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Antonín Sochor

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ADDITION OF INITIAL SEGMENTS I

Antonín SOCHOR

Abstract: In the alternative set theory, for every real initial segment $R \subseteq N$ there is either $\xi \in R$ with $R = \{\vartheta : (\exists \alpha \in R^+) \vartheta \leq \xi + \alpha\}$ or $\xi \in N - R$ with $R = \{\vartheta < \xi ; (\forall \alpha \in R^+) \vartheta + \alpha < \xi\}$ where $R^+ = \{\vartheta \in R ; (\forall \alpha \in R) \vartheta + \alpha + 1 \in R\}$. This result can be used in measure theory. More generally, we extend addition and subtraction to the system of all initial segments of N and we investigate properties of these operations. In particular, we describe the behaviour of these operations on all initial segments which are real classes. Further properties of these operations can be found in the following paper [S].

Key words: Alternative set theory, natural number, finite natural number, initial segment, real class, π-semiset, σ-semiset.

Classification: Primary 03E70
Secondary 03H15

We use the notions usual in the alternative set theory (AST; see [V]), in particular the symbols N and FN denote the class of all natural numbers and the class of all finite (in another terminology standard) natural numbers respectively. A class X is called a π-semiset (σ -semiset resp.) if there is a sequence $\{x_n ; n \in FN\}$ with

$X = \bigcap \{x_n ; n \in FN\}$ ($X = \bigcup \{x_n ; n \in FN\}$ resp.).

Complete subclasses of N are called initial segments and cuts are initial segments closed under the successor operation.

The most important axiom of AST is the prolongation axiom i.e. the statement

$(\forall F)((Fnc(F) \& dom(F)=FN) \rightarrow (\exists f)(Fnc(f) \& F \subseteq f))$.

Let us recall that every initial segment which is simultaneously π-semiset and σ-semiset is a set (cf. § 5 ch. II [V]; this statement is a consequence of the prolongation axiom) and that the sole cut which is a set is the empty set 0 (this assertion is implied by induction accepted for sets).

The system of real classes defined in [Č-V] plays an important role in

AST. In this paper we need only one property of real classes (proved in the cited article), namely that every initial segment which is simultaneously a subclass of a set and real, is either a π-semiset or a σ-semiset. However, let us remind at least that all sets, π-semisets and σ-semisets are real classes and that for every property $\Phi(z,Z_1,\dots,Z_n)$ in which only real classes are quantified and for all real classes $X_1,\dots,X_n$, the class

$$\{x;\ \Phi(x,X_1,\dots,X_n)\}$$

is real, too.

We use letters R, S, T and U to denote initial segments; the letters α, β ... denote natural numbers and the letters k, n and m are reserved for variables running through finite natural numbers.

Following a Zlatoš´s idea, we define for every initial segment R

$$R'=R\cup\{R\}$$

i.e. we put $R'=R$ for every nonempty cut and $\alpha'=\alpha+1$ for each $\alpha\in N$. Evidently R' is a nonempty initial segment and we have

$$\alpha\in R\equiv\alpha+1\in R'\ \&\ R=\{\vartheta;(\exists\alpha\in R')\vartheta<\alpha\}\&(R=S\equiv R'=S')\&$$
$$\&(R\subseteq S\equiv R'\subseteq S').$$

We are going to define addition and subtraction on the system of all initial segments; to avoid misunderstanding, we use for the operation of subtraction the symbol $\dot{-}$ because our operation extends subtraction defined on natural numbers, however, it does not extend subtraction defined on the class of integers (see (2)) which operation is denoted by the symbol - . (Let us mention that the symbol - is used in AST also to denote the difference of classes $X-Y=\{x\in X;x\notin Y\}$.)

For every two initial segments R, S, we define their sum by

$$R+S=\{\vartheta;(\exists\alpha\in R')(\exists\beta\in S')\vartheta<\alpha+\beta\}$$

and their difference by

$$R\dot{-}S=\{\vartheta;(\forall\beta\in S')\vartheta+\beta\in R\}=$$
$$=\{\vartheta;(\forall\beta\in S')(\exists\alpha\in R)\beta\leq\alpha\ \&\ \vartheta\leq\alpha-\beta\}.$$

The class $R\dot{-}R$ plays an important role in our investigation and we are going to denote it by the symbol R^+, i.e. we define (cf. (1) and (3c))

$$R^+=\{\vartheta\in R;(\forall\alpha\in R)\alpha+\vartheta+1\in R\}.$$

We say that an initial segment R is closed under the operation + iff

$$(\forall\alpha,\beta\in R)\alpha+\beta\in R.$$

If a cut R is closed under the operation + , then R' is also closed under the operation + , (because $0' = \{0\}$ is closed under +). Let us mention that $\{0\}$ is closed under the operation + , however, it is no cut (and $\{0\}' = \{0,1\}$ is not closed under +).

In the following we summarize some useful statements concerning the above defined operations starting with the trivial ones:

(1) a) $R \subseteq R+S$ and $R \dot{-} S \subseteq R$

because $0 \in S'$ for every S and since

$$\alpha \in R \rightarrow (\alpha < \alpha+1+0 \mathbin{\&} \alpha+1 \in R'). \quad \square$$

b) $R \subset S \rightarrow R \dot{-} S = 0$

because for $\beta \in S$ with $\beta \notin R$ and every $\vartheta \in N$ we have

$$\vartheta + \beta \geq \beta \notin R \mathbin{\&} \beta \in S'. \quad \square$$

(2) The operations + and $\dot{-}$ defined above extend the arithmetical addition and subtraction and, moreover, for $\xi \leq \zeta \in N$ we have

$$\xi \dot{-} \zeta = 0.$$

In fact for every ξ, $\zeta \in N$ we have

$$\{\vartheta ; \vartheta < \xi + \zeta\} = \{\vartheta ; (\exists \tau \leq \xi)(\exists \overline{\tau} \leq \zeta)\, \vartheta < \tau + \overline{\tau}\} =$$
$$= \{\vartheta ; (\exists \tau \in \xi')(\exists \overline{\tau} \in \zeta')\, \vartheta < \tau + \overline{\tau}\} = \xi + \zeta,$$

for every $\zeta \leq \xi \in N$ we have

$$\xi - \zeta = \{\vartheta ; \vartheta < \xi - \zeta\} = \{\vartheta ; \vartheta + \zeta < \xi\} = \{(\forall \tau \leq \zeta)\, \vartheta +$$
$$+ \tau < \xi\} = \{\vartheta ; (\forall \tau \in \zeta')\, \vartheta + \tau < \xi\} = \xi \dot{-} \zeta$$

and the last statement is a trivial consequence of (1b). $\square$

(3) a) $R+0=R=R \dot{-} 0$ and $0 \dot{-} R = 0 = 0^+$. $\square$

b) $(R+S)' = \{\vartheta ; (\exists \alpha \in R')(\exists \beta \in S')\, \vartheta \leq \alpha + \beta\}$

$(R \dot{-} S)' = \{\vartheta ; (\forall \beta \in S')\, \vartheta + \beta \in R'\}$ (assuming $S \subseteq R$)

$(R^+)' = \{\vartheta ; (\forall \alpha \in R')\, \vartheta + \alpha \in R'\}$.

