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Cofinal Families in Certain Function Spaces

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Respectfully Dedicated to Professor Miroslav Katětov
On the occasion of His 70th Birthday

Abstract. Relations of compact-covering numbers to dominating families are given for certain function spaces.

Key-words: dominating family, compact-covering number

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§0. Notation and Definitions.

The symbol ω denotes the least infinite cardinal. The symbols α, β, γ and λ denote (arbitary) infinite cardinals, and η, θ, ξ and ζ are ordinals.

The discrete space of cardinality α is again denoted α. The set α in its usual interval topology is written $\langle\alpha\rangle$. Since the (infinite) cardinal α has no last element, the space $\langle\alpha\rangle$ is not compact. Note that $\omega = \langle\omega\rangle$.

For an index set I the partial order $<$ is defined on the set ${}^I\alpha$ of functions from I into α by this rule:

$f < g$ if $f(i) < g(i)$ for all $i \in I$.

If I is an ordinal θ the pre-order $<^*$ is defined on ${}^I\alpha$ by

$f <^* g$ if there is $\zeta < \theta$ such that $f(\eta) < g(\eta)$ whenever $\zeta \leq \eta < \theta$.

A subset $\mathfrak{F}$ of ${}^I\alpha$ is said to be *dominating* in ${}^I\alpha$ if $\mathfrak{F}$ is $<$-cofinal in ${}^I\alpha$ (that is, if for all $g \in {}^I\alpha$ there is $f \in \mathfrak{F}$ such that $g < f$). And $\mathfrak{F}$ is *eventually dominating* in ${}^\theta\alpha$ if $\mathfrak{F}$ is $<^*$-cofinal in ${}^\theta\alpha$.

For each α and θ we write

$D({}^\theta\alpha) = \min\{|\mathfrak{F}| : \mathfrak{F} \subseteq {}^\theta\alpha, \mathfrak{F}$ is a dominating family$\}$, and

$E({}^\theta\alpha) = \min\{|\mathfrak{F}| : \mathfrak{F} \subseteq {}^\theta\alpha, \mathfrak{F}$ is an eventually dominating family$\}$.

(For $\alpha = \theta = \omega$, dominating and eventually dominating families in ${}^{\theta}\alpha$ are called 1-scales and 2-scales respectively by Hechler [10].)

For a space X we denote by κX, the *compact-covering number* of X, the least cardinality of a compact cover of X; that is,

$\kappa(X) = \min\{|\mathfrak{K}| : X = \cup \mathfrak{K} \text{ and each } K \in \mathfrak{K} \text{ is compact}\}$.

For a thorough survey of the literature about compact-covering numbers, and for a host of new results (with emphasis on product spaces), the reader may consult the doctoral dissertation of Baloglou [3]. Many of the results of [3] are given in [4]. I am pleased to acknowledge in addition the generous help of Dr. Baloglou in connection with the present paper: He has read several preliminary versions attentively and he has made many useful suggestions--mathematical, stylistic, historical and bibliographical.

§1. The Equality $D({}^{I}\alpha) = \kappa(<\alpha>^{I})$.

It is a consequence of the Baire category theorem that the space **P** of irrational numbers is not σ-compact; that is, $\kappa(\mathbf{P}) > \omega$. From the Continuum Hypothesis (CH) of course will follow the equality $\kappa(\mathbf{P}) = 2^{\omega}$, but it is a natural, intriguing problem to attempt to determine the value of $\kappa(\mathbf{P})$ using only the usual axioms of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). This problem was first addressed by M. Katětov [14]. He showed, using a new class of spaces introduced principally for this purpose--the so-called λ-spaces--that $\kappa(\mathbf{P})$ is equal to the number $D({}^{\omega}\omega)$. (Katětov adopted the symbol *b* to denote $D({}^{\omega}\omega)$. In subsequent years mathematical usage has converged to the symbol **d**; in the present paper we follow this modern convention and we write $\mathbf{d} = D({}^{\omega}\omega)$.) The details of Katětov's derivation of the formula $\kappa(\mathbf{P}) = \mathbf{d}$ need not concern us here, but it is worth noting that the result emerges as a special case of a very general theorem: two other cardinal invariants--the so-called compact character $k\chi$ and compact pseudocharacter $k\psi$--are shown to agree on λ-spaces, and the class of λ-spaces is shown to be closed under arbitrary products and to contain each well-ordered space and each locally compact, paracompact space; that $\kappa(\mathbf{P}) = \mathbf{d}$ is derived from the relation $k\chi(\mathbf{P}) = k\psi(\mathbf{P})$ and the familiar homeomorphism $\mathbf{P} = \omega^{\omega}$. As Katětov observes [14], the formula

