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ON POLYNOMIAL TIME DECIDABILITY OF INDUCED-MINOR-CLOSED CLASSES

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Dedicated to Professor Miroslav Katětov on his seventieth birthday

Abstract: It follows from recent results of Robertson and Seymour that for any minor-closed class of graphs $\mathcal{F}$ (i.e. $G \in \mathcal{F}$ and H minor of G implies $H \in \mathcal{F}$) there is a polynomially (in fact $O(|V(G)|^3)$) bounded algorithm for the membership problem of $\mathcal{F}$. We investigate this property for a weaker notion of induced-minor-closed classes. There is a linear algorithm if the class $\mathcal{F}$ consists of series-parallel graphs (i.e. those which contain no subdivision of K_4). However, for induced minor-closed classes in general this problem may be NP-hard or even algorithmically undecidable.

Key words: Tree-width, branch-width, minor, induced minor, series-parallel graph, well-quasi-ordering.

Classification: 68E10

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1. Introduction and statements of results. In this paper graphs are finite, without loops and multiple edges. A graph is a minor of another graph if the first can be obtained from a (not necessarily induced) subgraph of the second by edge contraction. A graph is an induced minor if the subgraph can be taken to be induced.

The following two outstanding results of Robertson and Seymour have important consequences.

(1.1) Theorem: Given any infinite sequence $G_1, G_2, \ldots$ of graphs, there are indices i, j such that $i < j$ and G_i is isomorphic to a minor of G_j.

(1.2) Theorem [RS3]: For a fixed graph H, there is a polynomially bounded algorithm to test, if the input graph G has a minor isomorphic to H.

An immediate corollary is that any minor-closed class $\mathcal{F}$ possesses a polynomial-time membership test. Indeed, by (1.1) there is a finite number of graphs $H_1,\dots,H_n$ such that $G \notin \mathcal{F}$ if and only if some H_i is isomorphic to a minor of G. So the test of membership to $\mathcal{F}$ can be reduced to n tests of minor containments, which can be done in polynomial time by (1.2). Let us note that this poses just the existence of an algorithm, but does not indicate how to construct such an algorithm, because neither (1.1) nor its proof can be used to construct $H_1,\dots,H_n$.

Theorems 1.1 and 1.2 suggest a general pattern (called a Robertson-Seymour poset by Fellows and Langston [FL]).

In this paper we investigate the induced minor relation from this point of view. This research is motivated by our earlier research [NT] and by the fact that at least one interesting class of graphs - namely string graphs (see e.g. [KGK]) - is induced minor closed (while it fails to be minor closed). We also solve a problem posed in [FL] concerning the complexity of the induced minor relation.

Some of the stones in this project are already known. We begin with an analogy of (1.1):

(1.3) Theorem [T]: Given any sequence $G_1,G_2,\dots$ of series-parallel graphs, there are indices i, j such that $i<j$ and G_i is isomorphic to a minor of G_j.

We thus aimed to show the corresponding analogue to (1.2). This is done in Section 3 by an application of an algorithm of Robertson and Seymour, in a slightly more general context than we need (for a definition of branch-width see Section 2):

(1.4) Theorem: For any fixed w and any fixed graph H, there is an $O(|V(G)|)$ algorithm to solve the following problem:
Instance: Graph G and a branch-decomposition (T,r) of G of width $\leq w$
Question: Is H isomorphic to an *induced minor* of G?

This is what we need, since series-parallel graphs have branch-width ≤ 2 and the corresponding branch decomposition can be found in linear time:

(1.5) Proposition: There is an $O(|V(G)|)$ algorithm which, for the input series-parallel graph G, finds its branch-decomposition of width ≤ 2.

We indicate the proof in Section 2, details can be found in [MT].

Combining (1.3), (1.4) and (1.5) we get our first result. In fact, when proving (1.4) we are able to prove the following "disjoint connecting induced subgraphsubgraph problem":

(1.6) Theorem: For any fixed w, k there is a polynomially bounded algorithm to solve the following problem:
Instance: Graph G of tree-width $\leq$ w and sets $Z_1,\dots,Z_t$ with $\sum|Z_i|\leq k$
Question: Does there exist induced subgraph of G consisting of t components $K_1,\dots,K_t$ such that $Z_i\subseteq V(K_i)$ $(i=1,\dots,t)$?