The statements are trivial consequences of the definitions, however, one has to distinguish whether the initial segments in question are sets or proper classes. $\square$

c) If $S \neq 0$, then

$R+S = \{\vartheta ; (\exists \alpha \in R')(\exists \beta \in S)\, \vartheta \leq \alpha + \beta\}$

$(R+S)' = \{\vartheta ; (\exists \alpha \in R')(\exists \beta \in S)\, \vartheta \leq \alpha + \beta + 1\}$

$R \dot{-} S = \{\vartheta ; (\forall \beta \in S)\, \vartheta + \beta + 1 \in R\}$

$(R \dot{-} S)' = \{\vartheta ; (\forall \beta \in S)\, \vartheta + \beta \in R\}$

$S^+ = \{\vartheta ; (\forall \beta \in S)\, \vartheta + \beta + 1 \in S\} = S \dot{-} S$ $\square$

d) If $R \neq 0 \neq S$, then

$R+S = \{\vartheta ; (\exists \alpha \in R)(\exists \beta \in S)\, \vartheta \leq \alpha + \beta + 1\}$. $\square$

(4) The operation + defined on the system of all initial segments is associative and commutative because of the associativity and commutativity

of the arithmetical addition (and because of (3b)). $\square$

(5) a) $S \subseteq R^+ \rightarrow R+S=R=R \dot{-} S$

because $S \subseteq R^+$ implies by (3b)

$$(\forall \beta \in S')(\forall \alpha \in R')\, \beta + \alpha \in R'$$

and thus we get

$$R+S \subseteq \{\vartheta ; (\exists \alpha \in R')\, \vartheta < \alpha\} = R$$

and furthermore (3b) implies also the formula

$$(\forall \alpha \in R)(\forall \tau \in (R^+)')\, \alpha + \tau \in R. \quad \square$$

b) $R^+ \subset S \rightarrow R \subset R+S$.

We have $(R^+)' \subset S'$ and thus there is $\beta \in S'$ with $\beta \notin (R^+)'$ and by (3b) we get

$$(\exists \alpha \in R')\, \beta + \alpha \notin R';$$

this shows $R' \subset (R+S)'$. $\square$

c) $(R^+ \subset S \,\&\, R \neq 0) \rightarrow R \dot{-} S \subset R$.

For $\beta \in S$ with $\beta \notin R^+$ there is $\alpha \in R$ (see (3c)) so that $\alpha + \beta + 1 \notin R$ and α being an element of R is no element of $R \dot{-} S$ because $\beta + 1 \in S'$. $\square$

d) If both R and S are cuts closed under the operation + , then

$$R+S=R \cup S.$$

According to (3b) we have

$(R+S)'=R' \cup S'=(R \cup S)'$ because both R' and S' are closed under the operation + . $\square$

(6) $(R \dot{-} S) \dot{-} T = R \dot{-} (S+T)$.

The formulae

$$\vartheta \in (R \dot{-} S) \dot{-} T$$
$$(\forall \tau \in T')\, \vartheta + \tau \in (R \dot{-} S)$$
$$(\forall \tau \in T')(\forall \beta \in S')\, \vartheta + \tau + \beta \in R$$
$$(\forall \delta \in (T+S)')\, \vartheta + \delta \in R$$
$$\vartheta \in (R \dot{-} (T+S))$$

are equivalent by (3b). $\square$

(7) If $R \subseteq T$ abd $S \subseteq U$, then

$$R+S \subseteq T+U$$

and

$$R \dot{-} U \subseteq T \dot{-} S. \quad \square$$

(8) a) If $S \subseteq R$, then $R \dot{-} S$ is the greatest T with $T+S \subseteq R$.

The formulae

$$(R \dot{-} S)+S \subseteq R$$
$$(\forall \vartheta \in (R \dot{-} S)')(\forall \beta \in S')\, \vartheta + \beta \in R'$$
$$(\forall \vartheta [(\forall \beta \in S')(\vartheta + \beta \in R') \rightarrow (\forall \beta \in S')\, \vartheta + \beta \in R')]$$

are equivalent; assuming

$$\tau \in T \subsetneq R \,\&\, \tau \notin R \dot{-} S,$$

we are able to find $\beta \in S'$ with $\tau + \beta \notin R$ and it is $\tau + \beta \in T+S$ because

$$\tau + 1 \in T' \,\&\, \tau + \beta < (\tau + 1) + \beta .$$

b) If $R \neq 0$, then $R+S$ is the smallest U with $U \dot{-} S \supseteq R$.

The formulae

$$(R+S) \dot{-} S \supseteq R$$

and

$$(\forall \alpha \in R)(\forall \beta \in S')(\exists \bar{\alpha} \in R')\, \alpha + \beta < \bar{\alpha} + \beta$$

are equivalent and the second one is valid (put $\bar{\alpha} = \alpha + 1$); supposing

$$\vartheta \notin U \,\&\, \vartheta \in R+S \,\&\, R \neq 0$$

we can find $\alpha \in R'$ and $\beta \in S'$ with

$$\vartheta < \alpha + \beta \,\&\, \alpha \neq 0$$

and then

$$\alpha - 1 \in R \,\&\, \alpha - 1 \notin U \dot{-} S. \quad \square$$

(9) a) The formulae

R is a cut

$R+FN=R$

$R \dot{-} FN=R$

are equivalent. To prove this assertion it is sufficient to realize that FN is the smallest nonempty cut. $\square$

b) For every ξ, $\xi + FN$ is the smallest cut containing ξ and $\xi \dot{-} FN$ is the maximal cut not containing ξ.