$k\chi(\mathbf{P}) = \kappa(\mathbf{P}) = \mathbf{d}$

shows not only that **P** is the union of **d**-many compact sets (and no fewer) but indeed that there is a family $\mathfrak{R}$ of compact subsets of **P** with $|\mathfrak{R}| = \mathbf{d}$ such that $\mathbf{P} = \cup\,\mathfrak{R}$ and such that for each compact $A \subseteq \mathbf{P}$ there is $K \in \mathfrak{R}$ such that $A \subseteq K$. In a subsequent paper [15], Katětov examined certain other naturally defined cardinal-valued topological invariants and showed that they, too, assume the value **d** on the space **P**. The formulas derived by Katětov [14], [15] are exposed, placed in a broad context, and extended by van Douwen [6](§8). The space **P** is carefully examined from a topological and set-theoretic point of view by Vaughan [20].

Because the present paper deals in part with the space ω^ω and with products of the form α^β for arbitrary (infinite) α and β, I have tried as a matter of historical curiosity to chase down in the literature the first statement of the homeomorphism theorem $\mathbf{P} \approx \omega^\omega$. It is difficult to assign credit and priority to any one mathematician for this result. Sierpiński [17](page 143) and Kuratowski [16](§14.V.3), for example, do not commit themselves on the matter. Were it not for the fact that the product topology was introduced and defined for the first time only in 1929 by Tychonoff [19], one would be tempted, following Engelking [7](page 348), to credit Baire [2] in 1909 for this homeomorphism. (Note in this connection the assertion of Engelking [7](page 121) that "Finite and countably infinite Cartesian products of metric spaces belonged to the topological folklore of the twenties.") In any event the representation of each irrational number as an infinite continued fraction is made explicit in Baire [1](§31), and in Baire [2] one finds the sentence "l'ensemble **P** se trouve décomposé en une infinité (dénombrable) d'ensembles partiels : $\mathbf{P}_1, \mathbf{P}_2, \ldots$, chacun de ceux-ci en une infinité d'ensembles partiels du deuxième ordre, etc. . . ." This sequence Δ_n of clopen partitions, each refining its predecessors and with the property $|\cap_n A_n| = 1$ whenever $A_n \in \Delta_n$ and $A_n \supseteq A_{n+1}$, is the phenomenon on the basis of which the homeomorphism $\mathbf{P} \approx \omega^\omega$ is readily constructed.

Generalizations to higher cardinals of the homeomorphism $\mathbf{P} \approx \omega^\omega$, together with applications, based on work of Hung and Negrepontis, are given in [5](§15).

The Katětov formula $\kappa(\mathbf{P}) = D({}^{\omega}\omega)$ was rediscovered independently by Hechler [10], [11], who showed that the cardinal number **d** arises in several other contexts not considered by Katětov [14], [15]. It is worthwhile to notice that Hechler's proof carries over directly to a more general result.

1.1. Theorem. Let α be an infinite cardinal and I an index set. Then $D({}^I\alpha) = \kappa(\langle\alpha\rangle^I)$.

Proof. ($\leq$) Given a compact cover $\mathfrak{K}$ of $\langle\alpha\rangle^I$, for each $K \in \mathfrak{K}$ define $f_K \in {}^I\alpha$ by

$$f_K(i) = (\max \pi_i[K]) + 1.$$

(That $\pi_i[K]$ has a largest element follows from the compactness of K and the continuity of the projection function π_i.) For every $g \in {}^I\alpha$ there is $K \in \mathfrak{K}$ such that $g \in K$, and from

$$g(i) = \pi_i(g) \leq \max(\pi_i[K]) < f_K(i) \text{ for each } i \in I$$

it follows that $g < f_K$.

