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# SOME REMARKS AND APPLICATIONS OF AN EXTENSION OF A LEMMA OF KY FAN

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**Abstract.** We apply a recent theorem of Lin, that is a generalization of a well known Lemma of Ky Fan, in order to obtain some extensions of results of Lin, Prolla and Park. It is pointed out that the theorem of Lin is equivalent to a fixed point theorem, here used to extend some fixed point theorems of Browder. Two coincidence theorems are established in the last Section generalizing known results of Browder, Komiya, Mehta and Tarafdar.

**Key words**: almost affine map, coincidence theorem, fixed point, Inward set.

**Classification**: 47H10, 49A40.

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**1. Introduction.** All topological vector spaces considered are real and are assumed tacitly to be separated.

In 1961, Fan [6, Lemma 1] established his infinite-dimensional generalization of the famous KKM-theorem [14]. Later he was able to relax the compactness condition of his famous Lemma in Theorem 4 of [8], that was used by Lin [16, Theorem 1] obtaining the following generalization of [6, Lemma 4].

**Theorem 1.** Let $X$ be a nonempty convex subset of a topological vector space. Let $A \subseteq X \times X$ be a subset such that

(a) for each $x \in X$, the set $\{y \in X : (x,y) \in A\}$ is closed in $X$;

(b) for each $y \in X$, the set $\{x \in X : (x,y) \notin A\}$ is convex or empty;

(c) $(x,x) \in A$ for each $x \in X$;

(d) $X$ has a nonempty compact convex subset $X_0$ such that the set $B = \{y \in X : (x,y) \in A$ for all $x \in X_0\}$ is compact.

Then there exists a point $y_0 \in B$ such that $X \times \{y_0\} \subseteq A$.

Lin [16] derived some consequences from Theorem 1: among the others, a generalization of a variational inequality of Browder [2] and Theorem 7 of Fan [8]. An aim is to apply Theorem 1 to extend Theorem 3 of [16] and a result of Prolla [19]. An equivalent formulation as a fixed point theorem is pointed out and is used to generalize some results of Browder [3].

Two coincidence theorems are also established: the first one is an immediate extension of a recent result of Komiya [15], the second one is a generalization of Theorem 2.2 of Mehta [17].

**2. Some applications.** Following Halpern and Bergman [12], we recall that the inward set of a subset X of a topological vector space E at a point $x \in X$ is defined as

$$I(X,x) = \{x+r\cdot(u-x)\in E: w\in X,\ r>0\}.$$

clS denotes the closure in E of a subset S of E. Following Prolla [19], a mapping $g : X \to E$, where X is a convex subset of a normed vector space E, is said to be almost affine on X if

$$||g(\lambda\cdot x_1+(1-\lambda)\cdot x_2)-y|| \leq \lambda\cdot||g(x_1)-y||+(1-\lambda)\cdot||g(x_2)-y||$$

for all $x_1,x_2 \in X$, $0<\lambda<1$ and $y \in E$. Clearly any affine mapping $g:X\to E$ is almost affine on X.

Using now Theorem 1, we unify Theorem 3 of [16] and the result of [19].

**Theorem 2.** Let X be a nonempty convex subset of a normed vector space E, $f,g:X\to E$ be continuous and g be almost affine on X.

(d) If X has a nonempty compact convex subset $X_0$ such that the set $B=\{y\in X: ||g(x)-f(y)|| \geq ||g(y)-f(y)||$ for all $x\in X_0\}$ is compact, then there exist a point $y_0 \in B$ such that

$$||g(y_0)-f(y_0)||=\min\{||w-f(y_0)||: w\in g(X)\}. \tag{1}$$

**Proof.** Let

$$A=\{(x,y)\in X\times X: ||g(x)-f(y)|| \geq ||g(y)-f(y)||\}.$$

Then the set $\{y\in X:(x,y)\in A\}$ is closed in X since g and f are continuous. Evidently $(x,x)\in A$ for each $x\in X$ and the set

$$C=\{x\in X:(x,y)\notin A\}=\{x\in X:||g(x)-f(x)||<||g(y)-f(y)||\}$$

is convex or empty. Indeed, since g is almost affine on X, we have that

$$||g(\lambda x_1+(1-\lambda)x_2)-f(y)||\leq$$
$$\lambda\cdot||g(x_1)-f(y)||+(1-\lambda)\cdot||g(x_2)-f(y)||<||g(y)-f(y)|| \quad \text{for } x_1,x_2\in C.$$

By Theorem 1, there exists a point $y_0\in B$ such that

$$||g(y_0)-f(y_0)||\leq||g(x)-f(y_0)||$$

for each $x\in X$ and hence (1).