These statemen ts are trivial consequences of a) and (1a),(2),(5d), (6) and (8). $\square$

At the end of this paper we are going to give some examples of cuts R, S such that $(R \dot{-} S)+S \subset R$, however, for every R, S if there is T with $T+S=R$, then $(R \dot{-} S)+S=R$; similarly there are R, S with $(R+S) \dot{-} S \supset R$, however, for every R, S , if there is U with $U \dot{-} S=R$, then $(R+S) \dot{-} S=R$.

The statement (10) which is an immediate consequence of (8)) gives us a description of couples R, S for which the equalities $(R \dot{-} S)+S=R$ and $(R+S) \dot{-} S=R$ are true; the question whether there is a better description of such couples is left as an open problem in this paper.

(10) a) If $R \dot{-} S \neq 0$, then $(R \dot{-} S)+S=R$ iff R is the smallest T with $T \dot{-} S \supseteq R \dot{-} S$.

b) If $R \neq 0$, then $(R+S) \dot{-} S=R$ iff R is the greatest U with $U+S \subseteq R+S$. $\square$

(11) $R \dot{-} S = \{\vartheta ; (\forall \gamma \notin R)(\vartheta < \gamma \,\&\, \gamma - \vartheta - 1 \notin S\}$.

If $S=0$, then the assertion is trivial; supposing $S \neq 0$ we can use (3c). If $\vartheta \in R \dot{-} S$ and $\gamma \notin R$, then $\vartheta < \gamma$ because $R \subseteq \gamma$ and the assumption

$\gamma-\vartheta-1\in S$ would imply $\gamma-\vartheta\in S'$ and thus it would imply

$$\gamma=\vartheta+(\gamma-\vartheta)\in R$$

according to the definition of $R\div S$. If $\vartheta\notin R\div S$, then there is $\beta\in S$ with $\vartheta+\beta+1\notin R$. Evidently

$$((\vartheta+\beta+1)-\vartheta)-1=\beta\in S$$

and therefore

$$\neg(\forall\gamma\notin R)(\vartheta<\gamma\,\&\,\gamma-\vartheta-1\notin S).\quad\square$$

As a trivial consequence we get

(12) $R^+=\{\vartheta\in R;(\forall\gamma\notin R)(\vartheta<\gamma\,\&\,\gamma-\vartheta-1\notin R)\}$. $\square$

(13) a) If $S\neq 0$, then

$$\xi+S=\{\vartheta;(\exists\beta\in S)\vartheta\leq\xi+\beta\}$$

and

$$(\xi+S)'=\{\vartheta;(\exists\beta\in S')\vartheta\leq\xi+\beta\}.\quad\square$$

b) If $S\subset\xi$, then

$$\xi\div S=\{\xi\div\delta;0<\delta\,\&\,\delta-1\notin S\}$$

and

$$(\xi\div S)'=\{\xi\div\delta;\delta\notin S\}.$$

If $\vartheta\in\xi\div S$, then

$$(\forall\beta\in S')\vartheta+\beta<\xi$$

which implies $\xi-\vartheta\notin S'$ and thence

$$\xi-\vartheta>0\,\&\,\xi-\vartheta-1\notin S.$$

On the other hand if $0<\delta\,\&\,\delta-1\notin S$, then $(\forall\beta\in S')\beta<\delta$ and hence

$$(\xi\div\delta)+\beta\leq\max(\beta,\xi-(\delta-\beta))<\xi$$

for every $\beta\in S'$. $\square$

(14) a) $R^+\subseteq R$ and R^+ is a cut closed under the operation $+$.

Really, if $\vartheta,\tau\in R^+$ and if $\alpha\in R$, then

$$\alpha+(\vartheta+\tau)+1<(\alpha+\vartheta+1)+\tau+1\in R$$

because $\alpha+\vartheta+1\in R$. If $R^+=0$, then it is a cut trivially; otherwise $0\in R^+$ i.e.

$$(\forall\alpha\in R)\ 0+\alpha+1\in R$$

which implies

$$(\forall\alpha\in R)(\alpha+1)+1\in R$$

i.e. $1\in R^+$ and hence R^+ is a cut. $\square$

b) $R^+=0$ iff R is a set.

If $R^+=0$, then either $R=0$ or there is $\alpha\in R$ with $\alpha+1\notin R$ and in the second case it is $R=\alpha+1$. $\square$

c) If a cut R itself is closed under the operation $+$, then $R=R^+$. $\square$

(15) $\vartheta+\tau\in R+S\longrightarrow(\vartheta\in R\vee\tau\in S)$.

If $\vartheta\notin R$ and $\tau\notin S$, then for every $\alpha\in R'$ and every $\beta\in S'$ we have $\alpha\leq\vartheta\,\&\,\beta\leq\tau$ and thus

$$\alpha+\beta\leq\vartheta+\tau$$

which implies $\vartheta+\tau\notin R+S$. $\square$

Let us note that the implication $\vartheta\notin R+S\rightarrow(\exists\gamma\notin R)(\exists\delta\notin S)\vartheta\geq\geq\gamma+\delta$ does not hold (e.g. let us choose $\xi\notin FN$ and put $R=\xi\dot{-}FN, S=FN$). However, the following result is available.

(16) If R is a nonempty cut, then

$$\vartheta\notin R+R\rightarrow(\exists\tau\notin R)\tau+\tau<\vartheta.$$

To prove our implication let us choose τ such that

$$2\tau<\vartheta\leq 2(\tau+1).$$

Supposing $\tau\in R$ we would get $\tau+1\in R$ which would imply

$$\vartheta\leq 2(\tau+1)\in R+R$$

and this assertion contradicts the assumption $\vartheta\notin R+R$. $\square$

In particular, if R is a cut closed under the operation + , then the implication

$$\vartheta\notin R\rightarrow(\exists\tau\notin R)\ 2\tau\leq\vartheta$$

is true.

Before we continue our list of properties of the operations + and we are going to state the main theorem of the paper.