($\geq$) Given a dominating family $\mathfrak{F} \subseteq {}^I\alpha$, for each $f \in \mathfrak{F}$ define $K_f \subseteq \alpha^I$ by

$$K_f = \prod_{i \in I} [0\,,f(i)],$$

where $[0\,, f(i)]$ is the "closed interval" of ordinals given by

$$[0\,, f(i)] = \{\xi < \alpha : \xi \leq f(i)\}.$$

The sets K_f are compact by the Tychonoff product theorem. To see that $\cup\,\{K_f : f \in \mathfrak{F}\} = \langle\alpha\rangle^I$ it is enough to note that for $g \in \alpha^I$ there is $f \in \mathfrak{F}$ such that $g < f$ (and hence $g \in K_f$). □

§2. The Equality $D({}^\beta\alpha) = E({}^\beta\alpha)$.

It is immediate from the definitions that $D({}^\beta\alpha) \geq E({}^\beta\alpha)$ for all infinite cardinals α and β. That equality holds in the case $\alpha = \beta = \omega$ was proved by Hechler [10], [11] by the following simple argument: Given an eventually dominating family $\mathfrak{F}$ in ${}^\omega\omega$, for $f \in \mathfrak{F}$ and $n < \omega$ define

$$\mathfrak{F}(f\,, n) = \{g \in {}^\omega\omega : g(k) = f(k) \text{ whenever } n < k < \omega\}$$

and set

$$\mathfrak{F}' = \cup\,\{\mathfrak{F}(f\,, n) : f \in \mathfrak{F}, n < \omega\}.$$

Then $\mathfrak{F}'$ is a dominating family, and from $|\mathfrak{F}| \geq \omega$ and $|\mathfrak{F}(f\,, n)| = \omega$ follows $|\mathfrak{F}'| = |\mathfrak{F}|$. Hence $D({}^\omega\omega) \leq E({}^\omega\omega)$.

It is easy to see that the analogous argument fails to show $D({}^\beta\alpha) = E({}^\beta\alpha)$ when α and β are allowed to assume arbitrary infinite values: Given an eventually dominating family $\mathfrak{F} \subseteq {}^\beta\alpha$

of course one may write

$$\mathfrak{I}(f, \zeta) = \{g \in {}^{\beta}\alpha : g(\eta) = f(\eta) \text{ whenever } \zeta < \eta < \beta\}$$

for $f \in \mathfrak{I}$ and $\zeta < \beta$, and then

$$\mathfrak{I}' = \cup\{\mathfrak{I}(f, \zeta) : f \in \mathfrak{I}, \zeta < \beta\},$$

but the relation

$$|\mathfrak{I}(f, \zeta)| = |\alpha^{\zeta}|$$

blocks the inference $|\mathfrak{I}| = |\mathfrak{I}'|$. Our aim in this section is to show that nevertheless the equality $D({}^{\beta}\alpha) = E({}^{\beta}\alpha)$ is valid. (Set theorists of our acquaintance have responded to this theorem with reactions ranging from "that is probably well-known" to "that is probably false." At our suggestion a version of this proof is given in [3]; we have been unable to locate a proof elsewhere.)

2.1. Lemma. Let α and β be infinite cardinals. Then $E({}^{\beta}\alpha) > \beta$.

Proof. Suppose that $\{f_{\zeta} : \zeta < \beta\}$ is an eventually dominating family in ${}^{\beta}\alpha$. Let $\{B_{\zeta} : \zeta < \beta\}$ be a partition of β into β-many (pairwise disjoint) subsets each of cardinality β, and define $g \in {}^{\beta}\alpha$ by

$$g(\eta) = f_{\zeta}(\eta) + 1 \text{ if } \eta \in B_{\zeta}.$$

Since each of the sets B_{ζ} is cofinal in β, the relation $g <^* f_{\zeta}$ is false for each $\zeta < \beta$. □

2.2. Lemma. Let α and β be infinite cardinals and let $\theta < \beta$. Then $D({}^{\theta}\alpha) \leq E({}^{\beta}\alpha)$.