The restriction to series-parallel graphs in (1.3) is essential, since otherwise it is not difficult to construct counterexamples, see [T]. It is asked in [T] whether such a counterexample can be constructed in such a way that no G_i has a minor isomorphic to K_5^- (the complete graph K_5 minus one edge). We answer this negatively in Section 4 as follows:

(1.7) Theorem: There exists an infinite sequence $G_1,G_2,\dots$ of graphs, such that each G_i is planar, contains no minor isomorphic to K_5^- and there are no indices i, j such that $i\neq j$ and G_i is isomorphic to an induced minor of G_j.

Finally we use this counterexample to prove

(1.8) Theorem: There exists an induced-minor-closed class $\mathcal{F}$ such that there is no algorithm to test the membership to $\mathcal{F}$.

It follows from the proof that $\mathcal{F}$ can be chosen in such a way that the decision is NP-complete ... etc.

2. Branch-width. A <u>separation</u> of a graph G is a pair (A,B) of subgraphs such that $V(A)\cup V(B)=V(G)$, $E(A)\cup E(B)=E(G)$ and $E(A)\cap E(G)=\emptyset$. The <u>order</u> of the separation (A,B) is $|V(A)\cap V(B)|$.

A <u>branch-decomposition</u> of a graph G=(V,E) is a pair (T,τ), where

(i) T is a tree and every vertex of T has valency 1 or 3,

(ii) τ is a bijection from E to the set of leaves of T (i.e. the set of vertices of T of valency 1).

For each $f\in E(T)$ the edges of G are divided into two sets, depending on which component of $T\setminus f$ contains $\tau(e)$. The order of f is the number of vertices of G incident with an edge from each set. The <u>width</u> of (T,τ) is the maximum order of edges of T, and the <u>branch-width</u> of G is the minimum width of all branch-decompositions of G. (If there are no branch -decompositions of G, then $|E(G)|\leq 1$ and G has branch-width 0 by convention.)

Let (T,τ) be a branch-decomposition of G with width $\leq$ w and assume that G has no isolated vertices. For a subgraph $S\subseteq T$, we denote by $\tau^{-1}(S)$ the subgraph of G formed by the edges e of G with $\tau(e)\in V(S)$ and their ends.

For $f \in E(T)$, let S,T be the two components of $T \setminus f$; then $(\tau^{-1}(S),\tau^{-1}(T))$ is a separation of G or order $\leq w$. This is in fact the only property of branch-width we need, but for the reader's convenience we list below some additional properties.

2.1. Proposition

(i) G has branch-width ≤ 1 iff G is a forest

(ii) G has branch-width ≤ 2 iff G has no K_4 minor

(iii) If H is a planar graph, then the set of all G with no minor isomorphic to H has bounded branch-width

(iv) If H is isomorphic to a minor of G, then the branch-width of H is at most the branch-width of G.

(v) There is an $O(|V(G)|^2)$ algorithm, which, for the input graph G of branch-width $\leq w$ finds a branch-decomposition of width $\leq 3w$.

Proof: see [RS1],[RS2],[RS3]. $\square$

Sketch of proof of Proposition (1.5): It is well known that every series-parallel graph can be reduced to the empty graph using the following reductions:

(i) removing a vertex of degree ≤ 1;

(ii) removing a vertex of degree 2 whose neighbours are already adjacent;

(iii) removing a vertex of degree 2 whose neighbours are not adjacent and joining these neighbours by an edge.

With a suitably chosen data structure, one can search and perform the reductions in a constant time per reduction; for details see [MT].

Now the branch-decomposition can be build-up in the reserved order. We make this precise for the case (iii), leaving the other ones to the reader:

Let G' be obtained according to (iii) from G by removing edges e_1,e_2, and adding edge e. Let (T',τ') be a branch-decomposition of G' of width ≤ 2. Let T be the tree obtained from T' by adding new vertices t_1 and t_2 which are joined to $\tau'(e)$ and let τ be defined by

$$\tau(f)=\begin{cases}\tau'(f) & \text{for } f \neq e\\ t_i & f=e_i.\end{cases}$$

Then (T,τ) is a branch-decomposition of G of width ≤ 2.

This build-up process takes a constant time at each step and hence $O(|V(G)|)$ in total. $\square$

3. Proofs of (1.4) and (1.6). The following definitions and results are principally those of Robertson and Seymour, modified to the induced case.

Let Z be a finite set. A Z-miniature is a pair (H, φ), where H is a graph and $\varphi : V(H) \to 2^Z$ satisfies $\varphi(v) \cap \varphi(v') = \emptyset$ for distinct $v, v' \in V(H)$. Two Z-miniatures (H_1, φ_1), $(H_2, \ _2)$ are isomorphic if there is an isomorphism of H_1 with H_2 taking φ_1 to φ_2.