Theorem. If R is a real class, then there is $\xi\in N$ so that

$$R=\xi+R^{+}\text{ or }R=\xi\dot{-}R^{+}.$$

Proof. We are going to assume $R^{+}\neq 0$, otherwise there is (by (14)) α with $R=\alpha=\alpha+R^{+}$. At first let us suppose that there is a sequence $\{\vartheta_n;n\in FN\}$ with

$$R=\bigcap\{\vartheta_n;n\in FN\}.$$

Put

$$\tau_n=\vartheta_n\dot{-}\vartheta_{n+1}.$$

We have $R+R^{+}\subseteq R\subseteq\vartheta_0$ (see (5)) and thus $R\subseteq\vartheta_0\dot{-}R^{+}$ according to (8). If $\tau_n\in R^{+}$ for all $n\in FN$, then for every n we have (cf. (14))

$$\sum_{k=0}^{n}\tau_k\in R^{+}.$$

For every $\gamma\notin R$ there is $n\in FN$ with $\vartheta_{n+1}\leq\gamma$ and consequently for this n the equality

$$\vartheta_0\leq\gamma+\sum_{k=0}^{n}\tau_n$$

holds. We have proved the implication

$$(\forall n\in FN)\ \tau_n \in R^+ \rightarrow R= \vartheta_0 + R^+$$

and therefore we are done under the assumption $(\forall n)\ \tau_n \in R^+$. Hence supposing R is a π-semiset we can also assume without loss of generality that $\tau_n \notin R^+$ for all $n\in FN$.

If $\nu\in \bigcap\{\tau_n : n\in FN\}$, then

$$(\forall n\in FN)\ \vartheta_{n+1} < \vartheta_n - \nu$$

and thus

$$(\forall \gamma \notin R)\ \gamma - \nu - 1 \notin R$$

and thence by (12) we get $\nu \in R^+$. We have proved

$$R^+= \bigcap\{\tau_n ; n\in FN\}$$

because we assume

$$(\forall n\in FN)\ \tau_n \notin R^+.$$

Using the last mentioned assumption, for every $n\in FN$ we can choose $\alpha_n\in R'$ with

$$\alpha_n + \tau_n \notin R .$$

R is supposed to be a π-semiset and it is no set because $R^+ \neq 0$, hence $R=R'$ is no σ-semiset, which proves

$$\bigcup\{\alpha_n ; n\in FN\}\subset R.$$

Therefore we are able to choose $\xi\in R$ with

$$(\forall n\in FN)\ \alpha_n < \xi .$$

Evidently $\xi + R^+ \subseteq R$ (we can use (5) and (7)). Let us suppose that there is $\alpha\in R$ with $\alpha \notin \xi + R^+$. In this case we have

$$\alpha - \xi \notin R^+= \bigcap\{\tau_n ; n\in FN\}$$

and hence there is $n\in FN$ with

$$\alpha - \xi \geq \tau_n$$

and furthermore we get

$$\alpha_n + \tau_n < \xi + (\alpha - \xi) = \alpha \in R$$

which contradicts the assumption $\alpha_n + \tau_n \notin R$. We have proved our statement for all π-semisets.

Now, let us assume that there is a sequence $\{\vartheta_n ; n\in FN\}$ with

$$R= \bigcup\{\vartheta_n , n\in FN\}.$$

If there is $n\in FN$ with $R= \vartheta_n + R^+$, then we are done and thus we can suppose without loss of generality that for every $n\in FN$,

$$0 \neq \tau_n = \vartheta_{n+1} - \vartheta_n \notin R^+.$$

For every $\nu\in \bigcap\{\tau_n , n\in FN\}$ we have

$$\vartheta_n + \nu + 1 \leq \vartheta_n + \tau_n = \vartheta_{n+1}$$

and therefore $\nu\in R^+$ according to the definition of R^+ (because $R= \bigcup\{\vartheta_n , n\in FN\}$). We have proved again

$$R^+= \bigcap\{\tau_n , n\in FN\}.$$

Since we are assuming $\tau_n \notin R^+$, for every $n \in FN$ we can choose $\alpha_n \in R$ with

$$\alpha_n + \tau_n + 1 \notin R.$$

R is supposed to be a σ-semiset and it is no set because

$$(\forall n \in FN)\ \vartheta_n < \vartheta_{n+1}$$

and therefore R is no π-semiset, which proves

$$R \neq \bigcap \{\alpha_n + \tau_n, n \in FN\}$$

and thence we are able to choose $\xi \notin R$ with

$$(\forall n \in FN)\ \xi < \alpha_n + \tau_n.$$

Evidently $R = R \underset{\cdot}{-} R^+ \subseteq \xi \underset{\cdot}{-} R^+$ according to (5) and (7). Supposing the existence of $\gamma \notin R$ with

$$(\forall \alpha \in R^+)\ \gamma + \alpha < \xi$$

we would get $\xi - \gamma \notin R^+$ and hence there would be $n \in FN$ with

$$\tau_n \leq \xi - \gamma,$$

however, the relation

$$\alpha_n + \tau_n \leq \gamma + (\xi - \gamma) = \xi$$

would give us a contradiction (we have $\alpha_n < \gamma$ because $\alpha_n \in R$ and $\gamma \notin R$). We have shown our statement for all σ-semisets.

If a segment R is a real class, then there are only three possibilities: either R is a π-semiset or R is a σ-semiset or R=N. Previously we dealt with two possibilities only, however, the remaining one is trivial: we have $N^+ = N$ and $N = 0 + N^+$. □

Let us note that the assumption of the reality of the cut R in the just proved theorem is essential. To show it we are going to construct a (non-real) cut R with $R^+ = FN$ such that there is no $\xi \in N$ with either $R = \xi + FN$ or $R = \xi \underset{\cdot}{-} FN$.

Let $\{\vartheta_\nu ; \nu \in \Omega\}$ be a decreasing sequence with

$$FN = \bigcap \{\vartheta_\nu ; \nu \in \Omega\}$$

and let $\preceq$ be a well-ordering of the universal class V. We shall construct by transfinite induction an increasing sequence $\{\alpha_\nu ; \nu \in \Omega\}$ and an increasing function $\nu \rightarrow \bar{\nu}$ defined on Ω in such a way that for every $\nu, \mu \in \Omega$ we have

$$(*) \quad \nu < \mu \rightarrow \alpha_\nu < \alpha_\mu < \alpha_\mu + \vartheta_{\bar{\mu}} \leq \alpha_\nu + \vartheta_{\bar{\nu}}\ .$$

We put $\alpha_0 = 0$. If α_τ is constructed ($\tau \in \Omega$), then we choose $\alpha_{\tau+1}$ as the smallest natural number α (in the sense of the well-ordering $\preceq$) such that there is $\sigma \in \Omega$ with

$$\alpha_{\tau} < \alpha < \alpha + \vartheta_{\sigma} \leq \alpha_{\tau} + \vartheta_{\tau} ;$$

such a choice is possible because the sequence $\{\vartheta_{\nu};\nu\in\Omega\}$ is supposed to be decreasing; we define $\overline{\tau+1}$ as the smallest $\sigma > \tau$ with the above property.