Proof. Let $\gamma = E({}^{\beta}\alpha)$, let $\{f_{\eta} : \eta < \gamma\}$ be an eventually dominating family in ${}^{\beta}\alpha$, and let $\{B_{\zeta} : \zeta < \beta\}$ be a partition of the set

$$\beta \setminus \theta = \{\xi < \beta : \xi \geq \theta\}$$

into β-many pairwise disjoint subsets each of cardinality $|\theta|$. For $\zeta < \beta$ let h_{ζ} be a one-to-one function from θ onto B_{ζ}, and for $\eta < \gamma$ and $\zeta < \beta$ define

$$f_{\eta,\zeta} : \theta \to \alpha$$

by

$$f_{\eta,\zeta} = f_{\eta} \circ h_{\zeta}.$$

Now write $\mathfrak{I} = \{f_{\eta,\zeta} : \eta < \gamma, \zeta < \beta\}$. From Lemma 2.1 we have

$|\mathfrak{I}| \leq \gamma \cdot \beta = \gamma,$

so to complete the proof it is enough to show that $\mathfrak{I}$ is a dominating family in ${}^{\theta}\alpha$. Given $f \in {}^{\theta}\alpha$, define $f' \in {}^{\beta}\alpha$ by

$f' \mid \theta = f$, and

$f' \mid B_\zeta = f \circ h_\zeta^{-1}$,

and find $\bar{\eta} < \gamma$ such that $f' <^* f_{\bar{\eta}}$ in ${}^{\beta}\alpha$. There is $\bar{\xi} < \beta$ such that

$f'(\xi) < f_{\bar{\eta}}(\xi)$ whenever $\bar{\xi} \leq \xi < \beta$,

and since $|\bar{\xi}| < \beta$ and $\{B_\zeta : \zeta < \beta\}$ is a pairwise disjoint family there is $\bar{\zeta} < \beta$ such that $\bar{\xi} \cap B_{\bar{\zeta}} = \varnothing$--that is, such that every $\xi \in B_{\bar{\zeta}}$ satisfies $\bar{\xi} \leq \xi$. It is then clear that $f < f_{\bar{\eta},\bar{\zeta}}$ in ${}^{\theta}\alpha$: Given $\xi < \theta$ we have

$h_{\bar{\zeta}}(\xi) > \bar{\xi}$

and hence

$f(\xi) = f'(h_{\bar{\zeta}}(\xi)) < f_{\bar{\eta}}(h_{\bar{\zeta}}(\xi)) = f_{\bar{\eta},\bar{\zeta}}(\xi),$

as required. □

2.3. Theorem. Let α and β be infinite cardinals. Then $D({}^{\beta}\alpha) = E({}^{\beta}\alpha)$.

Proof. Only the inequality $\leq$ requires proof. Let $\gamma = E({}^{\beta}\alpha)$, let $\{f_\eta : \eta < \gamma\}$ be an eventually dominating family in ${}^{\beta}\alpha$, and using Lemma 2.2 for $\theta < \beta$ let $\{f_{\theta,\zeta} : \zeta < \gamma\}$ be a dominating family in ${}^{\theta}\alpha$. Now for $\eta, \zeta < \gamma$ and $\theta < \beta$ define $f_{\eta,\theta,\zeta} : \beta \to \alpha$ by

$f_{\eta,\theta,\zeta} \mid \theta = f_{\theta,\zeta}$ and

$f_{\eta,\theta,\zeta} \mid (\beta \backslash \theta) = f_\eta$,

and set $\mathfrak{I} = \{f_{\eta,\theta,\zeta} : \eta, \zeta < \gamma, \theta < \beta\}$. From Lemma 2.1 we have

$|\mathfrak{I}| = \gamma \cdot \gamma \cdot \beta = \gamma,$

so it remains only to show that $\mathfrak{I}$ is a dominating family in ${}^{\beta}\alpha$. Given $f \in {}^{\beta}\alpha$ there is $\bar{\eta} < \beta$ such that $f <^* f_{\bar{\eta}}$ in ${}^{\beta}\alpha$, so there is $\bar{\theta} < \beta$ such that $f(\xi) < f_{\bar{\eta}}(\xi)$ whenever $\bar{\theta} \leq \xi < \beta$. There is $\bar{\zeta} < \gamma$ such that

$f \mid \bar{\theta} < f_{\bar{\theta},\bar{\zeta}}$ in ${}^{\theta}\alpha$, and it is then clear that

$$f < f_{\bar{\eta},\bar{\theta},\bar{\zeta}} \text{ in } {}^{\beta}\alpha.$$

Indeed if $\xi < \overline{\theta}$ then

$$f(\xi) < f_{\bar{\theta},\bar{\zeta}}(\xi) = f_{\bar{\eta},\bar{\theta},\bar{\zeta}}(\xi),$$

and if $\overline{\theta} \leq \xi < \beta$ then

$$f(\xi) < f_{\overline{\eta}}(\xi) = f_{\bar{\eta},\bar{\theta},\bar{\zeta}}(\xi). \square$$