Let G be a graph. A model in G is a set $\{G_i : i \in I\}$ of mutually disjoint, non-null connected induced subgraphs of G. If we collapse the vertices of G_i to a single vertex, the graph H formed by these vertices and edges $\{v_i, v_j\} \in E(H)$ iff there is $f_{ij} \in E(G)$ with one end in $V(G_i)$ and the other one in $V(G_j)$ is an induced minor of G, and every induced minor arises in this way. More generally, let $Z \subseteq V(G)$. Let H be obtained as above and define $\varphi : V(H) \to 2^Z$ by $\varphi(v_i) = Z \cap V(G_i)$ $(i \in I)$. Then (H, φ) is a Z-miniature. Any isomorphic Z-miniature is said to be a Z-miniature of the model $\{G_i : i \in I\}$. A Z-miniature is feasible on G if it is a Z-miniature of some model in G. The induced folio of G relative to Z is the set of all Z-miniatures which are feasible in G. A Z-miniature (H, φ) has detail $\leq \sigma$ if

(i) $|V(H)| \leq \sigma$, and

(ii) $\varphi(v) = \emptyset$ for at most σ vertices v of H.

The σ-folio of G relative to Z is the set of all feasible Z-miniature with detail $\leq \sigma$.

For fixed w, σ, ζ we shall give an algorithm with running time $O(|V(G)|)$ to solve the following problem:

INDUCED FOLIO WITH BRANCH-WIDTH $\leq w$

Instance: A graph G, a branch-decomposition (T, τ) of G of width $\leq w$, and a subset $Z \subseteq V(G)$ with $|Z| \leq \zeta$.

Question: What is the σ-folio of G relative to Z?

(We observe that, although the σ-folio is infinite, it is the union of finitely many isomorphism classes of Z-miniatures, and it suffices to output one member of each class. Indeed, the number of such classes is bounded by a function of σ and ζ.)

This algorithm can be used to prove both (1.4) and (1.6). For (1.4) we compute the σ-folio of G relative to $\emptyset$, where $\sigma = |V(H)|$ and for (1.6) we may assume that $Z_1, \dots, Z_k$ are mutually disjoint and we compute the 0-folio of G relative $Z_1 \cup \dots \cup Z_k$. In both cases we read off from the corresponding folio whether the desired configuration exists.

For the description of an algorithm for induced folio with branch-width $\leq w$ we shall need the following two propositions, the proofs of which we leave to the reader.

(3.1) If $Z' \subseteq Z \subseteq V(G)$ and $\sigma \geq 0$, the σ-folio of G relative to Z' is determined by a knowledge of the σ-folio of G relative to Z.

(3.2) Let (A,B) be a separation of the graph G, and let $Z \supseteq V(A) \cap V(B)$. For $\sigma \geq 0$, the σ-folio of G relative to Z is determined by a knowledge of the σ-folio of A relative to $V(A) \cap Z$ and of B relative to $V(B) \cap Z$.

The following is the algorithm (4.1) of [RS3], the method is originally due to Arnborg and Proskurowski [AP].

(3.3) Algorithm for induced folio with branch-width $\leq$ w

Input: A graph G, and a subset $Z \subseteq V(G)$ with $|Z| \leq \xi$, and a branch-decomposition (T, τ) of G of width $\leq$ w.

Output: The σ-folio of G relative to Z.

Since the effect of isolated vertices of G on the folio is clear we may assume (by deleting them) that G has no isolated vertices. Let z be a leaf of T, and number the edges of T as $f_1, \ldots, f_m$, where $m=|E(T)|$, so that the indexes on every path leaving z are decreasing. For $1 \leq i \leq m$, let S_i, T_i be the components of $T \setminus f_i$, with $z \in V(S_i)$. Let $A_i = \tau^{-1}(S_i)$, $B_i = \tau^{-1}(T_i)$. Since G has no isolated vertices, (A_i, B_i) is a separation of G of order $\leq$ w. Let $Z_i = (V(A_i) \cup Z) \cap V(B_i)$. We shall compute the σ-folio of B_i relative to Z_i for each i, by a recursion as follows. At the start of the i-th iteration, the σ-folio of B_j relative to Z_j has been determined for $1 \leq j < i$.