Let $\tau\in\Omega$ be a limit and let the sequence $\{\alpha_{\nu};\nu\in\Omega\}$ be constructed so that (*) holds for each $\nu,\mu\in\tau\cap\Omega$. The class $\tau\cap\Omega$ is at most countable and therefore there is an increasing sequence $\{\tau_n;n\in FN\}$ with

$$\bigcup\{\tau_n;n\in FN\}=\bigcup(\tau\cap\Omega).$$

By the prolongation axiom there are functions $f,g\subseteq N^2$ with

$$(\forall n\in FN)(f(n)=\alpha_{\tau_n} \,\&\, g(n)=\vartheta_{\overline{\tau}_n}).$$

We choose $\overline{\tau}\in\Omega-\bigcup\{\overline{\tau}_n;n\in FN\}$ and using (*) there is $\delta\notin FN$ so that

$$(\forall\mu\in\delta)(\forall\nu\in\mu)(g(\mu)\geq\vartheta_{\overline{\tau}} \,\&\, f(\nu)<f(\mu)<f(\mu)+g(\mu)\leq f(\nu)+g(\nu)).$$

For every $\nu\in\tau\cap\Omega$ there is $n\in FN$ with $\nu<\tau_n$ and thus

$$\alpha_{\nu}<\alpha_{\tau_n}<f(\delta)<f(\delta)+\vartheta_{\overline{\tau}}\leq f(\delta)+g(\delta)\leq\alpha_{\tau_n}+\vartheta_{\overline{\tau}_n}\leq\alpha_{\nu}+\vartheta_{\overline{\nu}}$$

i.e. we have shown

$$(\exists\alpha)(\forall\nu\in(\tau\cap\Omega))(\alpha_{\nu}<\alpha<\alpha+\vartheta_{\overline{\tau}}\leq\alpha_{\nu}+\vartheta_{\overline{\nu}})$$

and we choose α_{τ} as the smallest α (in the well-ordering $\preceq$) with the property in question.

Evidently

$$R=\{\alpha;(\exists\nu\in\Omega)\alpha\leq\alpha_{\nu}\}$$

is a cut because

$$(\forall\nu\in\Omega)(\alpha_{\nu}<\alpha_{\nu+1})\rightarrow(\forall\alpha\in R)\alpha+1\in R$$

and furthermore the formula

$$(\forall\nu\in\Omega)\ \vartheta_{\nu}\notin R^{+}$$

is implied by the condition (*) and therefore the equality

$$R^{+}=FN$$

is true.

The sequences

$$\{\alpha_{\nu};\nu\in\Omega\} \text{ and } \{\alpha_{\nu}+\vartheta_{\overline{\nu}};\nu\in\Omega\}$$

are monotonous and the equality

$$R=\{\alpha;(\forall\nu\in\Omega)\alpha<\alpha_{\nu}+\vartheta_{\overline{\nu}}\}$$

is a consequence of the condition (*), suitable choices and of the assumption

$$FN=\bigcap\{\vartheta_{\nu};\nu\in\Omega\}$$

Thus $R\subseteq\alpha_0+\vartheta_0$ is neither a π-semiset nor a σ-semiset and therefore it is no real class, hence it can be expressed neither in the form ξ +FN nor in the form $\xi \dot{-}$ FN because all classes expressible in these forms are real.

Our theorem shows that every real cut is either of the form ξ +R or of the form $\xi \dot{-}$ R where R is a cut closed under the operation + (cf. (14)).The following results deal with the uniqueness of these characteristics.

(17) Let R, S be cuts closed under the operation + and let $R \subseteq \xi$ and $S \subseteq \zeta$ & $R \neq 0 \neq S$.

a) If ξ +R= ζ +S, then

$$R=S \text{ and } \xi \dot{-} \zeta \in R \ \& \ \zeta \dot{-} \xi \in R$$

which implies $\frac{\xi}{\zeta} \doteq 1$.

Without loss of generality we can suppose $\zeta \leq \xi$ (which implies $\zeta \dot{-} \xi = 0 \in R$). Since $\xi \in \xi + R = \zeta + S$ we can fix $\beta \in S$ with

$$\xi \leq \zeta + \beta .$$

Under the assumption $\zeta \leq \xi$ the implication

$$\xi + R = \zeta + S \rightarrow R \subseteq S$$

is trivial. Supposing $R \subset S$ we can find $\gamma \in S$ so that $\gamma \notin R$. Evidently,

$$\xi + \gamma \notin \xi + R,$$

however, this formula contradicts the formula

$$\xi + \gamma \leq \zeta + \beta + \gamma \in \zeta + S.$$

We have proved R=S and consequently $\xi \dot{-} \zeta \leq \beta \in R$.

We want to show further the implication

$$(\xi \dot{-} \zeta \in R \ \& \ \zeta \dot{-} \xi \in R \ \& \ R \subseteq \xi \cap \zeta) \rightarrow \frac{\xi}{\zeta} \doteq 1.$$

Without loss of generality we can assume $\zeta \leq \xi$ because

$$\frac{\xi}{\zeta} \doteq 1 \text{ iff } \frac{\zeta}{\xi} \doteq 1.$$

It is $\xi - \zeta \in R$ and thus for every $n \in FN$ we have

$$n(\xi - \zeta) \in R$$

because R is supposed to be closed under the operation + and therefore $\zeta \notin R$ guarantees moreover

$$n(\xi - \zeta) < \zeta .$$

Thus we get

$$0 \leq n\left(\frac{\xi}{\zeta} - 1\right) \leq \frac{n(\xi - \zeta)}{\zeta} < \frac{\zeta}{\zeta} = 1$$

which proves $\frac{\xi}{\zeta} \doteq 1$.

b) If $\xi \dot{-} R = \zeta \dot{-} S$, then

$$R=S \text{ and } \xi \dot{-} \zeta \in R \ \& \ \zeta \dot{-} \xi \in R$$

which implies $\frac{\xi}{\zeta} \doteq 1$.