2.4. Remarks. The potential utility of the relation $D({}^{\beta}\alpha) = E({}^{\beta}\alpha)$ just established derives from the fact that each of the orders we have considered (that is, $<$ and $<^*$) has in appropriate situations conceptual advantages over the other. In topological contexts (see for example Theorem 1.1 above) the order $<$ and the cardinal D seem most natural, but in contexts with a strong set-theoretic flavor it is convenient to be able to discard or ignore small initial segments; see for example [9] and [13], which consider explicitly $E({}^{\omega}\omega)$ and $E({}^{\omega_1}\omega)$ in connection with forcing and infinitary combinatorics.

Let us say for an infinite cardinal λ, following Hechler [9], that a λ-*scale* in ${}^{\beta}\alpha$ is a $<^*$-cofinal subset of ${}^{\beta}\alpha$ which is order-isomorphic to λ in the order $<^*$. Hechler [9] attributes to Hausdorff [8] in 1907 the fact that the Continuum Hypothesis yields the existence of an ω_1-scale in ${}^{\omega}\omega$; see also Sierpiński [18](page 145) for this result. Both the existence and the non-existence of ω_1-scales are consistent with ZFC together with the denial of CH, and Martin's Axiom and the denial of CH imply the existence of a 2^{ω}-scale (see [9] or [12](Lemma 24.12)). As is pointed out in [3], these results on the existence of λ-scales become false if the definition is altered to refer to $<$ in place of $<^*$. Indeed, the existence of an uncountable, $<$-cofinal subset of ${}^{\omega}\omega$ which is well-ordered under $<$ (cf. Theorem 1.1 above) would imply that the space $\mathbf{P} = {}^{\omega}\omega$ is the union of a (strictly increasing) well-ordered set of compact subsets. This contradicts the topological statement, easily proved, that in a hereditarily separable space every strictly increasing chain of compact subsets has at most countable length.

§3. The Equality $\kappa(\alpha^{\beta}) = \kappa(<\alpha>^{\beta}) \cdot \sigma(\alpha,\beta)$.

Several results about compact-covering numbers of spaces of the form $<\alpha>^{\beta}$ are given in

[3], [4] by elementary computation. Examples: $\kappa(<\alpha>^\beta) = \kappa(<cf(\alpha)>^\beta)$ for all α and β; $\kappa(<\alpha>^\beta) = cf(\alpha)$ if $\beta < cf(\alpha.)$ In contrast, it is difficult to carry out concrete computations about the numbers $\kappa(\alpha^\beta)$. In view of the equalities

$D({}^\beta\alpha) = \kappa(<\alpha>^\beta)$ and $D({}^\beta\alpha) = E({}^\beta\alpha)$,

for example, it follows from remarks of Jech and Prikry [13] that it is unknown whether the inequality $\kappa(\omega^{\omega_1}) < 2^{\omega_1}$ is consistent with the axioms of ZFC; and Hechler [9], [11] has shown that the Katĕtov number $\mathbf{d} = \kappa(\omega^\omega)$ can take on essentially any value consistent with the constraints $\omega_1 \leq \mathbf{d} \leq 2^\omega$, $cf(\mathbf{d}) > \omega$. In the following theorem and corollary, mild generalizations of results proved in [3], [4], we give a relation of recursive type in which the space α^β is replaced by $<\alpha>^\beta$ and by spaces λ^β for $\lambda < \beta$.

3.1. Notation. For infinite cardinals α and β, we write

$\sigma(\alpha, \beta) = \sup\{\kappa(\prod_{\eta<\beta}\lambda_\eta) : \lambda_\eta < \alpha\}$.

We note that if $\alpha = \omega$ the spaces $\prod_{\eta<\beta}\lambda_\eta$ are compact and we have $\sigma(\alpha, \beta) = 1$.

3.2. Theorem. Let α and β be infinite cardinals. Then

$\kappa(\alpha^\beta) = \kappa(<\alpha>^\beta)\cdot\sigma(\alpha, \beta)$.

