g(b)=1$.

We have $f'g(a)=g'f(a)=1$ and hence $f'g(a \wedge b)=f'g(a) \wedge f'g(b)=1_B \neq 0_B$ so that $a \wedge b \neq 0$ and hence $d(a \vee b) \leq d(a)+d(b)$. Now $f(a \vee b)=1=f(1)$ and $g(a \vee b)=1=g(1)$ so that $(f \sqcup g)(a \vee b)=1$. Thus,

$$d(f \sqcup g) \leq d(a \vee b) \leq d(f)+d(g)+2\varepsilon$$

and $\varepsilon > 0$ was arbitrary. □

5.5. **Proposition:** Let d be a diameter on A. Let $f:A \longrightarrow \mathbf{2}$ be a point. Then for each $\varepsilon > 0$ there is an $a \in A$ such that $d(a) < \varepsilon$ and $f \sqsubseteq (a)$.

Proof: Since $\bigvee U^d_\varepsilon = 1$ we have $1=f(\bigvee U^d_\varepsilon)=\bigvee\{f(u) \mid d(u) < \varepsilon\}$. □

5.6. As a generalization of 4.1 we obtain

Proposition: Let d be a compatible diameter on A. Then, for a sublocale $f:A \longrightarrow B$, $d(f)=0$ iff $f=0$ or f is a point.

Proof: If f is a point then $d(f)=0$ by 5.5 and 5.3.3. Now let $d(f)=0$ and $f \neq 0$, i.e., $f(0) \neq f(1)$. Let $f(a) \neq 0$. Thus, $0 \neq f(\bigvee_{\varepsilon > 0} \alpha_\varepsilon a)=\bigvee f(\alpha_\varepsilon a)$ and hence there is an $\varepsilon > 0$ such that $f(\alpha_\varepsilon a) \neq 0$. Since $d(f)=0$, there is an x such that $f(x)=1$ and $d(x) < \varepsilon$. Thus, $f(x \wedge \alpha_\varepsilon a)=f(\alpha_\varepsilon a) \neq 0$, hence $x \wedge \alpha_\varepsilon a \neq 0$ and hence $x \leq a$. Thus, $f(a)=1$ and we conclude that B consists of 0 and 1 only. □

5.7. **Proposition:** Let d be a star-diameter on A. Then, for each sublocale $f:A \longrightarrow A'$, $d(cl(f))=d(f)$.

Proof: Let $d(f) < \delta$. Thus, there is an $a \in A$ such that $f(a)=1$ and $d(a) < < \delta$. Using the star property we see that for sufficiently small $\varepsilon > 0$ we still have $d(b) < \delta$ for $b=U_\varepsilon a$. Put $c=\bigvee\{u \mid u \in U_\varepsilon , u \wedge a=0\}$.

Then

$$b \vee c = 1 \text{ and } a \wedge c = 0.$$

Thus, $f(c)=f(a)\wedge f(c)=0$ and hence $c \not\leq c_f$. Consequently, $b \vee c_f = 1 = 1 \vee c_f$, i.e., $b \sim 1$ and hence $cl(f)(b)=1$. Thus, $d(cl(f)) < \delta$. $\square$
(Note that we have used the ($*$) only - see 4.4.)

Corollary: In the case of star-diameters we have

$$cl(f) \sqcap cl(g) \neq 0 \Rightarrow d(f \sqcup g) \leq d(f)+d(g).$$

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