Again we can suppose $\zeta \leq \xi$ and this assumption and the equality $\xi \div R = \xi \div S$ imply $S \subseteq R$ by (13). Assuming $S \subset R$ we can fix $0 < \delta \in R$ with $\delta - 1 \notin S$ and according to (13) we get

$$\zeta \div \delta \in \zeta \div S = \xi \div R.$$

For each $\alpha \in R$ we have $\alpha + \delta \in R$ and then

$$\zeta + \alpha \leq (\zeta \div \delta) + (\delta + \alpha) < \xi$$

and therefore the assumption $S \subset R$ implies $\zeta \in \xi \div R$ which contradicts $\zeta \notin \zeta \div S$. We have shown R=S. Furthermore we have $\zeta \notin \zeta \div R = \xi \div R$ and therefore there is $\alpha \in R$ such that $\xi \leq \zeta + \alpha$ i.e. $\xi - \zeta \in R$. □

(18) For every $\xi, \zeta \in N$ and for all nonempty cuts R, S closed under the operation + we have

$$\xi + R \neq \zeta \div S.$$

Let us assume R, S are nonempty cuts closed under the operation + and let $\xi + R = \zeta \div S$. We have $\xi < \zeta$ because

$$\xi \in \xi + R = \zeta \div S \ \& \ \zeta \notin \zeta \div S.$$

If $R \subset S$, then there is $\gamma \in S$ with $\gamma \notin R$. By (13)

$$\xi + \gamma \notin \xi + R = \zeta \div S.$$

Thus there is $\beta \in S$ with

$$\xi + \gamma + \beta \geq \zeta.$$

S is assumed to be closed under + and therefore $\gamma + \beta \in S$ which implies

$$\xi \notin \zeta \div S = \xi + R$$

- a contradiction.

If $S \subset R$, then we can fix $\alpha \in R$ such that $\alpha \notin S$. In this case

$$\zeta - \alpha \in \zeta \div S = \xi + R$$

is implied by (13); however, the last formula together with the assumption R is closed under the operation + guarantees

$$\zeta = (\zeta - \alpha) + \alpha \in \xi + R.$$

We have shown that our assumptions imply $\zeta \in \zeta \div S$, which is absurd.

We have proved R=S. If $\zeta - \xi \in R$, then $\zeta = \xi + (\zeta - \xi)$ would be an element of $\xi + R = \zeta \div S$, this proves $\zeta - \xi \notin R$. Thence we can choose $\tau \notin R$ with $2\tau < \zeta - \xi$. Furthermore we have

$$\xi + \tau \in \zeta \div R = \xi + R$$

(because $(\forall \alpha \in R)\, \xi + \tau + \alpha < \xi + 2\tau \leq \xi + (\zeta - \xi) = \zeta$), this contradicts (13). □

The above stated theorem (together with (14)) gives an importance to the results concerning initial segments of the form $\xi + R$ and $\xi \div R$ where R is a cut closed under the operation + (cf. e.g. the following results) because investigating initial segments of those forms we deal with all real initial segments.

(19) Let $\xi, \zeta \in N$ and let R and S be cuts closed under the operation + . Then

a) $$(\xi+R)+(\zeta+S) = \begin{cases} (\xi+\zeta)+R & \text{if } S \subseteq R \\ (\xi+\zeta)+S & \text{if } R \subseteq S \end{cases}$$

because using (4) and (5d) we have

$$(\xi+R)+(\zeta+S)=(\xi+\zeta)+(R+S)=(\xi+\zeta)+(R\cup S). \quad \square$$

b) If $S \subseteq \zeta$, then

$$(\xi+R)+(\zeta \dot{-} S) = \begin{cases} (\xi+\zeta)+R & \text{if } S \subset R \\ (\xi+\zeta) \dot{-} S & \text{if } R \subseteq S. \end{cases}$$

For every R we have

$$(\xi+R)+(\zeta \dot{-} S) \subseteq (\xi+R)+\zeta = (\xi+\zeta)+R \text{ by (1), (4) and (7).}$$

We have to prove the converse inclusion under the assumption $S \subset R$. Let us fix δ with

$$\delta \notin S \;\&\; \delta \leq \zeta \;\&\; \delta \in R'$$

(such a choice is possible because we assumed $S \subseteq \zeta$). We have $R \neq 0$ and therefore (cf. (13))

$$(\xi+\zeta)+R=\{\vartheta; (\exists \alpha \in R)\vartheta < \xi+\zeta+\alpha\}.$$

For every $\alpha \in R$ we have (using (3))

$$\xi+\zeta+\alpha \leq (\xi+\alpha+\delta)+(\zeta-\delta) \in (\xi+R)+(\zeta \dot{-} S),$$

because $\alpha+\delta \in R$ (and thus $\xi+\alpha+\delta \in \xi+R$) and because $\zeta-\delta \in (\zeta \dot{-} S)'$ according to (13). We have proved

$$S \subset R \rightarrow (\xi+\zeta)+R \subseteq (\xi+R)+(\zeta \dot{-} S).$$

Now let us assume $R \subseteq S$ and let

$$\vartheta \in (\xi+R)+(\zeta \dot{-} S).$$

There are $\tau \in (\xi+R)'$ and $\bar{\tau} \in (\zeta \dot{-} S)'$ so that $\vartheta < \tau+\bar{\tau}$ and thus according to (13) there are $\alpha \in R'$ and $\delta \notin S$ with $\delta \leq \zeta$ such that

$$\tau \leq \xi+\alpha \;\&\; \bar{\tau} = \zeta-\delta ,$$

however, using (13) again, we get

$$\vartheta < \tau+\bar{\tau} \leq \xi+\alpha+(\zeta-\delta)=(\xi+\zeta)-(\delta-\alpha) \in ((\xi+\zeta) \dot{-} S)',$$

because $\delta-\alpha \notin S$ (S being closed under the operation +). We have proved

$$(\xi+R)+(\zeta \dot{-} S) \subseteq (\xi+\zeta) \dot{-} S.$$

To prove the converse inclusion it is sufficient to realize that for every δ with $0 < \delta \notin S$ we have (cf. (13); $\delta-1 \notin S$ because S is a cut)

$$(\xi+\zeta) \dot{-} \delta \leq \xi+(\zeta \dot{-} \delta) \;\&\; \xi \in (\xi+R)' \;\&\; (\zeta \dot{-} \delta) \in (\zeta \dot{-} S) \;\cdot$$

and to use (3c). $\square$

c) If $R \subseteq \xi$ and $S \subseteq \zeta$, then

$$(\xi \dot{-} R)+(\zeta \dot{-} S)= \begin{cases} (\xi+\zeta)\dot{-} R & \text{if } S\subseteq R \\ (\xi+\zeta)\dot{-} S & \text{if } R\subseteq S. \end{cases}$$