(1) If $i \leq m$ and $|V(T_i)|=1$ then $|E(B_i)|=1$ and $|V(B_i)| \leq 2$. We determine the folio of Z_i in B_i and return to (1) for the next iteration.

(2) If $i \leq m$ and $|V(T_i)| > 1$, let f_j, f_k be the two edges of T_i with a common end with f_i. Then $j,k < i$, and (B_j, B_k) is a separation of B_i, and the σ-folios of B_j relative to Z_j and of B_k relative to Z_k have already been determined. Since $V(B_j) \cap V(B_k) \subseteq Z_j \cap Z_k$ we can determine the σ-folio of $B_j \cup B_k$ relative to $Z_j \cup Z_k$ from this information, by (3.2). Since $Z_i \subseteq Z_j \cup Z_k$, and $B_j \cup B_k = B_i$, the folio of B_i relative to Z_i can be determined, by (3.1). We return to (1) for the next iteration.

(3) If $i=m+1$, the folio of B_m relative to Z_m has been determined. We determine the σ-folio of A_m relative to $(Z \cup V(B_m)) \cap V(A_m)$ (this is easy, since $|V(A_m)| \leq 2$), and use (3.1) and (3.2) to determine the folio of Z in $A_m \cup B_m = G$, and stop.

Each iteration takes constant time, and there are $|E(T)|+1$ iterations. Thus the running time is $O(|E(G)|)=O(|V(G)|)$, since $|E(G)| \leq c \cdot |V(G)|$ for graphs of bounded branch-width (as one calculates using the branch-decomposition).

4. Proof of (1.7). An infinite sequence $G_1, G_2, \ldots$ of graphs is called <u>bad</u>, if there are no indices i, j such that $i < j$ and G_i is an induced minor of G_j.

Our aim is to construct a bad sequence $G_1, G_2, \ldots$ such that no G_i has a minor isomorphic to K_5^-. The i-th term of this sequence is depicted on Fig.1:

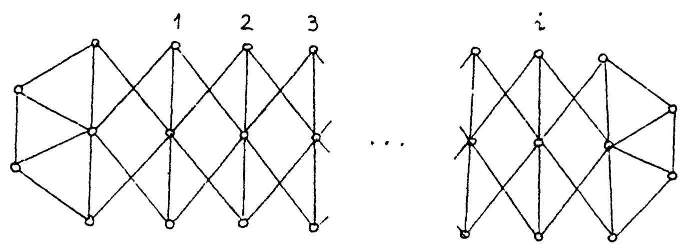

Fig. 1

These graphs are planar and contain no minor isomorphic to K_5^-. The latter statement follows from the fact that the graphs on Fig. 2 are obtained by pasting wheels together along edges

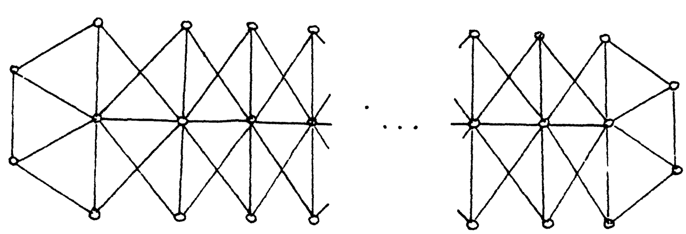

Fig. 2

On the other hand it is easy to see that these graphs are induced - minors incomparable. We just sketch the idea. So assume that for two such graphs, G_i and G_j, say G_i is an induced minor of G_j. It is easily seen that the endblocks of G_j must be retained. Let us call the vertices of degrees 6 or 7 inner and those of degree 3 outer. An inner vertex can neither be deleted, nor can be collapsed to form an outer vertex, since this would lead to a graph that can be made disconnected by removal of an outer vertex and one other vertex. But there must be a set $U \subseteq V(G_j)$ corresponding to an inner vertex $u \in V(G_i)$ such that $|U| > 1$, but then it follows that two inner vertices would be joined by an edge in G_i - a contradiction. $\square$

5. **Proof (1.8).** Let us take, say, the bad sequence $G_1 G_2, \ldots$ from (1.7) and a nonrecursive set $A \subseteq \omega$. We define $\mathcal{F}$ as a class of graphs which are induced minors of some G_i for $i \in A$. Clearly, $\mathcal{F}$ is induced-minor-closed. If there was an algorithm to test the membership to $\mathcal{F}'$, we would be able to test the membership to A simply by asking if $G_i \in \mathcal{F}'$ (since G_i can be effectively constructed from i). $\square$

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