**Remark 1.** (i) As is stressed by Lin [16], condition (d) of Theorem 2 can be replaced by the following condition:

(d') Let $X_0$ be a nonempty compact convex subset of X, K be a nonempty compact subset of X such that for each $y\in X-K$, there exist a point $x\in X_0$ such that

$$||g(x)-f(y)||<||g(y)-f(y)||.$$

Then the conclusion of Theorem 2 will be: there exists a point $y_0\in K$ such that (1) holds.

(ii) If $X=K=X_0$, (d') is automatically satisfied and if $g(X)=X$, Theorem 2 becomes the theorem of [19], where the author proved his result using the fixed point theorem of Bohenblust and Karlin [1].

(iii) If g=identity on X, Theorem 2 is Theorem 3 of [16].

**Remark 2.** If $g(X)$ is convex, then (1) implies that

$$||g(y_0)-f(y_0)||=\min\{||z-f(y_0)||:z\in \mathrm{cl}X'\},$$

where $X'=I(g(X),g(y_0))$. Indeed, following a well known argument (e.g. Park [18]), let z be a point of $X'-g(X)$.

Hence, there exist $u\in g(X)$ and $r>0$ such that $z=g(y_0)+r\cdot(u-g(y_0))$. We have $r>1$ otherwise the convexity of $g(X)$ should imply

$$r\cdot u+(1-r)\cdot g(y_0)=z\in g(X),$$

a contradiction. Thus $(1/r)\cdot z+(1-1/r)\cdot g(y_0)=u\in g(X)$ and (1) implies that

$$\begin{aligned}||g(y_0)-f(y_0)|| &\le ||u-f(y_0)||\\ &\le 1/r\cdot||z-f(y_0)||+(1-1/r)\cdot||g(y_0)-f(y_0)||,\end{aligned}$$

from which $||g(y_0)-f(y_0)||\le||z-f(y_0)||$ for each $z\in X'$ and then for each $z\in \mathrm{cl}X'$.

For g=identity on X, similar consideration can be made on Theorem 2 of [16], here enunciated for sake of completeness.

**Theorem 3.** Let X be a nonempty convex subset of a locally convex topological vector space E, $f:X\to E^*$ be continuous.

(d) If X has a nonempty compact convex subset $X_0$ such that the set $B=\{y\in X:(fy,y-x)\ge 0 \text{ for all } x\in X_0\}$ is compact,

then there exists a point $y_0\in B$ such that $(fy_0,y_0-z)\ge 0$ for all $z\in \mathrm{cl}I(X,y_0)$.

**Remark 3.** Theorem 3 generalizes (in the inward case) Theorem 2 of Park [18], which in turn strenghtens Browder [2, Theorem 2].

**3. Fixed point theorems.** Fan [7] pointed out the equivalence of his intersection Lemma to the fixed point theorem of Browder [2, Theorem 1].

Similarly, Theorem 2 is equivalent to the following result.

**Theorem 4.** Let X be a nonempty convex subset of a topological vector space and $T:X\to 2^X$ be a mapping such that

(a) for each $x\in X$, $T(x)$ is a nonempty convex subset of X;

(b) for each $y\in X$, $T^{-1}(y)=\{x\in X:y\in T(x)\}$ is open in X;

(c) X has a nonempty compact convex subset $X_0$ such that the set

$$B=X-\bigcup_{y\in X_0}T^{-1}(y)$$

is compact.

Then there exists a point $x_0\in X$ such that $x_0\in T(x_0)$.

Browder proved his Proposition 2 of [3] using Theorem 1 of [2]. In accordance to this idea, we now use Theorem 4 to generalize Proposition 2 of [3].