According to (4) we can assume $R \subseteq S$ and we get

$$(\xi \dot{-} R)+(\zeta \dot{-} S)\subseteq \xi+(\zeta \dot{-} S)=(\xi+\zeta)\dot{-} S$$

as a consequence of (3a), (7) and (19b). If $\delta \notin S$, then there is $\tau \notin S$ with $2\tau \leq \delta$ (cf. (16)) and

$$(\xi+\zeta)\dot{-}\delta \leq (\xi \dot{-} \tau)+(\zeta \dot{-} \tau)\ \&\ \xi \dot{-} \tau \in (\xi \dot{-} S)' \subseteq (\xi \dot{-} R)'\ \&$$
$$\&\ (\zeta \dot{-} \tau) \in (\zeta \dot{-} S)'$$

follows by (7) and (13). Therefore using (3b) and again (13), we obtain

$$((\xi+\zeta)\dot{-} S)' \subseteq ((\xi \dot{-} R)+(\zeta \dot{-} S))'$$

i.e.

$$(\xi+\zeta)\dot{-} S \subseteq (\xi \dot{-} R)+(\zeta \dot{-} S). \quad \square$$

d) If $\zeta+S \subseteq \xi+R$, then

$$(\xi+R)\dot{-}(\zeta+S) = \begin{cases} (\xi \dot{-} \zeta)+R & \text{if } S\subseteq R \\ (\xi \dot{-} \zeta)\dot{-} S & \text{if } R\subset S. \end{cases}$$

Since $\zeta \in (\zeta+S)' \subseteq (\xi+R)'$, we are able to fix $\bar{\alpha} \in R'$ with

$$\zeta \leq \xi + \bar{\alpha}.$$

If $\vartheta \in (\xi \dot{-} \zeta)+R$, then there is $\alpha \in R'$ with $\vartheta < (\xi \dot{-} \zeta)+\alpha$ and for every $\beta \in S'$ we have

$$\vartheta+(\zeta+\beta) < (\xi \dot{-} \zeta)+\alpha+\zeta+\beta \leq \xi+\bar{\alpha}+\alpha+\beta .$$

Assuming $S \subseteq R$ we get $\bar{\alpha}+\alpha+\beta \in R'$ which implies

$$\xi+\bar{\alpha}+\alpha+\beta \in (\xi+R)'$$

and this guarantees

$$\vartheta+(\zeta+\beta) \in \xi+R.$$

We have proved

$$(\xi \dot{-} \zeta)+R \subseteq (\xi+R)\dot{-}(\zeta+S).$$

The converse inclusion is trivial, since for every ϑ, the formula

$$(\forall \beta \in S')(\exists \alpha \in R')\ \vartheta+\zeta+\beta < \xi+\alpha$$

implies

$$(\exists \alpha \in R')\ \vartheta < (\xi \dot{-} \zeta)+\alpha$$

and thus the formula in question implies even the formula

$$\vartheta \in (\xi \dot{-} \zeta)+R.$$

Let us deal with the case $R \subset S$. By (13) every element of $(\xi \dot{-} \zeta)\dot{-} S$ is of the form $(\xi \dot{-} \zeta)\dot{-}\delta$ where $0 < \delta\ \&\ \delta-1 \notin S$. For such δ and every $\beta \in S'$ we have

$$((\xi \dot{-} \zeta) \dot{-} \delta)+(\zeta+\beta) \leq (\xi+\bar{\alpha}) \dot{-} (\delta-\beta) \leq \xi+\bar{\alpha}$$

(it is $\beta \leq \delta$). We have shown the inclusion

$$(\xi \dot{-} \zeta) \dot{-} S \subseteq (\xi+R) \dot{-} (\zeta+S).$$

If $\vartheta \notin (\xi \dot{-} \zeta) \dot{-} S$, then there is $\beta \in S'$ with

$$\vartheta+\beta \geq \xi \dot{-} \zeta$$

and thus

$$\vartheta+(\zeta+\beta) \geq \xi .$$

Choosing $\gamma \in S$ with $\gamma \notin R$ we get

$$\vartheta+(\zeta+\beta)+\gamma \geq \xi+\gamma \notin \xi+R.$$

Since $\zeta+(\beta+\gamma) \in \zeta+S$, we obtain

$$\vartheta \notin (\xi+R) \dot{-} (\zeta+S),$$

thus we have shown the inclusion we had to prove. $\square$

e) If $S \subseteq \zeta$ and if $(\zeta \dot{-} S) \subseteq \xi+R$, then

$$(\xi+R) \dot{-} (\zeta \dot{-} S) = \begin{cases} (\xi \dot{-} \zeta)+R, & \text{if } S \subseteq R \\ (\xi \dot{-} \zeta)+S, & \text{if } R \subseteq S. \end{cases}$$

At first let us prove (assuming $R \cup S \neq 0$) that there is $\tau \in R \cup S$ with

$$\zeta \leq \xi+\tau .$$

If $\xi < \zeta \,\&\, (\zeta-\xi) \notin R \cup S$, then there is $\vartheta \notin R \cup S$ such that

$$\xi+2\vartheta < \zeta$$

($R \cup S$ being a cut closed under the operation +) and therefore for each $\beta \in S'$ we have

$$\xi+\vartheta+\beta < \xi+2\vartheta < \zeta$$

and hence $\xi+\vartheta \in (\zeta \dot{-} S)$. Furthermore $\xi+\vartheta \notin \xi+R$ holds trivially and these facts contradict our assumptions. We have shown that

$$(\xi \dot{-} \zeta)+\zeta \leq \xi+\tau$$

where $\tau \in R \cup S$.