Proof. If $\alpha = \omega$ we have $\sigma(\alpha, \beta) = 1$ and $\alpha^\beta = <\alpha>^\beta$, so the statement is obvious. We assume therefore that $\alpha > \omega$.

($\geq$) It is enough to prove that $\kappa(\alpha^\beta) \geq \kappa(<\alpha>^\beta)$ and $\kappa(\alpha^\beta) \geq \sigma(\alpha, \beta)$. The first of these inequalities is obvious, since there is a continuous function from α^β onto $<\alpha>^\beta$. For the second, note that for each choice $\{\lambda_\eta : \eta < \beta\}$ of cardinals with $\lambda_\eta < \alpha$, the (discrete) space λ_η is closed in the (discrete) space α; thus $\prod_{\eta<\beta}\lambda_\eta$ is closed in α^β and we have

$\kappa(\alpha^\beta) \geq \kappa(\prod_{\eta<\beta}\lambda_\eta)$.

It follows that $\kappa(\alpha^\beta) \geq \sigma(\alpha, \beta)$, as required.

($\leq$) Given a compact subset K of $<\alpha>^\beta$, for $\eta < \beta$ let $\lambda_\eta = |\pi_\eta[K]|$. The set $\pi_\eta[K]$ now has two topologies--one inherited from $<\alpha>$ and the other discrete. The product space $\prod_{\eta<\beta}\pi_\eta[K]$ accordingly has two topologies; K is closed in the first, hence in the second, so K is covered by $\kappa(\prod_{\eta<\beta}\lambda_\eta)$-many compact subsets of $\prod_{\eta<\beta}\lambda_\eta$.

It follows from the preceding paragraph that for every compact subset K of $<\alpha>^\beta$ there is a family $\mathfrak{R}(K)$ of compact subsets of α^β such that $|\mathfrak{R}(K)| \leq \sigma(\alpha, \beta)$ and $K \subseteq \cup\, \mathfrak{R}(K)$. Taking $\{K_\xi : \xi < \gamma\}$ a compact cover of $<\alpha>^\beta$ with $\gamma = \kappa(<\alpha>^\beta)$ and defining

$$\mathfrak{R} = \cup_{\xi < \gamma} \mathfrak{R}(K_\xi),$$

we see that $\mathfrak{R}$ is a compact cover of α^β with

$$|\mathfrak{R}| \leq \gamma \cdot \sigma(\alpha, \beta) = \kappa(<\alpha>^\beta) \cdot \sigma(\alpha, \beta). \square$$

In the following corollary we denote as usual by α^+ the smallest cardinal greater than α.

3.3. Corollary. Let α and β be infinite cardinals. Then

(a) $\kappa((\alpha^+)^\beta) = \kappa(<\alpha^+>^\beta) \cdot \kappa(\alpha^\beta)$; and

(b) if $\beta < \mathrm{cf}(\alpha)$ then $\kappa(\alpha^\beta) = \kappa(<\alpha>^\beta) \cdot \sum_{\lambda < \alpha} \kappa(\lambda^\beta)$.

Proof. (a) It is clear in this case that

$$\sigma(\alpha^+, \beta) = \kappa(\alpha^\beta).$$

(b) It is enough to show that

$$\sigma(\alpha, \beta) = \sum_{\lambda < \alpha} \kappa(\lambda^\beta).$$

The inequality $\geq$ is clear. For each set $\{\lambda_\eta : \eta < \beta\}$ of cardinals with $\lambda_\eta < \alpha$ there is $\lambda < \alpha$ such that each λ_η satisfies $\lambda_\eta < \lambda$, and we have $\kappa(\prod_{\eta < \beta} \lambda_\eta) \leq \kappa(\lambda^\beta)$; the inequality $\leq$ is now immediate. $\square$

Corollary 3.3(a) and some of its consequences are recorded in [3], [4]. Let us notice in particular, writing as usual

$\omega_0 = \omega$ and

$\omega_{n+1} = (\omega_n)^+$ for $n < \omega$,

that the identities

$$\kappa(\omega_n{}^\omega) = \omega_n \cdot \mathbf{d}$$

are easily established by induction; in particular one has

$$\mathbf{d} = \kappa(\omega^\omega) = \kappa(\omega_1{}^\omega)$$

in (every model of) ZFC.

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