**Theorem 5**. Let $X$ be a nonempty convex subset of a topological vector space $E, f:X\to E$ and $p:C\times E\to\mathfrak{R}$ (reals) be continuous such that

(a) for each $x\in X$, the set $\{y\in E:p(x,y)<r\}$ is convex for each $r\in\mathfrak{R}$;

(b) $X$ has a nonempty compact convex subset $X_0$ such that the set $B=\{x\in X:p(x,y-f(x))\geq p(x,x-f(x))$ for all $y\in X_0\}$ is compact;

(c) for each $x\in X$ with $x\neq f(x)$, there exists a point $y\in X$ such that $p(x,y-f(x))<p(x,x-f(x))$.

Then $f$ has a fixed point in $C$.

**Proof**. If $f$ has no fixed points in $C$, then the set

$$T(x)=\{y\in X:p(x,y-f(x))<p(x,x-f(x))\}$$

is nonempty and convex by (a) and (c). Further, the continuity of $p$ and $f$ implies that

$$X-T^{-1}(y)=\{x\in X:p(x,y-f(x))\geq p(x,x-f(x))\}$$

is closed for each $y\in X$. Moreover, (b) implies that

$$B=\bigcap_{y\in X_0}[X-T^{-1}(y)]=X-\bigcup_{y\in X_0}T^{-1}(y).$$

By Theorem 4, there exists a point $x_0\in Tx_0$ for which $p(x_0,x_0-f(x_0))<p(x_0,x_0-f(x_0))$, a contradiction.

We now generalize Theorem 1 of Browder [3].

**Theorem 6**. Let $X$ be a nonempty convex subset of a topological vector space $E, f:X\to E$ and $p:X\times E\to\mathfrak{R}$ be continuous such that

(a') for each $x\in X, p(x,\cdot)$ is a convex function on $X$;

(b') as (b) of Theorem 5;

(c') for each $x\in X$ with $x\neq f(x)$, there exists $y\in I(X,x)$ such that $p(x,y-f(x))<p(x,x-f(x))$.

Then $f$ has a fixed point in $C$.

**Proof**. Clearly (a') implies (a) and, as in [3, p.4760], one proves that (c') implies (c). Hence the thesis follows from Theorem 5.

**4. Coincidence theorems**. Clearly Theorem 5 is a particular case of Theorem 3.1 of Mehta [17] or Theorem 1.2 of Tarafdar [21]. These authors proved their results using Theorem 4 of [8], but Tarafdar [22] gave a proof of his result independent of Theorem 4 of [8] using his fixed point theorem [20] and he also proved the equivalence of his result with the cited theorem of Fan.

Inspired by some recent papers, we like to point out here a coincidence theorem that unifies the fixed point theorems of Tarafdar [20, Theorem 1] and Komiya [15, Theorem 1]. We omit the proof since it is obtained, as in [15], using Lemma 1 of Ha [10] and Proposition 1 of Browder [4].

**Theorem 7**. Let X be a nonempty convex subset of a topological vector space E and let Y be a nonempty compact convex subset of a topological vector space F. Let $S:X\to 2^Y$ be an upper semicontinuous mapping and $T:Y\to 2^X$ be a mapping such that

(a) for each $x\in X$, $S(x)$ is a nonempty closed convex subset of Y;

(b) for each $y\in X$, $T(y)$ is a nonempty convex subset of X;

(c) for each $x\in X$, $T^{-1}(x)$ contains a relatively open subset $0(x)$ of $Y (0(x)$ could be empty for some $x)$;

(d) $\bigcup_{x\in X} 0(x)=Y$.

Then there exist points $x_0\in X, y_0\in Y$ such that $y_0\in S(x_0)$ and $x_0\in T(y_0)$.

**Remark 4**. (i) Note that if $T^{-1}(x)$ is open in Y for each $x\in X$, we have Theorem 1 of [15], which in turn generalizes Theorem 3 of [4].

(ii) If E=F, X=Y and $Sx=\{x\}$ for each $x\in X$, we deduce Theorem 1 of [20], which in turn includes Theorem 1 of [2].