For every $\alpha \in R'$, $\beta \in S'$ and every δ with $\delta \leq \zeta \,\&\, \delta \notin S$ we have

$$(\xi \dot{-} \zeta)+\alpha+\beta)+(\zeta-\delta) \leq ((\xi \dot{-} \zeta)+\zeta)+\alpha-(\delta-\beta) \leq \xi+\tau+\alpha-(\delta-\beta).$$

Evidently $(\delta-1)-\beta \geq 0$ i.e. $\delta-\beta > 0$. If $\tau \in R$, then $\tau+\alpha \in R'$. If $\tau \in S$, then $\delta-\beta-\tau \geq 0$ and therefore in both cases we get

$$((\xi \dot{-} \zeta)+\alpha+\beta)+(\zeta-\delta) \in (\xi+R)'$$

and using (13) we obtain the inclusion

$$(\xi \dot{-} \zeta)+(R \cup S) \subseteq (\xi+R) \dot{-} (\zeta \dot{-} S).$$

If $\vartheta \notin (\xi \dot{-} \zeta)+(R \cup S)$, then there is $\gamma \notin R \cup S$ so that

$$(\xi \dot{-} \zeta)+2\gamma < \vartheta$$

according to (16) and further we can choose $\delta > 0$ so that

$$\delta \notin S \;\&\; \delta \leq \zeta$$

(because $S \subseteq \zeta$) and thus we get

$$\vartheta+(\zeta \dot{-} \delta) > (\xi \dot{-} \zeta)+2\gamma+(\zeta-\delta) \geq \xi+\gamma \notin \xi+R.$$

To prove the inclusion

$$(\xi+R) \dot{-} (\zeta \dot{-} S) \subseteq (\xi \dot{-} \zeta)+(R \cup S)$$

it is sufficient to apply (13). □

f) If $R \subseteq \xi$ and if $\zeta+S \subseteq \xi \dot{-} R$, then

$$(\xi \dot{-} R) \dot{-} (\zeta+S)= \begin{cases} (\xi \dot{-} \zeta) \dot{-} R, & \text{it } S \subseteq R \\ (\xi \dot{-} \zeta) \dot{-} S, & \text{if } R \subseteq S. \end{cases}$$

Under the assumption $R \cup S=0$, our assertion is trivial. Assuming $R \cup S \neq 0$, let us realize at first that

$$\zeta+(R \cup S) \subseteq \xi$$

i.e. the formula

$$(\forall \alpha \in R')(\forall \beta \in S')\zeta+\alpha+\beta \leq \xi$$

is a consequence of $(\zeta+S) \subseteq \xi \dot{-} R$. The equalities

$$(\xi \dot{-} R) \dot{-} (\zeta+S)= \xi \dot{-} (R+(\zeta+S))= \xi \dot{-} (\zeta+(R+S))= \xi \dot{-} (\zeta+(R \cup S))=$$
$$=(\xi+0) \dot{-} (\zeta+(R \cup S))=(\xi \dot{-} \zeta) \dot{-} (R \cup S)$$

are consequences of (6),(4),(5d),(1) and (19d). □

g) If $R \subseteq \xi$, $S \subseteq \zeta$ and if $\zeta \dot{-} S \subseteq \xi \dot{-} R$, then

$$(\xi \dot{-} R) \dot{-} (\zeta \dot{-} S)= \begin{cases} (\xi \dot{-} \zeta) \dot{-} R, & \text{if } S \subset R \\ (\xi \dot{-} \zeta)+S, & \text{it } R \subseteq S. \end{cases}$$

If $S \subset R$, then the equalities

$$(\xi \dot{-} R) \dot{-} (\zeta \dot{-} S)= \xi \dot{-} (R+(\zeta \dot{-} S))= \xi \dot{-} (\zeta+R)=(\xi \dot{-} 0) \dot{-} (\zeta+R)=$$
$$=(\xi \dot{-} \zeta) \dot{-} R$$

hold according to (6),(19b),(1) and (19f). Supposing $R \subseteq S$ we get

$$(\xi \dot{-} R) \dot{-} (\zeta \dot{-} S)= \xi \dot{-} (R+(\zeta \dot{-} S))= \xi \dot{-} (\zeta \dot{-} S)=(\xi+0) \dot{-} (\zeta \dot{-} S)=$$
$$=(\xi \dot{-} \zeta)+S$$

by (6),(19b),(1) and (19e).

We have claimed that there are cuts R, S and $\overline{R}$, $\overline{S}$ such that

$$(R \dot{-} S)+S \neq R \text{ and } (\overline{R}+\overline{S}) \dot{-} \overline{S} \neq \overline{R},$$

using the last statement we can construct such cuts quite easily. If $T \subset U \subset \xi$ are cuts closed under the operation $+$, then putting

$R= \xi+T$	$S= \xi \dot{-} T$
$\overline{R}= \xi \dot{-} T$	$\overline{S}= \xi \dot{-} T$
$\tilde{R}= \xi+T$	$\tilde{S}=U$

we have

$$(R \dot{-} S)+S \neq R \qquad (R+S) \dot{-} S=R$$
$$(\overline{R} \dot{-} \overline{S})+\overline{S}=\overline{R} \qquad (\overline{R}+\overline{S}) \dot{-} \overline{S} \neq \overline{R}$$
$$(\widetilde{R} \dot{-} \widetilde{S})+\widetilde{S} \neq \widetilde{R} \qquad (\widetilde{R}+\widetilde{S}) \dot{-} \widetilde{S} \neq \widetilde{R}.$$

In fact, using (19) we get

$$((R \dot{-} S)+S=((\xi+T) \dot{-} (\xi \dot{-} T))+(\xi \dot{-} T)=T+(\xi \dot{-} T)=\xi \dot{-} T \neq R(R+S) \dot{-} S=$$
$$(R+S) \dot{-} S=((\xi+T)+(\xi \dot{-} T)) \dot{-} (\xi \dot{-} T)=(2\xi \dot{-} T) \dot{-} (\xi \dot{-} T)=\xi+T=R$$
$$(\overline{R} \dot{-} \overline{S})+\overline{S}=((\xi \dot{-} T) \dot{-} (\xi \dot{-} T))+(\xi \dot{-} T)=T+(\xi \dot{-} T)=\xi \dot{-} T=\overline{R}$$
$$(\overline{R}+\overline{S}) \dot{-} \overline{S}=((\xi \dot{-} T)+(\xi \dot{-} T)) \dot{-} (\xi \dot{-} T)=(2\xi \dot{-} T) \dot{-} (\xi \dot{-} T)=\xi+T \neq \overline{R}$$
$$(\widetilde{R} \dot{-} \widetilde{S})+\widetilde{S}=((\xi+T) \dot{-} U)+U=(\xi \dot{-} U)+U=\xi \dot{-} U \neq \widetilde{R}$$
$$(\widetilde{R}+\widetilde{S}) \dot{-} \widetilde{S}=((\xi+T)+U) \dot{-} U=(\xi+U) \dot{-} U=\xi+U \neq \widetilde{R}.$$

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Math. Institute of Czechoslovak Acad. of Sci., Žitná 25, 110 00 praha 1, Czechoslovakia

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