For convenience of the reader, we recall that a mapping $S:X\to 2^Y$ is upper semicontinuous iff $S^{-1}(B)=\{x\in X:S(x)\cap B\neq\emptyset\}$ is closed for any closed subset B of Y.

Mehta [17], proved in Theorem 2.2 that the compactness of the domain of the multifunction, required in the results of Browder and Tarafdar, can be weakened in locally convex topological vector spaces setting. His method of proof relies on a partition of unity argument to derive a continuous selection, whose fixed point is the fixed point of the assigned multifunction. The existence of this fixed point is guaranteed by a well known result of Himmelberg [13], that here we use to obtain the following coincidence theorem.

**Theorem 8.** Let X be a nonempty paracompact convex subset of a locally convex topological vector space E. Let D be a nonempty subset of E, $S:D\to 2^X$ be an upper semicontinuous mapping and $T:X\to 2^D$ be a mapping such that

(a) $\forall y\in D$, Sy is a nonempty closed convex subset of X;

(b) $\forall x\in X$, Tx is a nonempty convex subset of D;

(c) $S(D)=\bigcup_{y\in D} S(y)\subseteq C$, where C is a compact subset of X;

(d) for each $x\in X$, there exists a point $y\in D$ such that $x\in \operatorname{int}T^{-1}(y)$, where $\operatorname{int}T^{-1}(y)$ denotes the topological interior of $T^{-1}(y)$.

Then there exist points $\bar{x}\in X$, $\bar{y}\in D$ such that $\bar{x}\in S(\bar{y})$ and $\bar{y}\in T(\bar{x})$.

**Proof.** We adopt essentially the proof of Theorem 2.2 of [17] with some slight variants.

By putting $O(y)=\operatorname{int}T^{-1}(y)$ for any $y\in D$, we have that the family $\{O(y):y\in D\}$ is an open covering by (d) of the paracompact space X and let $\{f_y:y\in D\}$ be a partition of unity by continuous nonnegative real functions defined on X subordinate to this covering such that $\operatorname{supp}f_y \subseteq O(y)$ for each $y\in D$. We note that $\{\operatorname{supp} f_y:y\in D\}$ is a locally finite closed covering of X and $\sum_{y\in D} f_y(x)=1$ for each $x\in X$.

Define a continuous function $f:X\to D$ by setting for each $x\in X$,

$$f(x)=\sum_{y\in D} f_y(x)\cdot y.$$

If $x\in X$ and $f_y(x)\neq 0$, then $x\in \operatorname{supp}f_y\subseteq O(y)\subseteq T^{-1}(y)$, i.e. $y\in T(x)$ and $f(x)\in T(x)$ since $T(x)$ is convex for each $x\in X$ by (b), being $f(x)$ a convex combination of a finite number of points y of $T(x)$. We now define a mapping $H:X\to 2^X$ by putting $H(x)=S(f(x))$ for each $x\in X$. Since f is continuous and S is upper semicontinuous, H is upper semicontinuous. Further, $H(x)$ is a nonempty closed convex subset of X for each $x\in X$ by (a) and $H(X)\subseteq S(D)\subseteq C\subseteq X$. Since C is compact by (c), applying the fixed point theorem of Himmelberg [13], there exists a point $\bar{x}\in X$ such that $\bar{x}\in H(\bar{x})=S(\bar{y})$ where $\bar{y}=f(\bar{x})\in T(\bar{x})\subseteq D$.

**Remark 5.** (i) If D is a compact subset of X and $S(x)=\{x\}$ for each $x\in D$, Theorem 8 becomes Theorem 2.2 of [17].

(ii) As stressed by Mehta [17], any subset of a locally convex space is of Zima type [11]. Thus in Theorem 8 one may assume X to be of Zima type and E to be a topological vector space not necessarily locally convex, but in the proof one must use Theorem 5 of Hadžić [11, p.136] that generalizes the cited theorem of Himmelberg [13].

(iii) If C=X, then in the proof of Theorem 8 one may use the classical fixed point theorem of Fan [5] or Glicksberg[9].

Further coincidence theorems can be found in [4], [11] and [15].

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